# Representations of Finite Groups of Lie Type

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## 1 Introduction

Let $G$ be a connected reductive algebraic group over $\overline{\mathbb{F}}_p = \mathbb{K}$. Let $F$ be a Frobenius endomorphism with respect to some $\mathbb{F}_q$-structure on $G$. We would like to study the representations of the finite group $G^F = G(\mathbb{F}_q)$ over an algebraically closed field of characteristic zero. We can construct some representations of $G^F$ (namely, the Principal Series Representations) as follows:

Let $T$ be an $F$-stable maximal torus of $G$. Suppose $T$ is maximally split, i.e. there exists an $F$-stable Borel subgroup $B$ containing $T$. Then given a character $\theta$ of $T^F$, we consider it as a character of $B^F$ and induce it to $G^F$. Thus, given a maximally split maximal torus and a character $\theta$ as above, we can construct a representation of $G^F$. Moreover, this representation is irreducible if $\theta$ is in general position. However, we cannot use this simple procedure if the maximal torus is not maximally split.

In this article, we study the construction due to Deligne and Lusztig which, given any $F$-stable maximal torus $T$ and a character $\theta$ of $T^F$, produces a virtual character $R^G_{T,\theta}$ of $G^F$ using $l$-adic cohomology with compact support of a certain algebraic variety. Using this construction for a maximally split torus, we recover the Principal Series Representations. We will show that $\pm R^G_{T,\theta}$ is an irreducible character for $\theta$ in general position, and that all irreducible representations of $G^F$ occur in some virtual character $R^G_{T,\theta}$.

Finally, we will study the representations of $GL_2(\mathbb{F}_q)$ and $SL_2(\mathbb{F}_q)$ from the point of view of Deligne-Lusztig theory.

## 2 Notation and Preliminaries

We fix a prime $p$ and an algebraic closure $\mathbb{K}$ of $\mathbb{F}_p$. We fix $q$, a power of $p$. Let $X$ be a $\mathbb{K}$-scheme defined over $\mathbb{F}_q$. $F : X \to X$ will denote the corresponding (Geometric) Frobenius endomorphism. $F$ will be used to denote the Frobenius endomorphism of any $\mathbb{K}$-scheme defined over $\mathbb{F}_q$.

**Definition 2.1.** Let $G$ be an algebraic group over $\mathbb{K}$ defined over $\mathbb{F}_q$.
The Lang map, $L\colon G \to G$, is defined by $L : g \mapsto g^{-1}F(g)$.

**Proposition 2.2** (Lang's Theorem; [8], Thm. 2.4.)**.** *Suppose that $G$ is connected. Then the map $L : G \to G$ is surjective.*

From now on, unless mentioned otherwise, $G$ will denote a connected reductive algebraic group over $\mathbb{K}$ defined over $\mathbb{F}_q$. We are interested in the representations of the finite group $G^F$ of $\mathbb{F}_q$ points of $G$. We fix a pair $T_0 \subset B_0$ of an $F$-stable maximal torus of $G$ and an $F$-stable Borel subgroup containing it. It is a consequence of Lang's theorem that such a pair exists and that all such pairs are $G^F$ conjugate. Let $U_0$ be the unipotent radical of $B_0$.

Throughout this article we will use the terms virtual representation and virtual character interchangeably and we will often use the same symbol to denote both. We fix another prime $l \neq p$. We will consider characters of various finite groups with values in $\overline{\mathbb{Q}}_l$. Note that these characters take values in the subfield of $\overline{\mathbb{Q}}_l$ generated by the roots of unity. On this subfield there is a well defined operation of 'conjugation' which maps every root of unity to its inverse. We denote this operation by $\bar{\cdot}$. Thus we can define the inner product $(\cdot, \cdot)$ of two characters as if they were complex representations.

We will let $T$ denote an arbitrary $F$-stable maximal torus. We let $B$ be *any* Borel subgroup containing $T$, and let $B = TU$. We denote by $\hat{T}^F$ the group of irreducible characters $\mathrm{Hom}(T^F, \overline{\mathbb{Q}}_l^*)$. Let $X(T)$ and $Y(T)$ denote the dual free abelian groups $\mathrm{Hom}(T, \mathbb{G}_m)$ and $\mathrm{Hom}(\mathbb{G}_m, T)$ respectively.

**Proposition 2.3** ([1], Props. 3.2.2. and 3.2.3.).

$$T^F \cong Y(T)/(F-1)Y(T) \tag{1}$$

$$\hat{T}^F \cong X(T)/(F-1)X(T) \tag{2}$$

*Remark* 2.4. These isomorphisms depend upon the choice of an isomorphism[1] $\mathbb{K}^* \cong \mathbb{Q}_{p'}/\mathbb{Z}$ and an embedding $\mathbb{K}^* \hookrightarrow \overline{\mathbb{Q}}_l^*$. Let us fix such a pair once and for all.

