Complex reflection groups and Dynkin diagrams

by

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Abstract

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We want to study some complex hyperbolic reflection groups in the spirit of Weyl groups. The work evolved from studying one particular example: the reflection group of the complex Lorentzian Leech lattice L. We find that the reflection group can be generated by 26 complex reflections of order 3 that form the Coxeter diagram D given by the incidence graph of the projective plane over $\mathbb{F}_3$. There are two very interesting features.

(1) The same diagram D is known to give a presentation of the bimonster $M \wr \mathbb{Z}/2\mathbb{Z}$ (M is the largest sporadic simple group monster). Evidence suggest that there is an interesting connection between the reflection group of L and the bimonster.

(2) More central to the theme of the thesis is the fact that the diagram D behaves much like the "Dynkin diagram" for the reflection group of L. Much of the work follows this analogy. In particular we show that there is a unique point in the complex hyperbolic space of L that is fixed by the diagram automorphisms, which behaves like the Weyl vector for this diagram. The roots corresponding to the nodes of D have mirrors closest to the Weyl vector. These are the analogs of the simple roots.

We find a second example of the above phenomenon in the reflection group of

the quaternionic Lorentzian Leech lattice. Here the diagram turns out to be the incidence graph of the projective plane over $\mathbb{F}_2$. Then all the parallel results are true.

In the final chapter we describe a few more examples illustrating the phenomenon described in (2).

Professor Richard Borcherds
Dissertation Committee Chair

To

Amal Kiran Das

Contents

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Chapter 1

Introduction and background material

1.1 Summary of the work

In this thesis we study the reflection groups of four complex Lorentzian lattices and a quaternionic Lorentzian lattice. All but one of these examples (and many more) were studied by Allcock in [All1] where he showed that these groups are arithmetic. The paper [All1] is the main reference for this work.

For each of the complex or quaternionic Lorentzian reflection groups studied, we shall describe a finite set of reflections that form a "Coxeter-Dynkin diagram" for the group. We shall call these reflections the "simple reflections". The recurrent theme in the thesis is an analogy with the theory of Weyl groups and their Dynkin diagrams that is suggested by the above nomenclature. Other than being rather pretty we shall see that the diagrams and their analogy with Weyl groups can also be useful in understanding these infinite reflection groups. Now we describe the main results.

In each of the examples we shall see that the simple reflections generate the reflection group. For a couple of example in chapter 4 this was actually proved by

Allcock in [All1]. For the other three examples this is one of our main result and we use a computer calculation.

Let us call the mirrors of the simple reflections that form the diagram, the "simple mirrors". Though the diagrams are defined in a rather ad-hoc manner, in each of the example we shall see that there is some geometry associated with them. We show that the simple mirrors are equidistant from an unique point in the complex (or quaternionic) hyperbolic space. We call this point the "Weyl vector". We also show that the simple mirrors are precisely the mirrors that are closest to the Weyl vector. This is one of the other main result in this work.

This work actually evolved from studying one particular example, surely the prettiest and the most intriguing one, that is of special interest to us. This example, studied in detail in chapter 2, is of the complex Lorentzian Leech lattice. This was the biggest example studied by Daniel Allcock in [All1]. He showed that the reflection group is arithmetic. This reflection group acts on the 13 dimensional complex hyperbolic space. Thirteen is the largest dimension in which an example of arithmetic complex hyperbolic reflection group is known.

The starting point for this work is the curious observation that the diagram for the reflection group of this lattice happens to be the same Coxeter diagram appearing in a presentation of the bimonster. We find numerical evidence suggesting that these two groups are related in some way yet to be understood. This lead to a detailed investigation of this reflection group. In the course of proving that the simple reflections generate the reflection group, we observed the analogy with Weyl groups. Later we found a second example, for quaternionic Leech lattice where the same analogy goes though. This is described in chapter 3. In chapter 4 we study a couple of more complex Lorentzian lattices where, with some imagination, the same analogy goes though.

A few words about proofs. To prove that the simple reflections generate the reflection group we proceed in two steps. In step one we write down a finite set of mirrors such that reflections in them generate the group. To do this we use a computation similar to Conway's calculation of the reflection group of $II_{25,1}$ as adapted by Allcock in [All1] and some calculations about the Heisenberg group sitting as a subgroup of stabilizer of a norm zero vector, also following [All1].

Step two follows the analogy with Weyl groups. We take one of the mirrors from step one and try to reflect it in one of the simple mirrors to decrease its distance from the Weyl vector until we reach one of the simple mirrors. The reason we need a computer is that this does not always succeed though almost always it does.

Chapter 2 should be read first. Most definitions about lattices and reflection groups are given here. The same setup, notations and conventions mostly carry through in all the other examples. This leads to some repetition, because we have to repeat some of the definitions in chapter 3 again in the quaternionic context. But trying to write down the definitions in general in the beginning seem to make things rather confusing. So we decided to keep things this way.

Warning about a notational convention: Similar concepts are denoted by the same symbol in each of the example. This means that in each example, the lattice under study is called L, the diagram for its reflection group is denoted by D, et cetera. Thus we have to remember that the same symbol denotes different things when we are discussing different examples. This simplifies notation and brings out the parallel in the different examples easily. We hope this convention would not be confusing to the reader.

1.2 The background setting

In this section we shall try to put things into context to give some motivation for our work.

Real reflection groups have a very nice theory. But for a few exceptions in low dimension, the collection of irreducible Euclidean reflection groups coincide with the class of Weyl groups of complex simple Lie algebras. These are classified by the classical Dynkin diagrams. The diagrams are ubiquitous in various branches of mathematics. The Euclidean reflection groups have nice canonical presentations, the Coxeter presentations, that are encoded in the Dynkin diagram. One can get a lot of information about the Weyl group (and the Lie algebra or the Lie group) from the "diagram yoga".

