

ON THE STRAIGHTENING LAW FOR MINORS OF A MATRIX

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ABSTRACT. We give a simple new proof for the straightening law of Doubillet, Rota, and Stein using a generalization of the Laplace expansion of a determinant.

1. INTRODUCTION

The straightening law was proved in [5] by Doubillet, Rota, and Stein generalizing earlier work of Hodge [6]. Since then a number of other proofs have been given [4, 2, 1]. The object of the present paper is to offer yet another proof of this result based on a generalization of the Laplace expansion of a determinant. This proof has the advantage (to some!) of not requiring any significant amount of combinatorics, Young diagrams, etc. On the other hand, for the same reason, it does not show the interesting relations between the straightening law and invariant theory but these are very well covered in the above references and in [3]. For completeness, I have also included a proof of the linear independence of the standard monomials.

2. LAPLACE PRODUCTS

Let $X = (x_{ij})$ be an $m \times n$ matrix where $1 \leq i \leq m$ and $1 \leq j \leq n$. If $A \subseteq \{1, \dots, m\}$ and $B \subseteq \{1, \dots, n\}$ we define $X(A|B)$ to be the minor determinant of X with row indices in A and column indices in B . I will usually just write $(A|B)$ for $X(A|B)$ when it is clear what X is. I will write $|A|$ for the number of elements in A . We set $X(A|B) = 0$ if $|A| \neq |B|$.

I will write \tilde{A} for the complement $\{1, \dots, m\} - A$ and \tilde{B} for $\{1, \dots, n\} - B$. Also $\sum A$ will denote the sum of the elements of A .

Definition 2.1. If $m = n$, we define the Laplace product $X\{A|B\}$ to be $X\{A|B\} = (-1)^{\sum A + \sum B} (A|B)(\tilde{A}|\tilde{B})$.

If X is understood, I will just write $\{A|B\}$ for $X\{A|B\}$. This notation is, of course, for this paper only and is not recommended for general use.

The terminology comes from the Laplace expansion

$$(1) \quad \det X = \sum_{|S|=|B|} \{S|B\} = \sum_{|T|=|A|} \{A|T\}.$$

The following lemma explains the sign in Definition 2.1.

Lemma 2.2. *Let $y_{ij} = x_{ij}$ if (i, j) lies in $A \times B$ or in $\tilde{A} \times \tilde{B}$ and let $y_{ij} = 0$ otherwise. Then $X\{A|B\} = \det(y_{ij})$.*

Proof. Rearrange the rows and columns of $Y = (y_{ij})$ so that those with indices in A and B lie in the upper left hand corner. The resulting matrix has determinant $(A|B)(\tilde{A}|\tilde{B})$. The sign of the permutation of rows and columns is $(-1)^{\sum A + \sum B}$ by the next lemma. \square

Lemma 2.3. *Let $A = \{a_1 < \dots < a_p\}$ be a subset of $\{1, \dots, n\}$ and let $\{c_1 < \dots < c_q\} = \{1, \dots, n\} - A$. Then the sign of the permutation taking $\{1, \dots, n\}$ to $\{a_1, \dots, a_p, c_1, \dots, c_q\}$ is $(-1)^{\sum (a_i - i)}$.*

Proof. Move a_1 to position 1, then a_2 to position 2, etc. each time keeping the remaining elements in their given order. The number of transpositions used is $\sum (a_i - i)$. \square

The Laplace expansion (1) gives us a non-trivial relation between the Laplace products of X . This suggests looking for more general relations of the form

$$(2) \quad \sum_i a_i \{S_i|T_i\} = 0.$$

with constant a_i .

Since $\{A|B\}$ is multilinear in the rows of X it will suffice to check a relation (2) for the case in which the rows of X are all of the form $0, \dots, 0, 1, 0, \dots, 0$. Since $\{A|B\}$ is also multilinear in the columns of X , all terms of (2) will be 0 unless there is a 1 in each column. Therefore it will suffice to check a relation (2) for the case where X is a permutation matrix, $X = P(\sigma^{-1}) = (\delta_{\sigma i, j})$. To do this we first compute $\{A|B\}$ for this X .

