

**EXCERPT FROM
ON SOME ACTIONS OF STABLY ELEMENTARY MATRICES
ON
ALTERNATING MATRICES**

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ABSTRACT. This is an excerpt from a paper still in preparation. We show that there are examples of 2-stably elementary 3×3 matrices over a 4 dimensional ring, whose first row is not completable to an elementary matrix. We also give examples of 2-stably elementary matrices ρ of size $(2n - 1)$, with $(1 \perp \rho) \notin E_{2n}(A)Sp_{2n}(A)$.

1. INTRODUCTION

The leit motif of this paper is the following beautiful lemma of L.N. Vaserstein: Let ρ be an odd sized $(2n - 1)$, $n > 1$, invertible matrix which is such that $1 \perp \rho$ is an elementary matrix (i.e. a product of elementary generators $e_{ij}(\lambda)$, $i \neq j$). Then the first row $e_1\rho$ of ρ can be completed to an elementary matrix. Let $\psi_n \in \text{SL}_{2n}(\mathbb{Z})$ denote the standard alternating matrix of Pfaffian 1 obtained by taking a direct sum of n copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. One can even show above that $(1 \perp \rho)$ acts like an elementary matrix of size $(2n - 1)$ on ψ_n , i.e.

$$(1 \perp \rho)^\top \psi_n (1 \perp \rho) = (1 \perp \epsilon)^\top \psi_n (1 \perp \epsilon),$$

for some $\epsilon \in E_{2n-1}(R)$. We show that there are examples of 2-stably elementary matrices ρ , of size 3×3 , over a 4 dimensional ring A , whose first row is not completable to an elementary matrix. The main counter examples in this paper show that there is a commutative ring of dimension $2n - 1$, and a 2-stably elementary matrix ρ of size $(2n - 1)$, with $(1 \perp \rho) \notin E_{2n}(A)Sp_{2n}(A)$, for $(2n - 1) \equiv 3$ modulo 8. In particular, this means, that $(1 \perp \rho)$ does not act like an elementary matrix, in general.

2. SOME BASIC RESULTS FROM TOPOLOGY.

We recall here a few simple facts from topology. As usual if X and Y are topological spaces we let $[X, Y]$ be the set of homotopy classes of maps from X to Y and let $[(X, x_0), (Y, y_0)]$ be the set of homotopy classes preserving the base points. The following well-known result shows that we do not have to worry about base points in the cases to be considered.

Lemma 2.1. *Let X be a finite simplicial complex. If Y is simply connected or is a path connected topological group then $[(X, x_0), (Y, y_0)] \rightarrow [X, Y]$ is an isomorphism.*

Since this is usually only proved for the case where X is a sphere, using the action of the fundamental group on the higher homotopy groups, we will give a proof here.

Proof. To show the map is onto, suppose $f : X \rightarrow Y$ is given. Join $f(x_0)$ to y_0 by a path C and define a homotopy h of $f|_{\{x_0\}}$ by $h_t(x_0) = C(t)$. Extend this to a homotopy H of f by the homotopy extension theorem [9, Ch. I, Prop. 9.2]. Then $f = H_0$ is homotopic to H_1 which preserves the base points. Therefore the map is onto.

To show the map is injective, let $f, g : (X, x_0) \rightarrow (Y, y_0)$ be homotopic as maps $X \rightarrow Y$ by $h : X \times I \rightarrow Y$. If Y is a topological group let $H_t(x) = y_0 h_t(x_0)^{-1} h_t(x)$. Then $H : f \simeq g$ preserving the base point. This argument is taken directly from [9, Ch. IV, Prop. 16.10].

If Y is simply connected and $h : f \simeq g$ as above, then $t \mapsto h_t(x_0)$ is a loop at y_0 and is therefore contractible. Let σ_s , $0 \leq s \leq 1$, be a homotopy between this loop and the constant map y_0 so $\sigma_0(t) = h_t(x_0)$, $\sigma_1(t) = y_0$, and $\sigma_s(0) = \sigma_s(1) = y_0$. Define a homotopy k_s of $h|(X \times 0) \cup (x_0 \times I) \cup (X \times 1)$ by $k_s|(X \times 0) = f$, $k_s|(x_0 \times I) = \sigma_s$, and $k_s|(X \times 1) = g$. Extend this to a homotopy H_s of h . Then $H_1 : X \times I \rightarrow Y$ is a homotopy between f and g which preserves the base point. \square

In particular, $[S^m, Y] = \pi_m(Y)$ if Y is as in the lemma. The lemma holds more generally for a CW complex X since the homotopy extension theorem holds for such complexes [8, Ch. VII, Th. 1.4].