The finite complex reflection groups were classified back in the fifties by Shepherd and Todd. But a completely unified and combinatorial treatment like Weyl groups is not there. Coxeter found diagrams for these groups but they are rather ad-hoc and not nearly as useful as the Dynkin diagrams. Recently there have been many papers trying to understand these diagrams better, for example trying to show that the corresponding Artin groups (the group one gets if one forgets that the generating reflections have finite order but retains the other braiding type relations) are the fundamental groups for the quotients of the complements of the mirror arrangements by the reflection groups and that these quotients are $K(\pi, 1)$. Presently this seems to be proved except for some exceptional cases where the known diagrams behave poorly.

We mention the reference [BMR] for a survey of finite complex reflection groups and further references can be found there. In the nineties there was some activity trying to find an object, named spetses, analogous to an algebraic group such that the finite complex reflection group will be like the Weyl group of a spetses. But spetses are still not defined.

The other general context for our work is the study of discrete subgroups of Lie groups. Hyperbolic reflection groups provide a rich class of example of discrete subgroups of Lie groups of real rank one. They also provide a large class of examples for Gromov's theory of word hyperbolic groups.

Generators for real hyperbolic reflection groups can be constructed often in the same manner as for the finite Euclidean reflection groups using the fundamental domain obtained by taking a connected component of the complement of the mirrors of reflection in the hyperbolic space. Coxeter type presentations can also often be found by looking at how the mirrors forming the walls of the fundamental domain intersect with each other. Sometimes (though not always) the hyperbolic reflection groups are Weyl groups of infinite dimensional Lie algebras.

One major difficulty in studying complex reflection groups is that the mirrors are complex hyperplanes, hence real co-dimension two in the vector space on which the group is acting. Thus they do not chop up the space into chambers, so there is no naturally defined fundamental domain like the Weyl chamber. Since the pioneering work of Deligne and Mostow there has been a lot of activity in trying to construct and understand discrete subgroups of $PU(1,n)$, the discrete groups acting on complex hyperbolic space, deciding their arithmeticity, finding fundamental domains, trying to find generators and relations and so on. The surface groups, that is, the case of $PU(2,1)$ have been most studied. However even in this case, finding a fundamental domain or generators and relations explicitly seem to be hard. The reflection group of $E_8 \oplus E_8 \oplus H$ over $\mathbb{Z}[e^{2\pi i/3}]$ studied in chapter 4, is the largest of the arithmetic groups constructed by Mostow.

There is a mysterious relation between modular forms and hyperbolic reflection groups discovered by Borcherds. For real hyperbolic reflection groups, The moral in the paper [Bor] is that "the hyperbolic reflection groups are nice if there is a modular form with poles at cusps associated with it", which roughly means, that the Borcherds singular theta lift of the modular form gives an automorphic form for $O(1,n)$ with zeros and poles exactly along the mirrors of reflection.

For all but one of the reflection groups studied here, Allcock found automorphic forms on $U(1,n)$ related to them in the above sense. These also come from the singular theta lift of modular forms with poles at cusps. Similar modular forms can be written down for many other complex hyperbolic reflection groups but I could not find a method to show that the reflection groups are "nice" in some sense in the other examples.

Chapter 2

Complex Lorentzian Leech lattice and the bimonster diagram

2.1 Introduction

Let $\omega = e^{2\pi i/3}$ and $\mathcal{E} = \mathbb{Z}[\omega]$ be the ring of Eisenstein integers. Let Λ be the Leech lattice considered as a twelve dimensional negative definite complex Hermitian lattice over $\mathcal{E}$. In this chapter we study the automorphism group of the Lorentzian complex Leech lattice $L = \Lambda \oplus H$, where H is the lattice $\mathcal{E} \oplus \mathcal{E}$ with Gram matrix $\left(\begin{smallmatrix} 0 & \bar{\theta} \\ \theta & 0 \end{smallmatrix}\right)$ and $\theta = \omega - \bar{\omega}$. We call H the hyperbolic cell. L is a Hermitian Lorentzian lattice over $\mathcal{E}$; the underlying integer lattice is $II_{2,26}$. The roots of L are the vectors of minimal norm, which is -3 in our scaling. The reflection group $R(L)$ is generated by reflections of order 3 in the roots. Modulo multiplication by scalars $R(L)$ act faithfully on 13 dimensional complex hyperbolic space $\mathbb{C}H^{13}$. The group $R(L)$ was shown to be of finite index in the automorphism group of L by D. Allcock in [All1]. As a consequence the reflection group is arithmetic, and hence is finitely presented.

A detailed study of a specific object like the reflection group of L may call for some explanation. There are two themes explored in this chapter that suggest that the reflection group of L is interesting.

(1) We find numerical evidences that suggest a link between the reflection group of L and the group known as the bimonster. Let M be the monster, the largest sporadic simple group and $M \wr 2$ denote the wreath product of M with $\mathbb{Z}/2\mathbb{Z}$. Conway and Norton [CNS] conjectured two surprisingly simple presentations of the bimonster $M \wr 2$ which were proved by Ivanov, Norton [Iva] and Conway, Simons [CS]. The bimonster is presented on the set of 16 generators satisfying the Coxeter relations of the diagram M_{666} (see Fig.2.2) and on 26 generators satisfying the Coxeter relations of the incidence graph of the projective plane over $\mathbb{F}_3$ (Fig.2.3) respectively, with some extra relations. (The M_{666} diagram is also called Y_{555} in literature because the graph looks like the letter Y with 5 nodes on each hand. The name M_{666} was suggested to me by Prof. Conway and Prof. Mckay, because it is consistent with the names M_{333}, M_{244} and M_{236} for the affine Dynkin diagrams of E_6, E_7 and E_8. The triples $\{3,3,3\}$, $\{2,4,4\}$ and $\{2,3,6\}$ are the only solutions to $1/p + 1/q + 1/r = 1$ in positive integers.)