Lemma 2.4. *If $X = (\delta_{\sigma i, j})$ is a permutation matrix $P(\sigma^{-1})$ then $\{A|B\} = \text{sgn } \sigma$ if $\sigma A = B$ and $\{A|B\} = 0$ otherwise.*

This is immediate from Lemma 2.2 and the fact that $\det X$ becomes 0 if any entry 1 is replaced by 0.

Corollary 2.5. *A relation $\sum_i a_i \{S_i|T_i\} = 0$ between Laplace products (with constant a_i) holds if and only if for each σ in \mathcal{S}_n we have $\sum a_i = 0$ over those i with $\sigma S_i = T_i$.*

Theorem 2.6. *For given A and B we have*

$$\sum_{V \subseteq B} \{A|V\} = \sum_{U \supseteq A} \{U|B\}$$

Proof. For given σ the right hand side is $\text{sgn } \sigma$ if $\sigma A \subseteq B$ and otherwise is 0. The same is true of the left hand side. \square

We will show later that, for generic X , all linear relations between Laplace products are consequences of those in Theorem 2.6. In particular, we recover the Laplace expansion (1) by setting $A = \emptyset$ or by setting $B = \{1, \dots, n\}$.

Corollary 2.7. *For given A, B , and for $C \subseteq B$ we have*

$$\sum_{C \subseteq V \subseteq B} \{A|V\} = \sum_{\substack{U \supseteq A \\ W \subseteq C}} (-1)^{|W|} \{U|B - W\}$$

Proof. By Theorem 2.6, the right hand side is

$$\sum_{W \subseteq C} (-1)^{|W|} \sum_{V \subseteq B-W} \{A|V\} = \sum_{V \subseteq B} \sum_{W \subseteq C-V} (-1)^{|W|} \{A|V\}$$

which is equal to the left hand side since the inner sum is 0 unless $C \subseteq V$. \square

Recall that \tilde{A} here denotes the complement of A .

Corollary 2.8. *For given A, B ,*

$$\sum_{\substack{U \supseteq A \\ W \supseteq B}} (-1)^{|\tilde{W}|} \{U|W\} = \sum_{V \subseteq B} \{A|\tilde{V}\}$$

Proof. Set $B = \{1, \dots, n\}$ in Corollary 2.7, getting

$$\sum_{V \supseteq C} \{A|V\} = \sum_{\substack{U \supseteq A \\ W \subseteq C}} (-1)^{|W|} \{U|\tilde{W}\}$$

Replace C, V, W by $\tilde{B}, \tilde{V}, \tilde{W}$ where B is now the B given in Corollary 2.8. \square

3. STRAIGHTENING LAPLACE PRODUCTS

If $S = \{s_1 < \dots < s_p\}$ and $T = \{t_1 < \dots < t_q\}$ are subsets of $\{1, \dots, n\}$, we define a partial ordering $S \leq T$ as in [3] to mean $p \geq q$ and $s_\nu \leq t_\nu$ for all $\nu \leq q$. Equivalently $S \leq T$ if and only if $|S \cap \{1, \dots, r\}| \geq |T \cap \{1, \dots, r\}|$ for all $1 \leq r \leq n$. Note that $S \supseteq T$ implies $S \leq T$.

As above let \tilde{S} be the complement $\{1, \dots, n\} - S$. For want of a better terminology, I will say that S is good if $S \leq \tilde{S}$ and that S is bad otherwise.

The following theorem is our straightening law for Laplace products.

Theorem 3.1. *For any $\{A|B\}$ we have $\{A|B\} = \sum \pm \{A_i|B_i\}$ where $A_i \leq A$, $B_i \leq B$ and A_i and B_i are good.*

Proof. By induction on the set of pairs of subsets of $\{1, \dots, n\}$ partially ordered by $(A, B) \leq (C, D)$ if $A \leq C$ and $B \leq D$, it is sufficient to prove that if A is bad, then $\{A|B\} = \sum \pm \{A_i|B_i\}$ with $A_i < A$ and $B_i \leq B$, and, similarly, if B is bad then $\{A|B\} = \sum \pm \{A_i|B_i\}$ with $A_i \leq A$ and $B_i < B$. Each statement implies the other by transposing our matrix.