If X is a topological space let $C(X, \mathbb{R})$ be the ring of continuous real functions on X and let $C(X, \mathbb{C})$ be the ring of continuous complex functions on X . Clearly the space of mappings $X \rightarrow \mathrm{SL}_n(\mathbb{R})$ can be identified with $\mathrm{SL}_n(C(X, \mathbb{R}))$ and similarly for \mathbb{C} . The following is well-known in the case $A = C(X, \mathbb{R})$ [2, Ch. XIV, §5], [10, §7].

Lemma 2.2. *Let X be a compact Hausdorff space and let A be a subring of $C(X, \mathbb{R})$ containing \mathbb{R} and which is dense in $C(X, \mathbb{R})$. Assume that any a in A with no zeros on X is a unit of A . Then the maps*

$$\mathrm{SL}_n(A)/\mathrm{E}_n(A) \rightarrow \mathrm{SL}_n(C(X, \mathbb{R}))/\mathrm{E}_n(C(X, \mathbb{R})) \rightarrow [X, \mathrm{SL}_n(\mathbb{R})]$$

are isomorphisms. A similar statement holds for A in $C(X, \mathbb{C})$.

Proof. It suffices to show that $\mathrm{SL}_n(A)/\mathrm{E}_n(A) \rightarrow [X, \mathrm{SL}_n(\mathbb{R})]$ is an isomorphism since we can take $A = C(X, \mathbb{R})$. We use the following fact

(*) If $\sigma \in \mathrm{SL}_n(A)$ is sufficiently close to 1 then $\sigma \in \mathrm{E}_n(A)$.

This is proved in [10, Lemma 7.4]. It uses only the property that an element of A sufficiently near 1 is a unit.

The map is well defined since if σ and τ lie in $\mathrm{SL}_n(A)$ and are congruent modulo $\mathrm{E}_n(A)$ then $\sigma = \tau \prod e_{i_k j_k}(f_k)$, so $\sigma_t = \tau \prod e_{i_k j_k}(t f_k)$ gives a homotopy between σ and τ as maps $X \rightarrow \mathrm{SL}_n(\mathbb{R})$.

For the onto-ness, let $\sigma : X \rightarrow \mathrm{SL}_n(\mathbb{R})$ and write $\sigma = (f_{ij})$ with entries f_{ij} in $C(X, \mathbb{R})$. Let $\rho = (g_{ij})$ with entries g_{ij} in A very close to f_{ij} and let $\rho_t = (1-t)\sigma + t\rho$. This is still very close to σ for $0 \leq t \leq 1$ so $d_t = \det \rho_t$ is very close to $\det \sigma = 1$ and therefore is a unit of A . Let $\tau_t = \mathrm{diag}(d_t^{-1}, 1, \dots, 1)\rho_t$. This lies in $\mathrm{SL}_n(C(X, \mathbb{R}))$ and gives a homotopy of σ with $\tau = \tau_1 = \mathrm{diag}(d_1^{-1}, 1, \dots, 1)\rho$ which lies in $\mathrm{SL}_n(A)$.

For the injectivity, suppose σ and τ lie in $\mathrm{SL}_n(A)$ and are homotopic when considered as maps $X \rightarrow \mathrm{SL}_n(\mathbb{R})$. Let $h_t : X \rightarrow \mathrm{SL}_n(\mathbb{R})$ be a homotopy between them. Since X is compact we can find a sequence $0 = t_0 < t_1 < \dots < t_N = 1$ such that if $h_i = h_{t_i}$, the matrices $h_i^{-1}h_{i+1}$ are arbitrarily close to 1 so that, by (*), each of these matrices lies in $\mathrm{E}_n(C(X, \mathbb{R}))$. Therefore $\tau = \sigma \epsilon$ where $\epsilon = \prod_0^N (h_i^{-1}h_{i+1})$

lies in $E_n(C(X, \mathbb{R}))$. Write $\epsilon = \prod e_{i_k j_k}(f_k)$. Choose g_k in A very close to f_k . Then $\eta = \prod e_{i_k j_k}(g_k)$ is very close to ϵ . Now $\tau^{-1}\sigma\eta$ is very close to $\tau^{-1}\sigma\epsilon = 1$ and therefore lies in $E_n(A)$ and hence so does $\tau^{-1}\sigma$. \square

The following is a special case of a result of Vaserstein [13]. Recall that the stable dimension $\text{sdim } A$ is defined to be one less than the stable range of A . Bass' stability theorem says that $\text{sdim } A \leq \dim A$ for commutative noetherian rings A .