The basic observation, that led us to an investigation of the generators of the automorphism group of L is the following:

One can find 16 reflections in the complex reflection group of L and then extend them to a set of 26 reflections, such that they form exactly the same Coxeter diagrams appearing in the presentation of the bimonster mentioned above.

(However $\mathrm{Aut}(L)$ is an infinite group and the vertices of the diagram for $\mathrm{Aut}(L)$ have order 3 instead of 2 as in the usual Coxeter groups). Let D be the "Coxeter diagram" of the 26 reflections: it is a graph on 26 vertices (also called nodes) which correspond to reflections of order 3. Two of these reflections a and b either braid ($aba = bab$) or commute ($ab = ba$). The vertices corresponding to a and b are joined if and only if the reflections braid. Much of the calculations done here are aimed towards proving the following (see 2.5.7)

2.1.1 Theorem. *The 26 reflections of D generate the automorphism group of*

L. So the three groups, namely, the group R_1 generated by reflections in the 26 reflections of the diagram D, the reflection group $R(L)$ and the automorphism group $\mathrm{Aut}(L)$ *are equal.*

The group $PGL_3(\mathbb{F}_3)$ of diagram automorphism act on both $\mathrm{Aut}(L)$ and the bimonster. Analogs of the extra relations needed for the presentation of the bimonster are also available in the group $\mathrm{Aut}(L)$.

(2) The other theme that we pursue in this chapter is an analogy with the theory of Dynkin diagrams of Weyl groups. This will be a central theme in all the examples studied in this thesis. The incidence graph of the projective plane over $\mathbb{F}_3$ emerges as sort of a "Coxeter-Dynkin diagram" for the reflection group of L. There is a unique point $\bar{\rho}$ in the complex hyperbolic space $\mathbb{C}H^{13}$ fixed by the diagram automorphism, called the Weyl vector. Define the height of a root r by $\mathrm{ht}(r) = |\langle \bar{\rho}, r \rangle| / |\bar{\rho}|^2$. The analogy with Dynkin diagram is further substantiated by the following (see 2.6.1)

2.1.2 Proposition. *All the roots of L have height greater than or equal to one and the 26 nodes of D are the only ones with minimal height. This amounts to the statement that the mirrors of the 26 reflections of D are precisely the ones closest to the Weyl vector $\bar{\rho}$.*

I would like to remark upon the element of surprise in the above results. Suppose we are given the roots of L forming the M_{666} diagram and want to find the other 10 roots to get the 26 node diagram. Finding these roots amount to solving a system of highly over-determined system of linear and quadratic equations over integers. *This system of equations happens to be consistent.* This is only one of the many coincidences which suggest that the reflection group of L and the diagram D are worth exploring in detail. One should also emphasize that the proof of the theorem and the proposition mentioned above go through *because the numbers work out nicely in our favor.* For example, the square of the height as defined above is an positive element of $\mathbb{Z}[\sqrt{3}]$, so it is not even a-priori obvious that all the roots have height greater than or equal to one. What is more intriguing is that all the above remarks hold true for the example of the quaternionic lattice, studied in the next chapter too.

A few words of caution: though we use the terminology from the theory of Weyl groups like "simple roots" and "Weyl vector", how far the analogy with Dynkin diagram goes is not clearly understood. Some of the facts that are true for Weyl groups are certainly not true here. For example, one cannot reduce the height of the root always by a single reflection in one of the 26 roots of D. But experimentally it seems that one almost always can.

The connection between the bimonster and the automorphism group of L observed here is still based on numerical observations. It is possible though unlikely that this is just a coincidence. We do not yet know if the Coxeter relations coming from the diagram D along with the extra relators we found so far, following the analogy with the bimonster, suffice to give a presentation of the automorphism group of L.

This chapter is organized as follows. In section two we set up the notation for this chapter and sketch the steps of a computer calculation to find an explicit isomorphism from $3E_8 \oplus H$ to $\Lambda \oplus H$ over $\mathcal{E}$, that is used later. Section three describes the diagram D in detail and a new construction of L starting with D. Section four contains some preliminary result about the reflection group of L; in particular we give an elementary proof to show that the reflection group of L acts transitively on the roots. Sections five and six contain the main results, mentioned above. The calculations done here bear resemblance with the theory of Weyl groups. In section seven we mention some relations satisfied by the generators of $\mathrm{Aut}(L)$ that are analogous to those in the bimonster and state a conjecture by Daniel Allcock about a possible relation between the reflection group of L and the bimonster. The last two sections contain some numerical details and notes about computer calculations.

