ON MUNSHI'S PROOF OF THE NULLSTELLENSATZ

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ABSTRACT. We use an idea from Munshi's proof of Hilbert's Nullstellensatz to give a very short but less elementary proof using some basic facts about localization and integral extensions.

1. Introduction

In [5] Munshi gave a remarkable new and completely elementary proof of Hilbert's Nullstellensatz. In the present paper we give a simpler but less elementary version of his proof using a few basic facts of commutative algebra. A nice exposition of Munshi's work has been published by May [4]. The present version of the proof is actually closer to the more classical proof as presented in Grayson's paper [2] but, as we shall see, the use of the theorem proved by Munshi provides an essential simplification. All rings considered here will be commutative with unit.

2. Preliminary results

In this section we recall some well–known results from [3] for the reader's convenience. These results are also discussed in [2] and [4].

Following [3] we define a G–domain to be a domain R with the property that some localization $R_a = R[a^{-1}]$ is a field. This is equivalent to the property that the intersection of all non–zero prime ideals is non–zero but we will not need this here.

Lemma 2.1. *A polynomial ring $R[X]$ is never a G–domain.*

Proof. We can assume that R is a domain. Suppose that $R[X]_f$ is a field. Obviously $\deg f > 0$ so $1 + f$ is non–zero. Write $(1 + f)^{-1} = g/f^n$. Cross multiplying shows that $1 + f$ divides f^n but $f \equiv -1 \mod 1 + f$ which leads to the nonsensical conclusion that $1 + f$ is a unit. □

We write $U(R)$ for the group of units of a ring R.

Lemma 2.2. *Let $A \subseteq B$ be an integral extension. Then $A \cap U(B) = U(A)$.*

Proof. Let $a \in A \cap U(B)$. Write $ab = 1$ with $b \in B$ and find an equation $b^n + c_1 b^{n-1} + \cdots = 0$ with all c_i in A. Multiplying by a^{n-1} gives $b = -c_1 - c_2 a - \ldots$ so b is in A. □

Corollary 2.3. *Let R be a subring of a field K such that K is integral over R. Then R is also a field.*

3. The proof

Our basic result is essentially the same as Lemma 1.6 of [2].

Theorem 3.1. *Let R be domain and let M be a maximal ideal of the polynomial ring $R[X_1, \ldots, X_n]$. If $R \cap M = 0$ then there is a non–zero element a in R such that R_a is a field and $K = R[X_1, \ldots, X_n]/M$ is a finite extension of R_a.*

The only novelty in the present exposition is the use of the following consequence as an induction hypothesis. This is the main result of Munshi's paper [5] from which he deduces the Nullstellensatz. M.-c. Kang has pointed out that this result was previously proved by Nagata (with a different proof) in [6], generalizing an earlier result of Artin and Tate [1]. It also appears in Nagata's book [7, 14.10] with a historical note [7, p. 215]. The novelty of Munshi's proof lies, of course, in the *proof* of the result.

Corollary 3.2. *Let R be a domain and let M be a maximal ideal of the polynomial ring $R[X_1, \ldots, X_n]$. If R is not a G–domain then $R \cap M \neq 0$.*

Proof of Theorem 3.1. The case $n = 0$ is trivial. We use induction on n and so can assume that Corollary 3.2 holds for $n-1$ variables. Using Lemma 2.1 it follows that if $n \geq 1$ then $M \cap R[X_i] \neq 0$ for all i. Choose $f_i = a_i X_i^{n_i} + \ldots$ in $M \cap R[X_i]$ and let $a = \prod a_i$. The image x_i of X_i in K satisfies the monic equation $a_i^{-1} f_i = 0$ over R_a. Since the x_i generate K over R_a we see that K is integral over R_a. By Corollary 2.3, R_a is a field, and K is finite over it since it is integral and finitely generated over R_a □

4. Nullstellensatz

We can now deduce the Nullstellensatz in the following form.

Theorem 4.1. *Let F be a field, let A be a finitely generated F–algebra, and let M be a maximal ideal of A. Then A/M is finite over F.*

Proof. Since A is a quotient of a polynomial ring $B = F[X_1, \ldots, X_n]$ it is enough to prove the theorem for B but this is a special case of Theorem 3.1. □

For completeness we include some standard consequences. If M is a maximal ideal of a polynomial ring $A = F[X_1, \ldots, X_n]$ over a field F we can embed A/M in the algebraic closure $\bar{F}$ of F by Theorem 4.1. The X_i map to elements a_i of $\bar{F}$ so $(a_1, \ldots, a_n)$ is a zero of M in the sense that $f(a_1, \ldots, a_n) = 0$ for all f in M. Moreover, $M = \{f \in A \mid f(a_1, \ldots, a_n) = 0\}$. In particular if F is algebraically closed then $M = (X_1 - a_1, \ldots, X_n - a_n)$.

A commutative ring R is called a Jacobson ring if every prime ideal is an intersection of maximal ideals. By [3, 1–3, Ex. 9], a Jacobson ring is the same thing as a Hilbert ring but we will not need this fact here.

Theorem 4.2. *Let A be a finitely generated algebra over a field. Then A is a Jacobson ring.*

Proof. Let P be a prime ideal of A and let $a \in A - P$. We must find a maximal ideal Q of A such that $P \subseteq Q$ and $a \notin Q$. Let M be a maximal ideal of A_a containing P_a and let Q be the inverse image of M in A. Then $A/Q \subseteq A_a/M$. Since A_a/M is finite over F so is A/Q. Therefore A/Q is a field, showing that Q is maximal. It clearly has the required properties. □

It follows that if I is an ideal of A, then the radical of I is the intersection of all maximal ideals containing I. In particular, if A is a polynomial ring over a field F and $a \in A$ is 0 at all zeros of I in the algebraic closure $\bar{F}$ of F, then some power a^n lies in I. This is the usual statement of the Nullstellensatz.

References

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