Lemma 2.3 (Vaserstein). *Let X be a compact Hausdorff space and let A be a subring of $C(X, \mathbb{R})$ containing \mathbb{R} . Assume that any a in A with no zeros on X is a unit of A . Then $\text{sdim } \mathbb{C} \otimes_{\mathbb{R}} A \leq \lfloor \frac{1}{2} \text{sdim } A \rfloor$*

Vaserstein notes that under the hypothesis a row (c_1, \dots, c_n) is unimodular if and only if $\sum |c_n|^2$ is never 0 and hence is a unit. Write $c_n = a_n + ib_n$. Then $(a_1, b_1, \dots, a_{n-1}, b_{n-1}, |c_n|^2)$ is unimodular so if $2n - 1 \geq 2 + \text{sdim } A$ we can make $(a_1, b_1, \dots, a_{n-1}, b_{n-1})$ unimodular by adding multiples of $|c_n|^2$ to its entries. Therefore the same is true for (c_1, \dots, c_{n-1}) . The result follows since the condition on n is equivalent to $n \geq 2 + \lfloor \frac{1}{2} \text{sdim } A \rfloor$.

We conclude this section by recording the following classical result.

Lemma 2.4. *The following inclusions are homotopy equivalences.*

- (1) $\text{SU}(m) \subset \text{SL}_m(\mathbb{C})$
- (2) $\text{Sp}(m) \subset \text{Sp}_{2m}(\mathbb{C})$
- (3) $\text{SO}(m) \subset \text{SL}_m(\mathbb{R})$
- (4) $\text{U}(m) \subset \text{Sp}_{2m}(\mathbb{R})$

In fact, in each case, the larger group is the product of the smaller group with a Euclidean space.

Items (1) and (2) are given in [6, Ch. VI, §X]. For (3) and (4) we can use [7, Ch. IX, Lemma 4.3] from which (3) is clear. For (4) we also use [7, Ch. IX, Lemma 4.1(c)].

The inclusion $\text{U}(m) \subset \text{Sp}_{2m}(\mathbb{R})$ in (4) is obtained by observing that $\text{U}(m)$ preserves the (real) alternating form $\Im(z, w)$. Similarly, the inclusion $\text{U}(m) \subset \text{SO}(2m)$ is obtained by observing that $\text{U}(m)$ preserves $\Re(z, w)$ which is the usual inner product on \mathbb{R}^{2m} . Therefore we have a commutative diagram

$$\begin{array}{ccc} \text{U}(m) & \xrightarrow{\subset} & \text{SO}(2m) \\ \downarrow & & \downarrow \\ \text{Sp}_{2m}(\mathbb{R}) & \xrightarrow{\subset} & \text{SL}_{2m}(\mathbb{R}) \end{array}$$

showing that, up to homotopy, $\text{Sp}_{2m}(\mathbb{R}) \rightarrow \text{SL}_{2m}(\mathbb{R})$ looks just like $\text{U}(m) \rightarrow \text{SO}(2m)$.

3. THE MAIN COUNTEREXAMPLE

Let A be a noetherian ring. Let ρ be an element of $\text{SL}_{2n-1}(A)$ which is 2-stably elementary i.e. $I_2 \perp \rho \in E_{2n+1}(A)$. If $\dim A \leq 2n - 2$ then $\text{SL}_{2n}(A)/E_{2n}(A) \rightarrow \text{SK}_1(A)$ is an isomorphism by Vaserstein's stability theorem [12] and therefore $1 \perp \rho \in E_{2n}(A)$. Therefore, in the following counterexample, the dimension of A is as small as possible.