2.2 Notations and basic definitions

2.2.1 Notation

$\mathrm{Aut}(L)$	automorphism group of a lattice L
$\mathbb{C}H^n$	The complex hyperbolic space of dimension n
D	the 26 node diagram isomorphic to the incidence graph of $P^2(\mathbb{F}_3)$
$\mathcal{E}$	the ring $\mathbb{Z}[e^{2\pi i/3}]$ of Eisenstein integers
$\mathbb{F}_q$	finite field of cardinality q
$\mathrm{ht}(x)$	height of a vector x given by $\mathrm{ht}(x) = \lvert\langle\bar{\rho}, x\rangle\rvert / \lvert\bar{\rho}\rvert^2$
G	$G \cong PGL_3(\mathbb{F}_3)$ is the automorphism group of $P^2(\mathbb{F}_3)$
H	the hyperbolic cell over Eisenstein integers with Gram matrix $\left(\begin{smallmatrix} 0 & \bar{\theta} \\ \theta & 0 \end{smallmatrix}\right)$
$II_{m,n}$	the even unimodular integer lattice of signature (m, n).
l_i	the 13 lines of $P^2(\mathbb{F}_3)$ or the roots corresponding to them
L	the Lorentzian Leech lattice over Eisenstein integers: $L = \Lambda \oplus H$
L'	the dual lattice of L.
Λ	the Leech lattice as a negative definite Hermitian Lattice over Eisenstein Integers
M	the finite group of automorphisms generated by G and σ acting on L having a unique fixed point $\bar{\rho}$ in the complex hyperbolic space $\mathbb{C}H^{13}$
ϕ_x	ω-reflection in the vector x
Φ	a root system. (Φ_L is the set of roots of L)
r	a root
R_1	the group generated by reflections in the 26 roots of D
$R(L)$	reflection group of the lattice L
$\mathrm{rad}(D)$	the set of roots obtained by repeatedly reflecting the roots of D in each other
ρ_i	$(\rho_1, \cdots, \rho_{26}) = (x_1, \cdots, x_{13}, \xi l_1, \cdots, \xi l_{13})$
$\bar{\rho}$	the Weyl vector.(the average of the ρ_i's)
σ	an automorphism of L of order 12 that interchanges the lines and points

θ	$\sqrt{-3}$
$T_{\lambda,z}$	a translation, an element of the Heisenberg group
$\mathbb{T}$	Heisenberg group: automorphisms fixing $\rho = (0^{12}; 0, 1)$ and acting trivially on $\rho^{\perp}/\rho$
ω	$e^{2\pi i/3}$
$\wr$	the wreath product.
x_i	the 13 points of $P^2(\mathbb{F}_3)$ or the roots corresponding to them
ξ	$e^{-\pi i/6}$

2.2.2 **Basic definitions about complex lattices**

The following definitions are mostly taken from section 2 of [All1]. They are included here for completeness. All the definite lattices are going to be negative definite.

Let $\omega = e^{2\pi i/3}$. Let $\mathcal{E} = \mathbb{Z}[\omega]$ be the ring of Eisenstein integers. A *lattice* K over $\mathcal{E}$ is a free module over $\mathcal{E}$ with an $\mathcal{E}$-valued Hermitian form $\langle\ ,\ \rangle$, that is conjugate linear in the first variable and linear in the second. Let $V = K \otimes_{\mathcal{E}} \mathbb{C}$ be the complex inner product space underlying K. All lattices considered in this chapter are lattices over $\mathcal{E}$ unless otherwise stated. We also let $[\ ,\ ]$ be the real valued alternating form given by $[a, b] = (1/\theta)\,\mathrm{Im}\langle a, b\rangle$ where $\theta = \sqrt{-3}$. The *real form* or $\mathbb{Z}$-form of a complex lattice is the underlying $\mathbb{Z}$-module with the Euclidean inner product $Tr \circ \langle\ ,\ \rangle$, that is, twice the real part of the Hermitian form.

A *Gram matrix* of a lattice is the matrix of inner products between a set of basis vectors of the lattice. The *signature* of the Hermitian form or the lattice K is (m, n) if the Gram matrix of a basis of K has m positive eigenvalues and n negative eigenvalues. A lattice of rank n is called *Lorentzian* (resp. *negative definite*) if it has signature $(1, n-1)$ (resp. $(0, n)$). The *dual* K' of a lattice K is defined as $K' = \{v \in V : \langle v, y\rangle \in \mathcal{E} \forall y \in K\}$. Many of the lattices we consider satisfy $L \subseteq \theta L'$. So all the inner products between lattice vectors are divisible by θ.

A vector in the lattice K is called *primitive* if it is not of the form αy with $y \in K$ and α a non-unit of $\mathcal{E}$. The *norm* of a vector v is $|v|^2 = \langle v, v \rangle$.

A *complex reflection* in a vector r is an isometry of the vector space V that fixes $r^{\perp}$ and takes r to ϵr, where $\epsilon \neq 1$ is a root of unity in $\mathcal{E}$. It is given by the formula

$$\phi_r^{\epsilon}(v) = v - r(1-\epsilon)\langle r, v \rangle / |r|^2 \tag{2.1}$$

ϕ_r^{ϵ} is called an ϵ-reflection in the vector r. For any automorphism γ of V we have $\gamma \phi_r^{\epsilon} \gamma^{-1} = \phi_{\gamma r}^{\epsilon}$. Let $\phi_r := \phi_r^{\omega}$. By a *root* of a negative definite or Lorentzian lattice K, we mean a primitive lattice vector of negative norm such that a nontrivial reflection in it is an automorphism of K. The *reflection group* $R(K)$ of K is the subgroup of the automorphism group $\mathrm{Aut}(K)$ generated by reflections in roots of K. The lattice E_8, Λ and L appearing below all satisfy $\theta K' = K$. Λ has no root. For E_8 and L, the roots are the lattice vectors of norm -3. The ω-reflections of order 3 in these roots generate their reflection group.