Suppose first that $|A| = |B| < n/2$. Note both A and B are bad in this case. By Corollary 2.8 we have

$$\sum_{\substack{U \supseteq A \\ W \supseteq B}} \pm \{U|W\} = 0.$$

The other side in Corollary 2.8 is 0 since $|A| < n/2 < |V|$. One term of this sum is $\pm \{A|B\}$ while all other terms have the form $\pm \{U|W\}$ where $U < A$ and $W < B$.

In the remaining case $|A| = |B| \geq n/2$ we use an argument similar to that of Hodge [6]. Suppose B is bad. Let $B = \{i_1 < \dots < i_p\}$ and let $\tilde{B} = \{j_1 < \dots < j_q\}$. Since B is bad and $q \leq p$, we have $i_\nu > j_\nu$ for some ν which we choose minimal. Let $D = B \cup \{j_1, \dots, j_\nu\}$ and let $C = \{j_1 < \dots < j_\nu < i_\nu < \dots < i_p\}$. Apply

Corollary 2.7 with D in place of B . The left hand side is 0 since $|A| = p < |C| = p+1$ so we get

$$\sum_{\substack{U \supseteq A \\ W \subseteq C}} (-1)^{|W|} \{U|D - W\} = 0$$

The term with $U = A$ and $W = \{j_1, \dots, j_\nu\}$ is $\{A|B\}$. This is the only term of the form $\{U|B\}$ since $|U| = |B|$ and $U \supseteq A$. In the remaining terms we have $U \leq A$ since $U \supseteq A$. In these terms $D - W \neq B$ so if $W \subseteq \{j_1 < \dots < j_\nu\}$ then $D - W \supset B$ and so $D - W < B$. If W contains some i_μ then $D - W$ is obtained from B by removing some (and at least one) of the elements $\{i_\nu < \dots < i_p\}$ and replacing them by at least as many of the smaller elements $\{j_1 < \dots < j_\nu\}$. Therefore $D - W < B$. \square

4. THE STRAIGHTENING LAW FOR MINORS

We now use a simple trick to generalize Theorem 3.1 to the case of products of any two minors of a rectangular matrix $X = (x_{ij})$ where $1 \leq i \leq m$ and $1 \leq j \leq n$. Recall that $(S|T)$ is the minor of X with row indices in S and column indices in T . We set $(S|T) = 0$ if $|S| \neq |T|$. As above we write $(S', S'') \leq (T', T'')$ if $S' \leq T'$ and $S'' \leq T''$.

The following theorem is the straightening law for minors.

Theorem 4.1. *If $(S', T') \not\leq (S'', T'')$ then*

$$(S'|T')(S''|T'') = \sum \pm (S'_i|T'_i)(S''_i|T''_i)$$

where $(S'_i, T'_i) < (S', T')$ and $(S''_i, T''_i) \leq (S'', T'')$.

Proof. Let I' and I'' be disjoint sets in 1 - 1 correspondence with S' and S'' . Let $I = I' \cup I''$ and let $\varphi : I \rightarrow S' \cup S''$ map I' isomorphically onto S' and I'' isomorphically onto S'' . Define $\psi : J = J' \cup J'' \rightarrow T' \cup T''$ in the same way. Note $|I| = |J|$. Define a square matrix Y indexed by I and J by setting $y_{ij} = x_{\varphi(i)\psi(j)}$. Then, for $P \subseteq I$ and $Q \subseteq J$, we have $Y(P|Q) = X(\varphi(P)|\psi(Q))$ if $\varphi|P$ and $\psi|Q$ are injective while $Y(P|Q) = 0$ otherwise since two rows or columns will be equal.

Order I by setting $i_1 < i_2$ if $\varphi(i_1) < \varphi(i_2)$ or if $\varphi(i_1) = \varphi(i_2)$ with $i_1 \in I'$ and $i_2 \in I''$. Order J similarly. Then φ and ψ preserve order so if P and S are subsets of I on which φ is injective then $P \leq S$ implies $\varphi(P) \leq \varphi(S)$, and similarly for subsets of J .