The condition $(1 \perp \rho)^\top \psi_n(1 \perp \rho) = (1 \perp \epsilon)^\top \psi_n(1 \perp \epsilon)$ mentioned above is equivalent to $\rho\epsilon^{-1} \in \text{Sp}_n(A)$ or $\rho \in E_{2n-1}(A)\text{Sp}_{2n}(A)$

Theorem 3.1. *If $2n - 1 \equiv 3 \pmod{8}$, there is an affine domain A over \mathbb{R} of dimension $2n - 1$ and a 2-stably elementary element $\rho \in \mathrm{SL}_{2n-1}(A)$ such that $(1 \perp \rho) \notin \mathrm{E}_{2n}(A)\mathrm{Sp}_{2n}(A)$. Moreover, if $n = 2$, we can choose ρ in such a way that its first row is not completable to an elementary matrix or, equivalently, $e_1\rho$ is not elementarily equivalent to e_1*

We begin by offering a simpler example which has a weaker conclusion but works for all $2n - 1 \geq 3$. In the following theorem we have replaced the condition $\dim A = 2n - 1$ by $\mathrm{sdim} A = 2n$ and have replaced \mathbb{R} by \mathbb{C} .

Theorem 3.2. *If $n \geq 2$ there is a regular domain A , essentially of finite type over \mathbb{C} , of stable dimension $\leq 2n$, and a 2-stably elementary element $\rho \in \mathrm{SL}_{2n-1}(A)$ such that $(1 \perp \rho) \notin \mathrm{E}_{2n}(A)\mathrm{Sp}_{2n}(A)$. Moreover, if $n = 2$, we can choose ρ in such a way that its first row is not completable to an elementary matrix or, equivalently, $e_1\rho$ is not elementarily equivalent to e_1*

The proof of these theorems is topological.

We begin with the proof of Theorem 3.2 which is considerably simpler. We construct the example using a subring A of $C(S^{4n}, \mathbb{C})$, the ring of continuous complex functions on the $4n$ -sphere. We assume that A satisfies the conditions of Lemma 2.2.

The problem can be rephrased as follows: Find ρ in $\mathrm{SL}_{2n-1}(A)/\mathrm{E}_{2n-1}(A)$ whose image in $\mathrm{SL}_{2n+1}(A)/\mathrm{E}_{2n+1}(A)$ is 0 and whose image in $\mathrm{SL}_{2n}(A)/\mathrm{E}_{2n}(A)$ does not lie in the image of $\mathrm{Sp}_{2n}(A)$. By Lemma 2.2 and Lemma 2.1, we have $\mathrm{SL}_m(A)/\mathrm{E}_m(A) = [S^{4n}, \mathrm{SL}_m(\mathbb{C})] = \pi_{4n}(\mathrm{SL}_m(\mathbb{C}))$. We can replace $\mathrm{SL}_m(\mathbb{C})$ by $\mathrm{SU}(m)$ which has the same homotopy type by Lemma 2.4. The map $\mathrm{Sp}_{2n}(A) \rightarrow \mathrm{SL}_{2n}(A)/\mathrm{E}_{2n}(A) = [S^{4n}, \mathrm{SL}_{2n}(\mathbb{C})]$ factors as

$$\mathrm{Sp}_{2n}(A) \rightarrow \mathrm{Sp}_{2n}(C(S^{4n}, \mathbb{C})) \rightarrow [S^{4n}, \mathrm{Sp}_{2n}(\mathbb{C})] = \pi_{4n}(\mathrm{Sp}_{2n}(\mathbb{C})) \rightarrow \pi_{4n}(\mathrm{SL}_{2n}(\mathbb{C}))$$

so it will suffice to find a ρ whose image in $\pi_{4n}(\mathrm{SL}_{2n}(\mathbb{C}))$ is not in the image of $\pi_{4n}(\mathrm{Sp}_{2n}(\mathbb{C}))$. We can replace $\mathrm{Sp}_{2n}(\mathbb{C})$ by $\mathrm{Sp}(n)$ which has the same homotopy type as $\mathrm{Sp}_{2n}(\mathbb{C})$ by Lemma 2.4 and similarly replace $\mathrm{SL}_{2n}(\mathbb{C})$ by $\mathrm{SU}(2n)$.