The following notation is used for the lattices K with $K \subseteq \theta K'$. Say that two roots a and b are *adjacent* if $|\langle a, b \rangle| = \sqrt{3}$. By a direct calculation this amounts to requiring that the ω-reflections in a and b braid, that is, $\phi_a \phi_b \phi_a = \phi_b \phi_a \phi_b$. Let Φ be a set of roots; the same symbol also often denotes the graph whose set of vertices is Φ and two vertices are joined if the corresponding roots are adjacent. In such situations roots are often considered only up to units of $\mathcal{E}$. We call the set of roots *connected* if the corresponding graph is connected. Suppose we have a set of roots Φ such that the inner product of any two distinct roots in that set is zero, $-\omega\theta$ or its conjugate. In such situations we consider a directed graph on the vertex set Φ. We draw an arrow from a vertex a to a vertex b if $\langle b, a \rangle = -\omega\theta$. Both these kind of graphs are called the *root diagrams*. Which kind is being considered will be clear from whether the graph in question is directed or undirected.

Let Φ be a set of roots of L. (Here and in what follows, we often do not distinguish between two roots that differ by an unit of $\mathcal{E}$. This does not cause any problem because ω-reflection in two such roots are the same). Define $\Phi^{(n)} =$

$\{\phi_a(b)\colon a,b \in \Phi^{(n-1)}\}$, and $\Phi_{(n)} = \{\phi^{\pm}_{a_1} \cdots \phi^{\pm}_{a_k}(b)\colon a_i, b \in \Phi, k \leq n\}$ where $\Phi^{(0)} = \Phi$. It can be easily seen that $\bigcup_n \Phi^{(n)} = \bigcup_m \Phi_{(m)}$. We call this set the *radical* of Φ and denote it by $\mathrm{rad}(\Phi)$. The radical of Φ consists of the set of roots obtained from Φ, by repeated reflection in themselves. We make the following elementary observations: if $\Phi \subseteq \Phi'$ then $\mathrm{rad}(\Phi) \subseteq \mathrm{rad}(\Phi')$ and $\mathrm{rad}(\mathrm{rad}(\Phi)) = \mathrm{rad}(\Phi)$.

2.2.3 Lemma. *If* Φ *is connected then* $\mathrm{rad}(\Phi)$ *is also connected.*

Proof. It suffices to show that each $\Phi^{(n)}$ is connected. We use induction on n. Let $c \in \Phi^{(n+1)}$. There are two elements a and b in $\Phi^{(n)}$ such that $c = \phi_a(b)$, and a connected chain, $a = a_0, a_1, \cdots, a_n = b$. Then $a, \phi_a(b_1), \cdots \phi_a(b_n) = c$ is a connected chain from a to c in $\Phi^{(n+1)}$ ☐

2.2.4 **The complex Leech Lattice**

Below we set up some conventions about the complex lattices that are important for our purpose. A general reference for lattices is the book [SPLAG] of Conway and Sloane; we just state the definitions that we need from the book.

The complex Leech lattice Λ consists of the set of vectors in $\mathcal{E}^{12}$ of the form

$$\{(m + \theta c_i + 3z_i)_{i=1,\cdots,12} : m = 0 \text{ or } \pm 1, (c_i) \in \mathcal{C}_{12}, \sum z_i \equiv m \mod \theta\} \qquad (2.2)$$

where $\mathcal{C}_{12}$ is the ternary Golay code in $\mathbb{F}_3^{12}$. The codes we need are defined explicitly in 2.8.1. The Hermitian inner product $\langle u, v\rangle = -\frac{1}{3}\sum \bar{u}_i v_i$ is normalized so that the minimal norm of a nonzero lattice vector is -6. The automorphism group of the complex Leech lattice is $6 \cdot Suz$, a central extension of Suzuki's sporadic simple group Suz by $\mathbb{Z}/6\mathbb{Z}$. The set Λ as a $\mathbb{Z}$-module with the inner product $-\frac{2}{3}\,\mathrm{Re}\langle\ ,\ \rangle$ is the usual real Leech lattice with minimal norm 4. The definition of the complex Leech lattice in (2.2) is similar to the description of the real Leech lattice as the collection of vectors in $\mathbb{Z}^{24}$ of the form

$$\{(m + 2c_i + 4z_i)_{i=1,\cdots,24} : m = 0 \text{ or } 1, (c_i) \in \mathcal{C}_{24}, \sum z_i \equiv m \mod 2\}$$

where $\mathcal{C}_{24}$ is the binary Golay code in $\mathbb{F}_2^{24}$.

2.2.5 **The complex E_8 lattice and the hyperbolic cell**

The root lattice E_8 can also be considered as a negative definite Hermitian lattice over $\mathcal{E}$. It can be defined as the subset of $\mathcal{E}^4$ consisting of vectors v such that the image of v modulo θ belong to the ternary tetracode $\mathcal{C}_4$ in $\mathbb{F}_3^4 \simeq (\mathcal{E}/\theta\mathcal{E})^4$. The inner product $\langle u, v\rangle = -\sum \bar{u}_i v_i$ is normalized to make the minimal norm -3. The reflection group of this lattice is $6{\cdot}\mathrm{PSp}_4(3)$ and is equal to the automorphism group. This group has a minimal set of generators consisting of four ω-reflections, and the corresponding roots form the usual A_4 Dynkin diagram with vertices c, d, e, f. (*remember* : the vertices of this "complex diagram" correspond to reflections of order 3). This will be called the E_8 root diagram or simply, the E_8 diagram.

We can find a unique root b' that we call the analog of the "the lowest root" such that we have $b' + (2+\omega)c + 2d + (2+\omega)e + f = 0$. Then b', c, d, e, f form the affine E_8 diagram (See fig. 2.1), that looks like usual A_5 Dynkin diagram (but the vertices have order 3 instead of 2).