Now $(S'|T')(S''|T'') = Y\{I'|J'\}$. If I' is good, then $I' \leq \tilde{I}' = I''$ so $S' \leq S''$ and similarly for J . Since $(S', T') \not\leq (S'', T'')$, one of I' and J' , say I' is bad. By Theorem 3.1 we can write $\{I'|J'\} = \sum \pm \{I'_i|J'_i\}$ where $I'_i \leq I'$, $J'_i \leq J'$ and I'_i and J'_i are good. Since I' is bad, we have $I'_i < I'$. Let $I''_i = I - I'_i = \tilde{I}'_i$ and $J''_i = J - J'_i = \tilde{J}'_i$, and let $S'_i = \varphi(I'_i)$, $S''_i = \varphi(I''_i)$ and similarly for T . It follows that $(S'|T')(S''|T'') = \sum \pm (S'_i|T'_i)(S''_i|T''_i)$ where $(S'_i, T'_i) \leq (S', T')$ and $(S''_i, T''_i) \leq (S'', T'')$. We omit those terms where φ or ψ is not injective. The fact that $(S'_i, T'_i) < (S', T')$ follows from the next lemma with $K = I'_i$ since $I'_i < I'$. \square

Lemma 4.2. *Let K be a subset of I on which φ is injective. If $K < I'$, then $\varphi(K) < S'$.*

Proof. It is clear that $\varphi(K) \leq S'$. If $\varphi(K) = S'$ let $I' = \{i_1 < \cdots < i_p\}$, $S' = \{s_1 < \cdots < s_p\}$ and $K = \{k_1 < \cdots < k_p\}$. Since $K \neq I'$, we have $k_\nu \neq i_\nu$ for some ν . But i_ν lies in I' so k_ν is in I'' and therefore $k_\nu > i_\nu$ which is impossible since $K < I'$. \square

Remark 4.3. Since $I'_i \cup I''_i = I$, it is clear that $S'_i \cup S''_i = S' \cup S''$ and $T'_i \cup T''_i = T' \cup T''$ counting multiplicities.

5. STANDARD MONOMIALS

We say that a monomial $(A_1|B_1) \cdots (A_r|B_r)$ in the minors of a matrix X is standard if $A_1 \leq A_2 \leq \cdots \leq A_r$ and $B_1 \leq B_2 \leq \cdots \leq B_r$. The following is an easy consequence of Theorem 4.1.

Corollary 5.1. *Any polynomial in the entries of X is a linear combination of standard monomials in the minors of X .*

Proof. Since $x_{ij} = (i|j)$, it is clear that any such polynomial is a linear combination of monomials in the minors of X . We show that any monomial $(A_1|B_1) \cdots (A_r|B_r)$ is a linear combination of standard monomials by induction on r and on (A_1, B_1) in the finite partially ordered set of pairs of subsets of $\{1, \dots, m\}$ and $\{1, \dots, n\}$. By induction on r we can assume that $(A_2, B_2) \leq \cdots \leq (A_r, B_r)$. If $(A_1, B_1) \leq (A_2, B_2)$ we are done. If not, Theorem 4.1 shows that $(A_1|B_1)(A_2|B_2) = \sum \pm (C_i|D_i)(P_i|Q_i)$ where $(C_i, D_i) < (A_1, B_1)$ so we are done by induction on (A_1, B_1) . \square

Remark 5.2. It follows from Remark 4.3 that if we write $(A_1|B_1) \cdots (A_r|B_r)$ as a linear combination of standard monomials $(A_1^{(i)}|B_1^{(i)}) \cdots (A_{r_i}^{(i)}|B_{r_i}^{(i)})$ then $\bigcup_j A_j^{(i)} = \bigcup_j A_j$ and $\bigcup_j B_j^{(i)} = \bigcup_j B_j$ for all i , counting multiplicities. In other words the two sides have the same content in the sense of [2].