Our problem now appears as follows. Consider the diagram

$$\begin{array}{ccccc} \pi_{4n}(\mathrm{SU}(2n-1)) & \longrightarrow & \pi_{4n}(\mathrm{SU}(2n)) & \longrightarrow & \pi_{4n}(\mathrm{SU}(2n+1)) \\ & & \uparrow & & \\ & & \pi_{4n}(\mathrm{Sp}(n)) & & \end{array}$$

and find ρ in $\pi_{4n}(\mathrm{SU}(2n-1))$ whose image in $\pi_{4n}(\mathrm{SU}(2n+1))$ is 0 and whose image in $\pi_{4n}(\mathrm{SU}(2n))$ does not lie in the image of $\pi_{4n}(\mathrm{Sp}(n))$. The first part is no problem since $\pi_{4n}(\mathrm{SU}(2n+1))$ is stable and is 0 by Bott's calculations [4]. By [3], $\pi_{4n}(\mathrm{SU}(2n)) = \mathbb{Z}/(2n)!\mathbb{Z}$. The homotopy sequence of the fibration $\mathrm{SU}(2n-1) \rightarrow \mathrm{SU}(2n) \rightarrow S^{4n-1}$ gives us

$$\pi_{4n}(\mathrm{SU}(2n-1)) \rightarrow \pi_{4n}(\mathrm{SU}(2n)) \rightarrow \pi_{4n}(S^{4n-1}) = \mathbb{Z}/2\mathbb{Z}$$

Therefore the image of $\pi_{4n}(\mathrm{SU}(2n-1)) \rightarrow \pi_{4n}(\mathrm{SU}(2n))$ has order either $(2n)!$ or $(2n)!/2$ and is therefore greater than 2. But $\pi_{4n}(\mathrm{Sp}(n))$ is stable and is either 0 or $\mathbb{Z}/2\mathbb{Z}$ by [4]. This shows that the required ρ exists. For example, we could take ρ to have order 3.

For the last part of the theorem we want to choose ρ so that $\text{Row}_1(\rho) = e_1\rho$ is not elementarily completable or, equivalently such that $e_1\rho \neq e_1\epsilon$ for any $\epsilon \in E_{2n-1}(A)$. In other words, we want to find a ρ in $\text{SL}_{2n-1}(A)/E_{2n-1}(A)$ whose image in $\text{SL}_{2n+1}(A)/E_{2n+1}(A)$ is trivial and such that the image $\text{Row}_1(\rho)$ in $\text{Um}_{2n-1}(A)/E_{2n-1}(A)$ is non-trivial. As above we can restate this as follows: Find ρ in $\pi_{4n}(\text{SU}(2n-1))$ whose image in $\pi_{4n}(\text{SU}(2n+1))$ is 0 and whose image in $\pi_{4n}(S^{4n-3})$ is non-zero. We use here the fact that $\text{Row}_1 : \text{SL}_{2n-1}(\mathbb{C}) \rightarrow \mathbb{C}^{2n-1} - \{0\}$ restricts to $\text{SU}(2n-1) \rightarrow S^{4n-3}$. The first part is clear since $\pi_{4n}(\text{SU}(2n+1)) = 0$. The homotopy sequence of the fibration $\text{SU}(2n-2) \rightarrow \text{SU}(2n-1) \rightarrow S^{4n-3}$ gives $\pi_{4n}(\text{SU}(2n-2)) \rightarrow \pi_{4n}(\text{SU}(2n-1)) \rightarrow \pi_{4n}(S^{4n-3}) \rightarrow \pi_{4n-1}(\text{SU}(2n-2))$. We restrict to the case $n = 2$ where the required homotopy groups are well-known. Since $\text{SU}(2) \cong S^3$, our sequence becomes $\pi_8(S^3) \rightarrow \pi_8(\text{SU}(3)) \rightarrow \pi_8(S^5) \rightarrow \pi_7(S^3)$. Now $\pi_7(S^3) = \mathbb{Z}/2\mathbb{Z}$ and $\pi_8(S^5) = \mathbb{Z}/24\mathbb{Z}$ [9, Ch. XI, Th. 16.4, Th. 17.1]. It follows that there is a ρ in $\pi_8(\text{SU}(3))$ mapping to an element of order 3 in $\pi_8(S^5)$ as required. Since $\pi_8(S^3) = \mathbb{Z}/2\mathbb{Z}$ [9, Ch. XI, §18], any ρ of order 3 in $\pi_8(\text{SU}(3))$ will do.