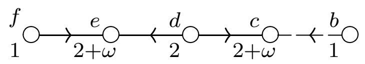


Figure 2.1: The affine E_8 diagram with the balanced numbering

Let $\bar{E}_8$ be the singular lattice $E_8 \oplus \mathcal{E}_0$, $\mathcal{E}_0$ being the one dimensional $\mathcal{E}$-lattice with zero inner product. Consider E_8 sitting in $\bar{E}_8$. Then the roots $b = b' + (0^4, 1), c, d, e, f$ generate the affine reflection group of $\bar{E}_8$. All this is very reminiscent of the theory of Euclidean reflection groups.

Let H be the Lorentzian lattice of signature (1,1), given by $\mathcal{E} \oplus \mathcal{E}$ with inner product $\langle u, v\rangle = u^* \left(\begin{smallmatrix} 0 & \bar{\theta} \\ \theta & 0 \end{smallmatrix}\right) v$, where u^* denotes the conjugate transpose of u. We call H the hyperbolic cell.

The L denote the Lorentzian $\mathcal{E}$-lattice $\Lambda \oplus H$. The minimum norm of a nonzero vector of L is -3, $L = \theta L'$ and the discriminant of L is 3^7. Elements of L are often written in the form $(\lambda; \alpha, \beta)$, where λ is a vector of length 12. $\lambda = (0, \cdots, 0)$ is abbreviated as $\lambda = 0^{12}$.

As a $\mathbb{Z}$-lattice, with the inner product $\frac{2}{3}Re \circ \langle \ , \ \rangle$, both L and $3E_8 \oplus H$ are even, unimodular and of signature $(2, 26)$. Hence they are both isomorphic to $II_{2,26}$. The following argument, that I learnt from D. Allcock, show that they are isomorphic over $\mathcal{E}$ too. We only briefly sketch the argument, because we find an explicit isomorphism later anyway.

2.2.6 Lemma. *There is at most one indefinite Eisenstein lattice T in a given signature satisfying $T = \theta T'$.*

Proof. Suppose T is an Eisenstein lattice with $T = \theta T'$. Then the Hermitian inner product defines a symplectic form on the $\mathbb{F}_3$ vector space T'/T. If we choose a Lagrangian or maximal isotropic subspace of T'/T and take the corresponding enlargement of T under pull-back, then we get a unimodular Eisenstein lattice. Therefore we can get all lattices T with $T = \theta T'$ by starting with a unimodular lattice and reversing the process. In the indefinite case there is only one unimodular lattice denoted by $\mathcal{E}^{1,n}$. (This follows from the uniqueness of indefinite unimodular integer lattice of given signature and type). So there is no choice about where to start. $\mathcal{E}^{1,n}/\theta\mathcal{E}^{1,n}$ is a vector space over $\mathbb{F}_3$, with a bilinear form on it; this bilinear form has isotropic subspace of rank half the dimension of the vector space. Reversing the process above amounts to passing to the sub-lattice of $\mathcal{E}^{1,n}$ corresponding to a maximal isotropic subspace of $\mathcal{E}^{1,n}/\theta\mathcal{E}^{1,n}$. Now, $\mathrm{Aut}(\mathcal{E}^{1,n})$ acts on $\mathcal{E}^{1,n}/\theta\mathcal{E}^{1,n}$ as the full orthogonal group of the form (just take the reflections of $\mathcal{E}^{1,n}/\theta\mathcal{E}^{1,n}$ given by reducing reflections of $\mathcal{E}^{1,n}$). Since this orthogonal group acts transitively on the maximal isotropic subspace, there is essentially only one way to construct T, so T is unique. ◻

The above proof is non-constructive. But we need an explicit isomorphism between $\Lambda \oplus H$ and $3E_8 \oplus H$ to perform calculations involving generators of automorphism group of the lattice; because the first set of generators we are able to write down are in the co-ordinate system $\Lambda \oplus H$ (and this uses special properties of the Leech lattice), while the roots of D that are of interest to us are naturally given in the other co-ordinate system $3E_8 \oplus H$. We give below the steps for this computation.

2.2.7 calculation

What we need from the following computation are the bases of $\Lambda \oplus H$ from which we get the change of basis matrix. These are given in 2.8.2 (see the matrices E_1 and E_2). The computer codes for these computation are contained in **isom.gp** and **isom2.gp**.

- (a) We start by making a list of all the vectors in the first shell of the Eisenstein Leech lattice using co-ordinates given in Wilson's paper [Wil2] (except that our co-ordinate for the ternary Golay code are different).
- (b) We find a regular 23 simplex, all whose 24 vertices are on the first shell of Λ. In other words, we find a set Δ of 24 minimal norm vectors in Λ such that the difference of any two distinct elements of Δ is also a vector of minimal norm.
- (c) We list the 4 tuples of vectors δ_i, $i = 1, \cdots, 4$ inside Δ such that there are 4 roots of L of the form $(\delta_i; 1, *)$ making an E_8 diagram. Each 4-tuple of vector generates a sub-lattice of L isomorphic to E_8.
- (d) In this list, we look for two E_8 lattices that are orthogonal to each other. (Both steps (c) and (d) amounts to checking a set of linear conditions among the entries of the matrix $([\delta_i, \delta_j])_I$: the imaginary parts of the inner products between the elements of Δ).
- (e) Next, choose one such subspace $K_1 \cong E_8 \oplus E_8$. If z is a primitive norm zero element perpendicular to K_1 then it is of type $3E_8$, because $z^\perp/z$ is a Niemeier lattice containing the root system $E_8 \oplus E_8$ and $3E_8$ is the only such.