Now let us consider all pairs $(T, \theta)$ where $T$ is an $F$-stable maximal torus and $\theta \in \hat{T}^F$. $G^F$ acts on all such pairs by conjugation. Note that in view of 2.3, we may regard $\theta$ as a character of $Y(T)$. Also, due to the norm map, $N : T^{F^n} \to T^F$, $\theta$ determines a character $\theta \circ N$ of $T^{F^n}$.

**Proposition 2.5** ([1], Prop. 4.1.3.). *Let $(T, \theta)$ and $(T', \theta')$ be two pairs as above. Then the following are equivalent:*

*(i) There is an element $g \in G$ which transforms $T$ to $T'$ and $\theta$, regarded as a character of Y(T), to $\theta'$, regarded as a character of $Y(T')$.*

*(ii) For some $n > 0$, there exists $g \in G^{F^n}$ which transforms $T$ to $T'$ and $\theta \circ N$ to $\theta \circ N'$.*

We define the two pairs $(T, \theta)$ and $(T', \theta')$ to be *geometrically conjugate* if they satisfy the above conditions.

We let $W(T)$ denote the Weyl group $N(T)/T$. We will denote $W(T_0)$ just by $W$. More generally, let $T$, $T'$ be two $F$-stable maximal tori. We define $N(T, T') = \{g \in G | g^{-1}Tg = T'\}$.
Define $W(T, T') = T\backslash N(T, T') = N(T, T')/T'$.
Note that since $T$, $T'$ are $F$-stable, $N(T, T')$ will also be $F$-stable and we will have an induced action of $F$ on $W(T, T')$. Then using Lang's Theorem, we observe that $W(T, T')^F$ can be identified with $T^F\backslash N(T, T')^F$ or with $N(T, T')^F/T'^F$.

Note that the torus ${}^gT_0$ is $F$-stable if and only if $g^{-1}F(g) \in N(T_0)$. So with any $F$-stable maximal torus ${}^gT_0$ we can associate an element of the Weyl group $W$, but this is well defined only up to $F$-conjugacy in $W$. Thus, we get a map from the set of $G^F$-conjugacy classes of $F$-stable maximal tori to the set of $F$-conjugacy classes in $W$. Again it follows from Lang's Theorem that:

**Proposition 2.6** ([8], pg. 15). *We have a bijection between $G^F$-conjugacy classes of $F$-stable maximal tori and $F$-conjugacy classes in $W$. If the torus $T$ corresponds to the class of $w \in W$, then the $\mathbb{F}_q$-structure on $T$ is isomorphic to the $\mathbb{F}_q$-structure on $T_0$ corresponding to the 'twisted Frobenius', i.e.* $\mathrm{ad}(w)F$. *We donote by $T(w)$ the torus which has this $\mathbb{F}_q$ structure.*

**Definition 2.7.** Let $T$ be an $F$-stable torus. Then we have a decomposition $T = T_dT_a$, where $T_d$ is a maximal $\mathbb{F}_q$-split subtorus of $T$. Then $\sigma(G) := \dim(T_{0d})$ and $\epsilon_G := (-1)^{\sigma(G)}$.

If $X$ is an algebraic variety over $\mathbb{K}$ we consider the $i^{th}$ $l$-adic cohomology group with proper support $H_c^i(X, \overline{\mathbb{Q}}_l)$ and we will usually denote this just by $H_c^i(X)$.
Let $g$ be an automorphism of $X$ of finite order. Then we have an action of $g$ on $H_c^i(X)$.

**Definition 2.8.** We define the *Lefschetz number* of $g$ on X as

$$\mathcal{L}(g, X) = \sum_i (-1)^i Tr(g, H_c^i(X)) \tag{3}$$

This is in fact an integer independent of the choice of $l$.([2], Prop. 3.3.)

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[1] $\mathbb{Q}_{p'}$ is the additive group of all rational numbers whose denominators are prime to $p$.

# 3 $l$-adic cohomology with compact support

We will make use of some formal properties of $l$-adic cohomology with compact support of algebraic varieties and of the Lefschetz numbers of automorphisms of finite order. The proofs of these properties can be found in [5] and [2].

**Proposition 3.1.** *Suppose* $f\colon X \to Y$ *is a morphism of algebraic varieties over* $\mathbb{K}$ *such that for each* $y \in Y$, $f^{-1}(y)$ *is isomorphic to the affine space* $\mathbb{K}^n$ *for some fixed* $n$. *Let* $g, g'$ *be automorphisms of* $X$, $Y$ *respectively such that* $fg = g'f$. *Then* $\mathcal{L}(g, X) = \mathcal{L}(g', Y)$.