Finally, we choose A as follows. Consider S^{4n} as the set of $x \in \mathbb{R}^{4n+1}$ satisfying $\sum_0^{4n} x_i^2 = 1$. Let $B = \mathbb{R}[x_0, \dots, x_{4n}]/(\sum_0^{4n} x_i^2 - 1)$ be the ring of real polynomial functions on S^{4n} . Let $S \subset B$ be the set of such functions with no zeros on S^{4n} and let $C = B_S = B[S^{-1}]$. Let $A = \mathbb{C} \otimes_{\mathbb{R}} C = \mathbb{C}[x_0, \dots, x_{4n}]/(\sum_0^{4n} x_i^2 - 1)[S^{-1}]$. Then A is dense in $C(S^{4n}, \mathbb{C})$ by the Stone–Weierstrass theorem. It is clearly regular and $\text{sdim } A \leq \frac{1}{2} \text{sdim } C \leq 2n$ by Lemma 2.3.

Proof of Theorem 3.1. For this proof we use homotopy groups with coefficients [5]. Let Y be a Moore space $Y = e^m \cup_d S^{m-1}$. The group $\pi_m(X, x_0; \mathbb{Z}/d)$ is the group of homotopy classes of basepoint preserving maps $(Y, y_0) \rightarrow (X, x_0)$. As above we will ignore the basepoints using Lemma 2.1. The homotopy sequence of a fibration also holds for these groups which are related to the ordinary homotopy groups by an exact sequence

$$(1) \quad \dots \rightarrow \pi_m(X) \xrightarrow{d} \pi_m(X) \rightarrow \pi_m(X, \mathbb{Z}/d) \rightarrow \pi_{m-1}(X) \xrightarrow{d} \pi_{m-1}(X) \rightarrow \dots$$

For the proof of theorem 3.1, we let $d = 2$ and let $Y = e^{2n-1} \cup_2 S^{2n-2}$. Let $A \subseteq C(Y, \mathbb{R})$ be as in Lemma 2.2. As above it will suffice to find ρ in $[Y, \text{SO}(2n-1)] = \pi_{2n-1}(\text{SO}(2n-1), \mathbb{Z}/2)$ such that ρ maps to 0 in $\pi_{2n-1}(\text{SO}(2n+1), \mathbb{Z}/2)$ and such that the image of ρ in $\pi_{2n-1}(\text{SO}(2n), \mathbb{Z}/2)$ does not lie in the image of $\pi_{2n-1}(\text{U}(n), \mathbb{Z}/2)$ (using Lemma 2.4).

Let $T : S^{2n-1} \rightarrow \text{SO}(2n)$ be the characteristic map of the fibration

$$(2) \quad \text{SO}(2n) \rightarrow \text{SO}(2n+1) \rightarrow S^{2n}$$

defined in [11, §23.2]. Let π be the projection in the bundle

$$(3) \quad \text{SO}(2n-1) \rightarrow \text{SO}(2n) \xrightarrow{\pi} S^{2n-1}$$

and let $\Sigma : \pi_{2n-1}(S^{2n-1}) \rightarrow \pi_{2n}(S^{2n})$ be the suspension. The diagram

$$(4) \quad \begin{array}{ccc} \pi_{2n}(S^{2n}) & \xrightarrow{\partial} & \pi_{2n-1}(\text{SO}(2n)) \\ \Sigma \uparrow \approx & \nearrow T_* & \downarrow \pi_* \\ \pi_{2n-1}(S^{2n-1}) & \xrightarrow{2} & \pi_{2n-1}(S^{2n-1}) \end{array}$$

commutes since the northwest triangle commutes by [11, §23.2] and the southeast triangle commutes by [11, Th. 23.4]. It follows that in the diagram

$$(5) \quad \begin{array}{ccccc} \pi_{2n}(S^{2n}) & \xrightarrow{\partial} & \pi_{2n-1}(\mathrm{SO}(2n)) & \xrightarrow{\pi_*} & \pi_{2n-1}(S^{2n-1}) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_{2n}(S^{2n}, \mathbb{Z}/2) & \xrightarrow{\partial} & \pi_{2n-1}(\mathrm{SO}(2n), \mathbb{Z}/2) & \xrightarrow{\pi_*} & \pi_{2n-1}(S^{2n-1}, \mathbb{Z}/2) \end{array}$$

the composition of the upper maps is multiplication by 2 so the same is true of the composition of the lower maps which is therefore 0. The generator ι of $\pi_{2n}(S^{2n}, \mathbb{Z}/2)$ maps to an element $\alpha = \partial\iota$ of $\pi_{2n-1}(\mathrm{SO}(2n), \mathbb{Z}/2)$, and $\pi_*\alpha = 0$ in $\pi_{2n-1}(S^{2n-1}, \mathbb{Z}/2)$. The homotopy sequence