- (f) We list all the vectors in the first shell of the Leech lattice such that they are at minimal distance from the 8 distinguished vectors forming the diagram of $K_1 \cong E_8 \oplus E_8$. We find that there are 8 such vectors. It is easy to find an E_8 diagram amongst these eight. So we get 12 vectors forming $3E_8$ diagram inside $L = \Lambda \oplus H$.

- (g) Now calculate the orthogonal complement of the $3E_8$ inside L, which is a hyperbolic cell H. It remains to find two norm zero vector inside H, that have inner product θ. This gives an explicit isomorphism from $\Lambda \oplus H$ to $3E_8 \oplus H$.

2.2.8 The complex hyperbolic space

We recall some basic facts about the complex hyperbolic space.

Let $\mathbb{C}^{1,n}$ be the $n+1$ dimensional complex vector space with inner product $\langle u, v\rangle = -\bar{u}_1 v_1 - \cdots - \bar{u}_n v_n + \bar{u}_{n+1} v_{n+1}$. The underlying inner product space of L is isomorphic to $\mathbb{C}^{1,13}$. The *complex hyperbolic space* $\mathbb{C}H^n$ is defined to be the set of complex lines of positive norm in $\mathbb{C}^{1,n}$. A vector in $\mathbb{C}^{1,n}$ and the point it determines in $\mathbb{C}H^n$ are denoted by the same symbol. Similar convention is adopted for hyperplanes. Let

$$c(u,v)^2 = |\langle u, v\rangle|^2 / |u|^2 |v|^2 \tag{2.3}$$

The distance $d(u,u')$ between two points u and u' in $\mathbb{C}H^n$ is given by the formula (see [Gol])

$$\cosh(d(u,u'))^2 = c(u,u')^2 \tag{2.4}$$

A vector v of negative norm determines a totally geodesic hyperplane $v^\perp$ in $\mathbb{C}H^n$. If r is a root of L, $r^\perp$ or its image in $\mathbb{C}H^{13}$ is called the *mirror* of a reflection in r.

If u is a point in $\mathbb{C}H^n$ and v is a vector of negative norm in $\mathbb{C}^{1,n}$ then the distance between u and the hyperplane $v^\perp$ is given by

$$\sinh^2(d(u, v^\perp)) = -c(u,v)^2 \tag{2.5}$$

We sketch a proof of (2.5) because we could not find a reference. Scale v so that $|v|^2 = -1$. Then $a = u + v\langle v, u\rangle$ is the intersection of the line joining u and v and the hyperplane $v^\perp$ in $\mathbb{C}H^n$. We claim that a minimizes the distance from u to $v^\perp$. Then (2.5) follows from (2.4).

Proof of the claim: We note that $|a|^2 = \langle a, u\rangle = |u|^2 + |\langle u, v\rangle|^2$. Let a' be any point in $\mathbb{C}H^n$ lying on $v^\perp$. We need to minimize $|\langle u, a'\rangle|^2/|a'|^2$. There is a basis of $v^\perp$ consisting of vectors $a, w_2, \cdots, w_n$ where the span of the w_i's is negative definite. So we can write $a' = a + w$ with w a vector of negative norm in $v^\perp$. We find that $|a'|^2 = |a|^2 + 2\operatorname{Re}\langle u, w\rangle + |w|^2$ and

$$\frac{|\langle u, a'\rangle|^2}{|a'|^2} = \frac{|(|a|^2 + \langle u, w\rangle)|^2}{|a'|^2} = \frac{|a|^4 + 2|a|^2\operatorname{Re}\langle u, w\rangle + |\langle u, w\rangle|^2}{|a'|^2}$$
$$= |a|^2 + \frac{|\langle u, w\rangle|^2 - |w|^2|a|^2}{|a'|^2}$$

The second term in the last expression is positive since $|a'|^2 > 0$ and and $|w|^2 < 0$. So it is minimized when $w = 0$, that is $a' = a$.

The distance between two hyperplanes $v^\perp$ and $v'^\perp$ is zero if $c(v, v')^2 \leq 1$ and otherwise is given by

$$\cosh^2(d(v^\perp, v'^\perp)) = c(v, v')^2 \tag{2.6}$$

The reflection group of L acts by isometries on the complex hyperbolic space $\mathbb{C}H^{13}$. The results about the complex hyperbolic space given here, hold in the context of the next chapter, for the quaternionic hyperbolic space too.

2.3 The root diagram of L

2.3.1 The diagram of 26 roots

As above, let L be direct sum of the Leech lattice and hyperbolic cell. In view of the isomorphism in 2.2.6 this is the same as $3E_8 \oplus H$. In this section we use this later co-ordinate system. There are 16 roots of L called $a, b_i, c_i, d_i, e_i, f_i, i = 1, 2, 3$, that form the M_{666} diagram (See Fig. 2.2). The three hands correspond to the three copies of E_8. The reflections in the roots c_i, d_i, e_i, f_i generate the reflection group of the three copies of E_8. Along with b_i they form the "affine E_8 diagrams".