**Proposition 3.2.** *Let* $X$ *be an algebraic variety over* $\mathbb{K}$ *and* $g$ *an automorphism of finite order. Suppose* $X$ *is a finite disjoint union of locally closed subschemes* $X_j$ *which are stable under* $g$. *Then*

$$\mathcal{L}(g, X) = \sum \mathcal{L}(g, X_j) \tag{4}$$

*Suppose in addition that* $\bigcup_{j=1}^{k} X_j$ *is closed in* $X$ *for all* $k$ *and that* $G$ *is a finite group of automorphisms of* $X$ *leaving each* $X_j$ *invariant. Let* $\theta$ *be an irreducible character of* $G$ *and* $H_c^i(X_j)_\theta$ *be the* $\theta$ *isotypic subspace of* $H_c^i(X_j)$. *Then if* $H_c^i(X_j)_\theta = 0$ *for all* $i, j$ *we have* $H_c^i(X)_\theta = 0$ *for all* $i$.

**Proposition 3.3.** *Suppose* $X$ *is a disjoint union of closed subsets* $X_j$ *and that* $G$ *is a finite group of automorphisms of* $X$ *permuting the* $X_j$ *transitively. Let* $H$ *be the stabilizer of* $X_1$. *Then the virtual character* $g \mapsto \mathcal{L}(g, X)$ *of* $G$ *is induced by the virtual character* $h \mapsto \mathcal{L}(h, X_1)$ *of* $H$.

**Proposition 3.4.** *Suppose that* $G$ *is a finite group of automorphisms of* $X$ *such that the strict quotient* $X/G$ *exists.(This holds for* $X$ *affine.) Then*

$$H_c^i(X/G) \cong H_c^i(X)^G \tag{5}$$

**Proposition 3.5** (Künneth formula)**.** *Let* $X_1, X_2$ *be algebraic varieties. Then*

$$H_c^k(X_1 \times X_2) \cong \bigoplus_{i+j=k} H_c^i(X_1) \otimes H_c^j(X_2) \tag{6}$$

*Moreover, if* $g_1, g_2$ *are automorphisms of finite order of* $X_1, X_2$ *respectively, then*

$$\mathcal{L}(g_1 \times g_2, X_1 \times X_2) = \mathcal{L}(g_1, X_1)\mathcal{L}(g_2, X_2) \tag{7}$$

**Proposition 3.6** ([2], Thm. 3.2.)**.** *Let* $g$ *be an automorphism of* $X$ *of finite order and let* $g = su = us$ *where order of* $s$ *is prime to* $p$ *and order of* $u$ *is a power of* $p$. *Then*

$$\mathcal{L}(g, X) = \mathcal{L}(u, X^s) \tag{8}$$

**Proposition 3.7** ([2], Cor. 6.5.)**.** *Suppose a connected algebraic group* $G$ *acts on a variety* $X$. *Then each element of* $G$ *acts trivially on* $H_c^i(X)$.

# 4 The Steinberg Representation

We now define the Steinberg representation of the group $G^F$. The Frobenius endomorphism induces a permutation of the set $I$ of simple roots(w.r.t. $T_0 \subset B_0$). For an $F$-stable subset $J \subset I$, let $J/F$ denote the set of all $F$-orbits in $J$.

**Definition 4.1.** The Steinberg representation is defined as the virtual representation given by

$$St = St_G = \sum_{\substack{J \subset I \\ F(J) = J}} (-1)^{|J/F|} Ind_{P_J{}^F}^{G^F}(1). \tag{9}$$

This is in fact an irreducible representation:

**Proposition 4.2** ([1], Props. 6.2.2. and 6.2.3.)**.**

$$(St, St) = 1 \tag{10}$$

$$(Ind_{B_0{}^F}^{G^F}(1), St) = 1 \tag{11}$$

*Hence the Steinberg character is an irreducible character of* $G^F$*.*

**Theorem 4.3** ([1], Thm. 6.5.9.)**.** *Let* $s \in G^F$*. Then*

$$St(s) = \begin{cases} \epsilon_G \epsilon_{C^0(s)} |C(s)^F|_p & \textit{if } s \textit{ is semisimple} \\ 0 & \textit{else.} \end{cases}$$

## 5 The construction of Deligne and Lusztig

Let $G$, $T$, $B$, $U$ be as in section 2. Consider the closed subset $X = X_{T,B} = L^{-1}(U)$ of $G$. Then $X$ is an affine variety. Note that $G^F \times T^F$ acts on $X$ by $(g,t)h = ght$. This makes $H_c^i(X)$ into a left $G^F$-module and a right $T^F$-module. Let $\theta \in \hat{T}^F$. We see that the subspace $H_c^i(X)_\theta$ of elements of $H_c^i(X)$ on which $T^F$ acts by $\theta$ is $G^F$-stable.