$$\cdots \rightarrow \pi_{2n-1}(\mathrm{SO}(2n-1), \mathbb{Z}/2) \rightarrow \pi_{2n-1}(\mathrm{SO}(2n), \mathbb{Z}/2) \rightarrow \pi_{2n-1}(S^{2n-1}, \mathbb{Z}/2) \rightarrow \cdots$$

of the fibration (3) shows that α is the image of an element ρ of $\pi_{2n-1}(\mathrm{SO}(2n-1), \mathbb{Z}/2)$. The homotopy sequence

$$\cdots \rightarrow \pi_{2n}(S^{2n}, \mathbb{Z}/2) \rightarrow \pi_{2n-1}(\mathrm{SO}(2n), \mathbb{Z}/2) \rightarrow \pi_{2n-1}(\mathrm{SO}(2n+1), \mathbb{Z}/2) \rightarrow \cdots$$

of the fibration (2) shows that $\alpha (= \partial\iota)$, and therefore also ρ , maps to 0 in $\pi_{2n-1}(\mathrm{SO}(2n+1), \mathbb{Z}/2)$. It remains to show that α does not lie in the image of $\pi_{2n-1}(\mathrm{U}(n), \mathbb{Z}/2) \rightarrow \pi_{2n-1}(\mathrm{SO}(2n), \mathbb{Z}/2)$ if $2n-1 \equiv 3 \pmod{8}$.

The homotopy sequence of the fibration $\mathrm{U} \rightarrow \mathrm{SO} \rightarrow \mathrm{U}/\mathrm{SO}$ and Bott's calculations [4, 1.7] show that $\pi_m(\mathrm{U}) \rightarrow \pi_m(\mathrm{SO})$ is onto for $m \equiv 3 \pmod{8}$ and is therefore an isomorphism since both groups are \mathbb{Z} [4]. Since $\pi_{2n-1}(\mathrm{U}) = \mathbb{Z}$ for $n \geq 2$ and $\pi_{2n-2}(\mathrm{U}) = 0$, (1) shows that $\pi_{2n-1}(\mathrm{U}, \mathbb{Z}/2) = \pi_{2n-1}(\mathrm{U})/2$. Moreover, the map from (1) for $\mathrm{U}(n)$ to (1) for U is an isomorphism in these dimensions so $\pi_{2n-1}(\mathrm{U}(n), \mathbb{Z}/2) \rightarrow \pi_{2n-1}(\mathrm{U}, \mathbb{Z}/2)$ is an isomorphism. Consider the diagram

$$\begin{array}{ccccc} \pi_{2n-1}(\mathrm{U}(n), \mathbb{Z}/2) & \xrightarrow{\approx} & \pi_{2n-1}(\mathrm{U}, \mathbb{Z}/2) & \xleftarrow{\approx} & \pi_{2n-1}(\mathrm{U})/2 \quad \text{---} \quad \mathbb{Z}/2 \\ \downarrow & & \downarrow & & \approx \downarrow \\ \pi_{2n-1}(\mathrm{SO}(2n), \mathbb{Z}/2) & \longrightarrow & \pi_{2n-1}(\mathrm{SO}, \mathbb{Z}/2) & \longleftarrow & \pi_{2n-1}(\mathrm{SO})/2 \end{array}$$

Since α maps to 0 in $\pi_{2n-1}(\mathrm{SO}, \mathbb{Z}/2)$, we see that if α lies in the image of $\pi_{2n-1}(\mathrm{U}(n), \mathbb{Z}/2)$, then $\alpha = 0$ so we need to show that α is non-trivial.

The homotopy sequence of the fibration (2) gives us

$$(6) \quad 0 \rightarrow \pi_{2n}(S^{2n}) \xrightarrow{\partial} \pi_{2n-1}(\mathrm{SO}(2n)) \rightarrow \pi_{2n-1}(\mathrm{SO}(2n+1)) \rightarrow 0$$

since diagram (4) shows that ∂ is injective and $\pi_{2n-1}(S^{2n}) = 0$. Now $\pi_{2n-1}(\mathrm{SO}(2n+1)) = \pi_{2n-1}(\mathrm{SO}) = \mathbb{Z}$ if $2n-1 \equiv 3 \pmod{8}$ by Bott's calculations [4] so (6) splits and we get a diagram in which the top map is a split monomorphism

$$\begin{array}{ccc} \pi_{2n}(S^{2n})/2 & \longrightarrow & \pi_{2n-1}(\mathrm{SO}(2n))/2 \\ \approx \downarrow & & \downarrow \\ \pi_{2n}(S^{2n}, \mathbb{Z}/2) & \longrightarrow & \pi_{2n-1}(\mathrm{SO}(2n), \mathbb{Z}/2). \end{array}$$

The right hand vertical map is injective by (1). The generator of $\pi_{2n}(S^{2n})/2$ maps to ι which maps to α so it follows that $\alpha \neq 0$.