To conclude, we give a proof of the following theorem which is rather similar to the proof in [4] but which uses no combinatorial constructions. By a generic matrix we mean one whose entries are indeterminates.

Theorem 5.3. *If X is a generic matrix, the standard monomials in the minors of X are linearly independent.*

Proof. We specialize X to a matrix of the form $X = YZ$ where Y is a generic $m \times N$ matrix, Z is a generic $N \times n$ matrix and N is sufficiently large. By the classical Binet–Cauchy theorem we have

$$(3) \quad X(A|B) = \sum_S Y(A|S)Z(S|B).$$

This just expresses the functoriality of the exterior product:

$$(4) \quad \bigwedge(X) = \bigwedge(Y) \bigwedge(Z)$$

. We write $x_{ij} = \sum_{\nu=1}^N y_i^{(\nu)} z_j^{(\nu)}$. Order the indeterminates as follows:

$$y_1^{(1)} > \cdots > y_m^{(1)} > z_1^{(1)} > \cdots > z_n^{(1)} > y_1^{(2)} > \cdots > y_m^{(2)} > \cdots$$

and order the monomials in these indeterminates lexicographically with respect to this order. If $A = \{a_1 < \cdots < a_p\}$ and $S = \{s_1 < \cdots < s_p\}$, then

$$Y(A|S) = \sum_{\sigma} \pm y_{a_{\sigma 1}}^{(s_1)} \cdots y_{a_{\sigma p}}^{(s_p)}.$$

The leading form of $Y(A|S)$ is clearly $y_{a_1}^{(s_1)} \cdots y_{a_p}^{(s_p)}$. Similarly, if $B = \{b_1 < \cdots < b_p\}$, then the leading form of $Z(S|B)$ is $z_{b_1}^{(s_1)} \cdots z_{b_p}^{(s_p)}$. The various terms of (3) have leading forms $y_{a_1}^{(s_1)} \cdots y_{a_p}^{(s_p)} z_{b_1}^{(s_1)} \cdots z_{b_p}^{(s_p)}$. Of these, the one with $s_i = i$ for all i is clearly the largest. Therefore, the leading form of $X(A|B)$ is $y(A)z(B)$ where $y(A) = y_{a_1}^{(1)} \cdots y_{a_p}^{(p)}$ and $z(B) = z_{b_1}^{(1)} \cdots z_{b_p}^{(p)}$. It follows that the leading form of $(A_1|B_1) \cdots (A_r|B_r)$ is $y(A_1) \cdots y(A_r)z(B_1) \cdots z(B_r)$.

Now if $A_1 \leq A_2 \leq \cdots \leq A_r$, and if $N \geq |A_1|$, we can recover A_1 from $M = y(A_1) \cdots y(A_r)$ as follows. Let $A_i = \{a_{i1} < \cdots < a_{ip_i}\}$. Then $M = \prod_j y_{a_{j1}}^{(1)} \prod_j y_{a_{j2}}^{(2)} \cdots$. Since $a_{1i} \leq a_{2i} \leq \cdots$, we see that a_{1i} is the least c such that $y_c^{(i)}$ occurs in M .

Since $M/y(A_1) = y(A_2) \cdots y(A_r)$ we see that M determines A_2, A_3 , etc. for $N \geq |A_1|$. Similarly $z(B_1) \cdots z(B_r)$ determines B_1, B_2 , etc. Therefore the leading forms of the standard monomials $(A_1|B_1) \cdots (A_r|B_r)$ with $N \geq |A_1|$ and $N \geq |B_1|$ are all distinct and the theorem follows since N can be arbitrarily large. \square

Corollary 5.4. *For a generic square matrix X , all linear relations between the Laplace products of X are consequences of those in Theorem 2.6.*

Proof. By Theorem 3.1 the space of all Laplace products of X is spanned by those of the form $\{A|B\}$ with A and B good. For these $\{A|B\} = \pm(A|B)(\tilde{A}|\tilde{B})$ is a standard monomial so these $\{A|B\}$ are linearly independent. Since the only relations needed to prove Theorem 3.1 are those of Theorem 2.6, the result follows. \square

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