**Definition 5.1.** $R_T^G(\theta)$ is the following virtual representation of $G^F$ :

$$R_T^G(\theta) = \sum_{i \geq 0} (-1)^i H_c^i(X_{T,B})_\theta \tag{12}$$

We will often denote this by $R_{T,\theta}^G$, or just by $R_{T,\theta}$ whenever this is not likely to cause any confusion. We will see later that this really does not depend on our choice of $B$.

**Proposition 5.2** ([1], Prop. 7.2.3.)**.** *Let* $g \in G^F$*. Then*

$$R_{T,\theta}(g) = \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t^{-1}) \mathcal{L}((g,t), X) \tag{13}$$

This follows from basic properties of representations of finite groups.

We will now show that we recover the Principal Series Representations if $T$ is maximally split. This will follow from the following:

**Proposition 5.3** ([8], Thm. 6.24.)**.** *Suppose* $T$ *lies in an* $F$*-stable parabolic subgroup* $P$ *of* $G$*. Let* $M$ *be the (unique) Levi subgroup containing* $T$*. Thus* $M$ *is* $F$*-stable. Then we can consider the generalized character* $R_{T,\theta}^M$ *of* $M^F$ *as a generalized character of* $P^F$*. Then we have*

$$R_{T,\theta}^G = Ind_{P^F}^{G^F}(R_{T,\theta}^M) \tag{14}$$

We make use of Propositions 3.3 and 3.1.

**Corollary 5.4.** *If* $T$ *is contained in an* $F$*-stable Borel subgroup* $B$*, then we may consider* $\theta$ *as a character of* $B^F$*. Then*

$$R_{T,\theta}^G = Ind_{B^F}^{G^F}(\theta) \tag{15}$$

We only need to observe that $R_{T,\theta}^T = \theta$.

We will now state a character formula for $R_{T,\theta}^G$. We begin by introducing the Green functions.

**Proposition 5.5** ([8], pg. 71.). *Suppose $u \in G^F$ is nilpotent. Then $R^G_{T,\theta}(u)$ is an integer independent of $\theta$. We define the Green functions $Q_T = Q^G_T$ on the unipotent elements of $G^F$ by this common integer value, namely $Q_T(u) = R_{T,1}(u)$.*

This follows from Propositions 3.6 and 5.2 and the fact that a rational algebraic integer is an integer.
We will now express $R_{T,\theta}$ in terms of $\theta$ and certain Green functions.

**Theorem 5.6** (Character Formula; [1], Thm. 7.2.8.). *Let $g \in G^F$, and suppose $g = su = us$ is the Jordan decomposition of $g$. Then*

$$R_{T,\theta}(g) = \frac{1}{|C^0(s)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} \theta(x^{-1}sx) Q^{C^0(s)}_{xTx^{-1}}(u) \tag{16}$$

It follows from Propositions 5.2 and 3.6 that

$$R_{T,\theta}(su) = \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t^{-1}) \mathcal{L}(u, X^{(s,t)}) \tag{17}$$

Then we can break up $X^{(s,t)}$ as a disjoint union of $|C(t)^F : C^0(t)^F|$ closed subvarieties each isomorphic to $Y_t = X \cap C^0(t)$, and then use 3.2.

# 6 Orthogonality Theorems

Now we come to the important orthogonality relations between the Deilgne Lusztig virtual characters.

**Theorem 6.1** ([8], Thm. 6.12.). *If $(T, \theta^{-1}), (T', \theta')$(as in section 2) are not geometrically conjugate, then*

$$(H^i_c(X)_\theta \otimes H^j_c(X')_{\theta'})^{G^F} = 0 \tag{18}$$

*for all i, j. ($X' = L^{-1}(U')$ where $U'$ is chosen such that $T'U'$ is a Borel subgroup.)*

In view of the Künneth formula, we have to show that $(H^k_c(X \times X')_{(\theta,\theta')})^{G^F} = 0 \; \forall \; k$, i.e. we have to show that $H^k_c((X \times X')/G)_{(\theta,\theta')} = 0$.

Note that $R_{T,\theta^{-1}}$ is the dual of $R_{T,\theta}$, say by 5.2. Hence we have the following:

**Corollary 6.2.** *If $(T, \theta), (T', \theta')$ are not geometrically conjugate then the virtual representations $R_{T,\theta}$ and $R_{T',\theta'}$ are disjoint, i.e. have no irreducible components in common.*

**Theorem 6.3** ([8], Thm. 6.14.).

$$(R_{T,\theta}, R_{T',\theta'}) = |\{w \in W(T, T')^F | {}^{\dot{w}}\theta' = \theta\}| \tag{19}$$

*In particular, $(R_{T,\theta}, R_{T',\theta'}) = 0$ if $(T, \theta)$, $(T', \theta')$ are not $G^F$-conjugate.*

*Remark* 6.4. This does not mean that $R_{T,\theta}$ and $R_{T',\theta'}$ are disjoint, since they are *virtual* representations.