For the last statement we want, as above, to find ρ in $\pi_{2n-1}(\mathrm{SO}(2n-1), \mathbb{Z}/2)$ such that ρ maps to 0 in $\pi_{2n-1}(\mathrm{SO}(2n+1), \mathbb{Z}/2)$ but whose image in $\pi_{2n-1}(S^{2n-2}, \mathbb{Z}/2)$ (which is represented by $\mathrm{Row}_1(\rho) = \pi_*(\rho)$) is non-zero. This map occurs in the homotopy sequence

$$(7) \quad \cdots \rightarrow \pi_{2n-1}(\mathrm{SO}(2n-2), \mathbb{Z}/2) \rightarrow \pi_{2n-1}(\mathrm{SO}(2n-1), \mathbb{Z}/2) \rightarrow \pi_{2n-1}(S^{2n-2}, \mathbb{Z}/2) \rightarrow \pi_{2n-2}(\mathrm{SO}(2n-2), \mathbb{Z}/2) \rightarrow \cdots$$

of $\mathrm{SO}(2n-2) \rightarrow \mathrm{SO}(2n-1) \rightarrow S^{2n-2}$. As above we consider the case $n = 2$ where this homotopy sequence becomes

$$0 = \pi_3(\mathrm{SO}(2), \mathbb{Z}/2) \rightarrow \pi_3(\mathrm{SO}(3), \mathbb{Z}/2) \xrightarrow{\pi_*} \pi_3(S^2, \mathbb{Z}/2) \rightarrow \pi_2(\mathrm{SO}(2), \mathbb{Z}/2) = 0$$

so π_* is an isomorphism and any non-zero ρ will have a non-zero image.

This concludes the proof except for the choice of A which can be done as follows.

Let $E^{2n-1} = \{x \in \mathbb{R}^{2n-1} \mid \|x\| = \sum x_i^2 \leq 1\}$. Map the boundary $S^{2n-2} = \{x \in \mathbb{R}^{2n-1} \mid \|x\| = \sum x_i^2 = 1\}$ to S^{2n-2} by a map η of degree 2 and let $Y = E^{2n-1} \cup_2 S^{2n-2}$ be the quotient of $E^{2n-1} \sqcup S^{2n-2}$ obtained by identifying x with ηx for points x on the boundary. If we think of \mathbb{R}^{2n-1} as $\mathbb{C} \times \mathbb{R}^{2n-3}$ we can choose $\eta(z, x_3, \dots, x_{2n-1}) = (z^2, x_3, \dots, x_{2n-1})$. In real coordinates $\eta(x_1, x_2, x_3, \dots, x_{2n-1}) = (x_1^2 - x_2^2, 2x_1x_2, x_3, \dots, x_{2n-1})$. Therefore Y is the quotient of E^{2n-1} obtained by identifying $(x_1, x_2, x_3, \dots, x_{2n-1})$ with $(-x_1, -x_2, x_3, \dots, x_{2n-1})$ for points on the boundary S^{2n-2} . Let B be the sub- \mathbb{R} -algebra of $C(E^{2n-1}, \mathbb{R})$ generated by $x_3, \dots, x_{2n-1}, x_1^2, x_2^2, x_1x_2, (1 - \sum x_i^2)x_1$, and $(1 - \sum x_i^2)x_2$. These functions all factor through Y so we can regard B as a subalgebra of $C(Y, \mathbb{R})$. It separates points of Y and so is dense in $C(Y, \mathbb{R})$ by the Stone-Weierstrass Theorem. Let S be the set of elements of B with no zero on Y and define $A = B_S$. This clearly has the required properties. We can then replace A by B_s with $s \in S$ chosen so that ρ is in $\mathrm{SL}_{2n-1}(A)$ and is 2-stably elementary. \square

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