**Theorem 6.5** ([8], Thm. 6.15.). *We have the following relations for the Green functions*

$$\frac{1}{|G^F|} \sum_{\substack{u \in G^F \\ u \text{ unipotent}}} Q_T(u) Q_{T'}(u) = \frac{|N(T, T')^F|}{|T^F||T'^F|} \tag{20}$$

**Definition 6.6.** We say that a character $\theta \in \hat{T}^F$ is in general position, or is regular if it is not fixed by any non-trivial element of $W(T)$.

**Corollary 6.7.** *If $\theta$ is regular then $\pm R_{T,\theta}$ is irreducible.*

**Corollary 6.8** ([8], Cor. 6.18.)**.** *$R_{T,\theta}$ is independent of the choice of $U$ which was used to define the variety $X$.*

We now find the dimensions of the $R_{T,\theta}$. For this we will need some properties of the the Steinberg representation.

**Theorem 6.9** ([8], Thm. 6.21; [1], Thm. 7.5.1.)**.**

$$dim(R_{T,\theta}) = Q_T(1) = \epsilon_G \epsilon_T \frac{|G^F|}{|U_0^F||T^F|} = \epsilon_G \epsilon_T \frac{|G^F|}{St_G(1)|T^F|} \tag{21}$$

**Corollary 6.10** ([8], pg. 90; [1], Prop. 7.5.3.)**.**

$$R_{T,\theta}(s) = \frac{\epsilon_T \epsilon_{C^0(s)}}{|T^F| St_{C^0(s)}(1)} \sum_{\substack{g \in G^F \\ g^{-1}sg \in T^F}} \theta(g^{-1}sg) \tag{22}$$

**Corollary 6.11** ([8], Cor. 6.22.)**.**

$$Ind_{T^F}^{G^F}(\theta) = \epsilon_G \epsilon_T R_{T,\theta} \otimes St_G \tag{23}$$

The following theorem now shows that every irreducible representation of $G^F$ occurs as a constituent of some $R_{T,\theta}$.

**Theorem 6.12** ([8], Thm. 6.23.)**.** *The regular representation of $G^F$ is equal to $\frac{1}{|G^F|_p} \sum_T \sum_{\theta \in \hat{T}^F} \epsilon_G \epsilon_T R_{T,\theta}$, where we sum over all $F$-stable maximal tori $T$.*

**Corollary 6.13.** *Every irreducible representation of $G^F$ occurs as a constituent of some $R_{T,\theta}$.*

**Definition 6.14.** Let $\chi$ be an irreducible representation of the group $G^F$. We say that $\chi$ is *unipotent* if it occurs in some virtual character $R_{T,1}$ for some $F$-stable maximal torus $T$.

*Remark* 6.15. By 6.2 we see that a unipotent representation cannot occur in $R_{T,\theta}$ if $\theta \neq 1$.

# 7 An alternative description

In this section, we will describe some slightly different varieties, which also could be used to define the Deligne-Lusztig virtual characters. To do this we need the notion of relative positions of Borel subgroups. We have an action of $G$ on the set $G/B_0 \times G/B_0$ of pairs of Borel subgroups. Then the set of orbits $G\backslash(G/B_0 \times G/B_0)$ can be identified with the Weyl group $B_0\backslash G/B_0$, namely given a pair $(B, B')$ of Borel subgroups, there is a unique element in its orbit of the form $(B_0, {}^{\dot{w}}B_0)$ for $w \in W$. In this case we say that $B$ and $B'$ are in relative position $w$.

**Proposition 7.1** ([2], Lem. 1.13.)**.** *Let $J$ be the set of pairs $(T, B)$, where $T$ is an $F$-stable maximal torus and $B$ any Borel subgroup containing $T$. Then the map $h : (T, B) \mapsto$ relative position of $B$ and $FB$, induces a bijection*

$$h : G^F\backslash J \xrightarrow{\sim} W. \tag{24}$$

*Remark* 7.2. Note that this also induces the bijection of Prop. 2.6.

**Definition 7.3.** For $w \in W$, $X(w) \subset G/B_0$ is the locally closed subscheme of $G/B_0$ consisting of all Borel subgroups $B$ of $G$ such that $B$ and $FB$ are in relative position $w$.

**Theorem 7.4** ([1], Props. 7.7.10, 7.7.11.; [2], Defn. 1.17.)**.** *Let $h(T,B) = w$. Then we have*

$$X(w) \cong X_{T,FB}/(T^F(U \cap FU)). \tag{25}$$

*Hence we have*

$$R_{T,1}(g) = \mathcal{L}(g, X(w)) \tag{26}$$

*for all $g \in G^F$.*

The quotient $G/U_0$ is a $T_0$-torsor over $G/B_0$. The fibre $E(B)$ over a Borel $B$ is

$$E(B) = \{g \in G | {}^g B_0 = B\}/U_0. \tag{27}$$

Suppose $B, B'$ are two Borel subgroups in relative position $w$. Then we can define a map $E(B) \xrightarrow{\cdot \dot{w}} E(B')$, namely if ${}^g(B_0, {}^{\dot{w}}B_0) = (B, B')$, then map the class of $g \in E(B)$ to the class of $g\dot{w} \in E(B')$.(See [2], 1.7.) For a Borel $B$ the Frobenius map induces a map $F : E(B) \to E(FB)$. For $B \in X(w)$ put

$$E(B, \dot{w}) = \{u \in E(B) | F(u) = u\dot{w}\}. \tag{28}$$

Consider the torus $T(w)$, with the $\mathbb{F}_q$-structure given by the twisted Frobenius(2.6). Then $E(B, \dot{w})$ is a $T(w)^F$-torsor. The $E(B, \dot{w})$ form the fibres of a map $\pi : \tilde{X}(\dot{w}) \to X(w)$, with $\tilde{X}(\dot{w}) \subset E$ a $T(w)^F$-torsor over $X(w)$. Also the action of $G$ on $E$ restricts to an action of $G^F$ on $\tilde{X}(\dot{w})$. Up to isomorphism, the $G^F$-equivariant $T(w)^F$-torsor $\tilde{X}(\dot{w})$ over $X(w)$ is independent of the lifting $\dot{w}$ of $w$. Now suppose $\theta$ is a character of $T(w)^F$. Then as before, we can define a virtual character

$$R^G_\theta(w) = \sum_{i \geq 0} (-1)^i H^i_c(\tilde{X}(\dot{w}))_\theta \tag{29}$$

**Proposition 7.5** ([2], Defn. 1.17, Prop. 1.19.)**.** *Let $h(T,B) = w$. Then the $G^F$-equivariant $T(w)^F$-torsor $\tilde{X}(\dot{w})$ over $X(w)$ is isomorphic to the $G^F$-equivariant $T^F$-torsor $X_{T,FB}/(U \cap FU)$ over $X_{T,FB}/(T^F(U \cap FU))$. Since $U \cap FU$ is an affine space, we conclude that $R^G_{T,\theta} \cong R^G_\theta(w)$.*

# 8 Examples: $GL_2$ and $SL_2$

Let $G = GL_2$ with the standard $\mathbb{F}_q$-structure. Then we let $T_0$ be the group of diagonal matrices and $B_0$ the group of upper triangular matrices. ${T_0}^F \cong \mathbb{F}^*_q \times \mathbb{F}^*_q$. A character of ${T_0}^F$ is the same as a pair of characters $(\chi_1, \chi_2)$ of $\mathbb{F}^*_q$. The Weyl group $W$ is $S_2$ with trivial action of $F$. The non-trivial element $s \in W$ acts by $(\chi_1, \chi_2) \mapsto (\chi_2, \chi_1)$. Hence if $\chi_1 \neq \chi_2$ we have a regular character. Now we have one more $G^F$-conjugacy class of $F$-stable maximal tori. Let $\epsilon$ be a generator of the multiplicative group $\mathbb{F}^*_q$. Let $T = T(s)$ be the maximal torus consisting of matrices of the type $\begin{pmatrix} x & y \\ \epsilon y & x \end{pmatrix}$. We have $T^F \cong \mathbb{F}^*_{q^2}$. The non-trivial element of $W(T)$ acts on $\hat{T}^F$ by $\theta \mapsto \theta^q$. In this case the Deligne-Lusztig variety $\tilde{X}(s)$ is given by the curve $-(xy^q - x^q y)^{q-1} = 1$ in $\mathbb{A}^2$. $T^F \cong \mathbb{F}^*_{q^2}$ acts on this curve by scalar multiplication. This curve is a disjoint union of $q-1$ smooth irreducible curves. We can compute the characters $R_{T,\theta}$ by considering the action of $G^F$ on the disjoint union of the $q-1$ projective closures of the $q-1$ components. We get the following character values for the Deligne-Lusztig characters and the Steinberg character:

| | $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ | $\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$ | $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ <br> $x \neq y$ | $\begin{pmatrix} x & y \\ \epsilon y & x \end{pmatrix} = \zeta$ <br> $y \neq 0$ |
|---|---|---|---|---|
| $\chi \circ det$ | $\chi(x^2)$ | $\chi(x^2)$ | $\chi(xy)$ | $\chi(x^2 - \epsilon y^2)$ |
| $St_G$ | $q$ | $0$ | $1$ | $-1$ |
| $St^\chi$ | $q\chi(x^2)$ | $0$ | $\chi(xy)$ | $-\chi(x^2 - \epsilon y^2)$ |
| $R_{T_0,(\chi_1,\chi_2)}$ | $(q+1)\chi_1(x)\chi_2(x)$ | $\chi_1(x)\chi_2(x)$ | $\chi_1(x)\chi_2(y) + \chi_1(y)\chi_2(x)$ | $0$ |
| $-R_{T,\theta}$ | $(q-1)\theta(x)$ | $-\theta(x)$ | $0$ | $-(\theta(\zeta) + \theta(\zeta)^q)$ |

We have $q-1$ characters $\chi$ of $\mathbb{F}_q^*$. These give $(q-1)$ 1-dimensional characters of $G^F$, after composing with the determinant. Also, tensoring the Steinberg character with these 1-dimensional characters we get $(q-1)$ $q$-dimensional irreducible characters denoted by $St^\chi$. From the table we see that we have the following decompositions of the Deligne-Lusztig virtual characters into irreducible factors.

$$R_{T_0,(\chi,\chi)} = \chi \circ det + St^\chi. \tag{30}$$

$$R_{T_0,(\chi_1,\chi_2)} = R_{T_0,(\chi_2,\chi_1)} \text{ are irreducible when } \chi_1 \neq \chi_2. \tag{31}$$

$$-R_{T,\theta} = -R_{T,\theta^q} \text{ are irreducible when } \theta \neq \theta^q. \tag{32}$$

$$R_{T,\chi\circ N} = \chi \circ det - St^\chi \text{ where } N : \mathbb{F}_{q^2}^* \to \mathbb{F}_q^* \text{ is the norm map.} \tag{33}$$

Thus we have all irreducible characters of $G^F$.

Now let us consider the group $G = SL_2$. Let $T_0, B_0$ be the standard maximal torus and Borel subgroup. Then we have ${T_0}^F \cong \mathbb{F}_q^*$. The action of the non-trivial element of the Weyl group on $\hat{T_0}^F$ is given by $\chi \mapsto \chi^{-1}$. Let $T = T((1,2))$ be the intersection of the corresponding torus of $GL_2$ with $SL_2$. Here the Deligne-Lusztig variety is the curve $C$ given by $xy^q - x^q y = 1$ in $\mathbb{A}^2$. $T^F$ is the subgroup of $\mathbb{F}_{q^2}^*$ consisting of norm 1 elements. Again the non-trivial element of $W(T)^F$ acts by inversion. Let $\nu$ and $\tau$ be the non-trivial quadratic characters of $T^F$ and ${T_0}^F$ respectively. Let $\alpha = \frac{1}{2} + \frac{1}{2}\sqrt{\tau(-1)q}$. Here we get the following character values:

| | $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ $a = \pm 1$ | $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ $a = \pm 1$ | $\begin{pmatrix} a & \epsilon \\ 0 & a \end{pmatrix}$ $a = \pm 1$ | $\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$ $x \neq \pm 1$ | $\begin{pmatrix} x & y \\ \epsilon y & x \end{pmatrix} = \zeta$ $x \neq \pm 1$ |
|---|---|---|---|---|---|
| $St_G$ | $q$ | $0$ | $0$ | $1$ | $-1$ |
| $R_{T_0,(\chi,1)}$ | $(q+1)\chi(a)$ | $\chi(a)$ | $\chi(a)$ | $\chi(x) + \chi(x)^{-1}$ | $0$ |
| $W'$ | $\frac{(q+1)}{2}\tau(a)$ | $\alpha\tau(a)$ | $(1-\alpha)\tau(a)$ | $\tau(x)$ | $0$ |
| $W''$ | $\frac{(q+1)}{2}\tau(a)$ | $(1-\alpha)\tau(a)$ | $\alpha\tau(a)$ | $\tau(x)$ | $0$ |
| $-R_{T,\theta}$ | $(q-1)\theta(a)$ | $-\theta(a)$ | $-\theta(a)$ | $0$ | $-(\theta(\zeta) + \theta(\zeta)^{-1})$ |
| $X'$ | $\frac{(q-1)}{2}\nu(a)$ | $-\alpha\nu(a)$ | $(\alpha-1)\nu(a)$ | $0$ | $-\nu(\zeta)$ |
| $X''$ | $\frac{(q-1)}{2}\nu(a)$ | $(\alpha-1)\nu(a)$ | $-\alpha\nu(a)$ | $0$ | $-\nu(\zeta)$ |

Here we have

$$R_{T_0,1} = 1 + St. \tag{34}$$

$$R_{T,1} = 1 - St. \tag{35}$$

$$R_{T_0,\chi} = R_{T_0,\chi^{-1}} \text{ are irreducible if } \chi^2 \neq 1. \tag{36}$$

$$-R_{T,\theta} = -R_{T,\theta^{-1}} \text{ are irreducible if } \theta^2 \neq 1. \tag{37}$$

$$R_{T_0,\tau} = W' + W'' \text{ where } \tau^2 = 1, \tau \neq 1. \tag{38}$$

$$R_{T,\nu} = -X' - X'' \text{ where } \nu^2 = 1, \nu \neq 1. \tag{39}$$

In case $\nu$ and $\tau$ are the non-trivial quadratic characters of $T^F$ and ${T_0}^F$ respectively, then from 6.3 we know that the corresponding Deligne-Lusztig characters must break up into two irreducible components. Since we already know most part of the character table of $SL_2(\mathbb{F}_q)$, we can compute the characters of the components(See [4] pg. 72). We have written the characters of the corresponding components in the table above. We can in fact explicitly describle how $R_{T_0,\tau}$ and $R_{T,\nu}$ break up into irreducibles. To do this, we will describe a non-trivial self-intertwiner for each of $R_{T_0,\tau}$ and $R_{T,\nu}$. Let $V = \{f : {\mathbb{F}_q}^2 \backslash \{0\} \to \overline{\mathbb{Q}}_l\}$. The group $G^F$ acts on this space. All the Principal Series Representations occur in this space. Let $V_\chi = \{f \in V | f(tx) = \chi(t^{-1})f(x), \forall t \in {\mathbb{F}_q}^*\}$. Then $V_\chi$ is a sub-representation isomorphic to $R_{T_0,\chi}$. Define the map $\Lambda : V \to V$ by $\Lambda f(a,b) = \sum_{bx-ay=1} f(x,y)$. This is infact a $G^F$-linear isomorphism that maps $V_\chi$ to $V_{\chi^{-1}}$. In case $\chi = \tau$ we get the non-trivial self-intertwiner for $R_{T_0,\tau}$.

We know that all the $R_{T,\theta}$'s occur as the $\theta$ isotypic parts of the first cohomology with compact support, of the curve $C$. We have the Frobenius map $F : C \to C$. It can be seen that this induces a map on $H^1_c(C)$ that takes the $\theta$ isotypic parts to the $\theta^{-1}$ isotypic parts, explicitly giving the isomporphism between $R_{T,\theta}$ and $R_{T,\theta^{-1}}$. Again when $\theta = \nu$, this gives us the non-trivial self-intertwiner for $R_{T,\nu}$.

We can also construct the four remarkable representations $W', W'', X', X''$ from the Weil representations[2] of $SL_2(\mathbb{F}_q)$. The Weil representation acts on the space $W$ of $\overline{\mathbb{Q}}_l$ valued functions on $\mathbb{F}_q$. We have the stable subspaces $W_+$ and $W_-$ of even and odd functions respectively. Then we see that in fact $W_-$ is a cuspidal representation of dimension $(q-1)/2$ and hence it must be isomorphic to one of $X'$ or $X''$, say $X'$. We can also see then, that $W_+$ must be irreducible and in fact isomorphic to $W''$. So the Weil representation splits up as $W'' + X'$. The other Weil representation of $SL_2(\mathbb{F}_q)$ breaks up as $W' + X''$.

# 9 Further Questions

Although we have seen that all irreducible representations of $G^F$ occur in some Deligne-Lusztig virtual characters and that 'most' of these are irreducible, we have not said anything about the multiplicities with which the irreducible representations of $G^F$ occur in the reducible Deligne-Lusztig characters. In case of $GL_2$ and $SL_2$ we used some *ad hoc* methods to decompose the Deligne-Lusztig virtual characters into irreducibles. In general, with each irreducible representation we can associate an $F^*$-stable semisimple conjugacy class in $G^*$, where $G^*$ is the dual reductive group with Frobenius map $F^*$. This is very closely related to 6.2(See [2] or [1] Section 3.2). The irreducible representations that correspond to the semisimple element $1 \in {G^*}^{F^*}$ are the unipotent representations that we defined earlier. These are the representations that occur in some $R_{T,1}$. The question of computing the multiplicities of all irreducible representations in the Deligne-Lusztig virtual characters can be reduced to that of computing the multiplicities of the unipotent representations(of possibly smaller and possibly disconnected reductive subgroups)(See [6] and [3]). In studying these multiplicities the Weyl group and its associated Iwahori-Hecke algebra come into play([6]).

Finally, there is the question of determing the complete character table of $G^F$. The irreducible characters of $G^F$ of course form an orthonormal basis of the space of class functions. There is another orthonormal basis consisting of what are known as 'almost characters'. The transformation between these two bases is known explicitly. Hence the problem of computing the character table is equivalent to that of computing the values of almost characters. The framework for attacking this problem is provided by Lusztig's theory of character sheaves (developed in [7]), via the sheaf to function correspondence.

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