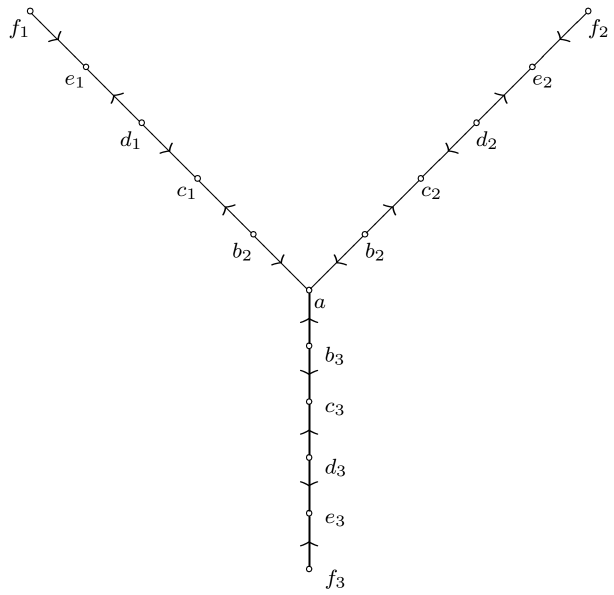


Figure 2.2: The M_{666} diagram

The remaining node a is called the "hyperbolizing node". ϕ_a fixes $3E_8$ and acts on the hyperbolic cell. These 16 roots can be uniquely extended to a diagram D of 26 roots. D is the incidence graph of the projective plane $P^2(\mathbb{F}_3)$ (See Fig. 2.3). $P^2(\mathbb{F}_3)$ has 13 points and 13 lines. There is one vertex in D for each point of $P^2(\mathbb{F}_3)$ and one for each line of $P^2(\mathbb{F}_3)$. Two vertices are joined if and only if the point is on the line.

We use the following shorthand borrowed from [CS], to draw the 26 node diagram (see Fig. 2.3). The index i runs from 1 to 3. A single vertex labeled a_i stands for three vertices a_1, a_2 and a_3. A single edge between a_i and b_i means that a_i and b_j are connected only if the indices i and j are equal. A double edge between a_i and f_i mean that a_i and f_j are connected if and only if the indices i and j are not equal. Both these root diagrams are directed, where the arrows always go from the "lines" to the "points" of $P^2(\mathbb{F}_3)$.

It was a surprise for me to find that there are these 26 roots with specified inner product that fit into a 14 dimensional lattice L. The actual co-ordinates used for

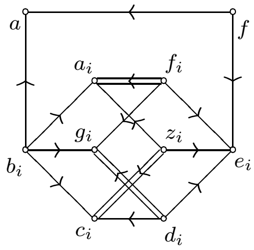


Figure 2.3: The 26 node diagram $D \cong \mathrm{Inc}(P^2(\mathbb{F}_3))$

the calculation are given below:

$$\begin{array}{rlllll}
r[1] = & a = [& & , & 1 & , \omega^2] \\
r[1+i] = & c_i = [& (-\theta\omega^2, \ , \ , \ )_i \ , & & , & , \ ] \\
r[4+i] = & e_i = [& (\ , -\theta\omega^2, \ , \ )_i \ , & & , & , \ ] \\
r[7+i] = & a_i = [& , & (\ , \ , \ , -\theta\omega^2)_{jk} \ , & \omega & , \omega^2] \\
r[10+i] = & g_i = [& (\ , \ , \ , -\theta\omega^2)_i \ , & (\ , \ , \theta\omega^2, -\theta\omega^2)_{jk} \ , & 2\omega & , 2\omega^2] \\
r[14] = & f = [& (\ , 1, 1, -2)^3 & , & -2+\omega & , -\theta] \\
r[14+i] = & f_i = [& (\ , 1, 1, 1)_i \ , & , & & , \ ] \\
r[17+i] = & b_i = [& (1, \ , 1, -1)_i \ , & , & -1 & , \ ] \\
r[20+i] = & z_i = [& (\ , 1, 1, -2)_i \ , & (1, \ , 1, -1)_{jk} \ , & -\theta\omega^2 & , -\theta] \\
r[23+i] = & d_i = [& (1, 1, -1, \ )_i \ , & , & & , \ ]
\end{array}$$

An E_8 vector with subscript i means we put it in place of the i-th E_8, while the subscript jk means that we put it at the j-th and k-th place. The indices i, j and k are in cyclic permutation of $(1, 2, 3)$. Blank spaces are to be filled with zero. Let $R_1 \subseteq R_2$ be the subgroups of $R(L)$ generated by ω-reflections in these 16 and the 26 roots respectively.

2.3.2 Lemma. *We have* $\mathrm{rad}(M_{666}) = \mathrm{rad}(D)$ *and hence* $R_1 = R_2$.

Proof. Take any A_{11} sub-diagram in M_{666} and add an extra vertex to make an affine A_{11} diagram. For example, the A_{11} Dynkin diagram

$(f_1, e_1, d_1, c_1, b_1, a, b_2, c_2, d_2, e_2, f_2)$ is completed to a 12-gon by adding the vertex a_3. The graph D is closed under this operation of "completing A_{11} to 12-gon" (see [CS]). Let $y = (y_1, \cdots, y_{12})$ be any "free" 12-gon in D. In the Conway-Simons presentation of the bimonster on 26 generators, there are the relations called "deflating a 12-gon" (see [CS]). For any 12-gon y in D let deflate(y) be the relation

$$(y_1 y_2 \cdots y_{10}) y_{11} (y_1 y_2 \cdots y_{10})^{-1} = y_{12}$$

To prove the lemma we just check that the reflections making the diagram D also satisfy the above relations. For the 12-gon mentioned above the relation deflate(y) follows from

$$\phi_{f_1} \phi_{e_1} \phi_{d_1} \phi_{c_1} \phi_{b_1} \phi_a \phi_{b_2} \phi_{c_2} \phi_{d_2} \phi_{e_2} (f_2) = \omega^2 a_3$$

Using the diagram automorphisms (see section 5) we see that the relation deflate(y) holds for the other 12-gons in D too. □

2.3.3 Linear relations among the roots of D

Let $x_1, \cdots, x_{13}$ denote the thirteen points of $P^2(\mathbb{F}_3)$ or the roots of L corresponding to them and $l_1, \cdots, l_{13}$ denote the 13 lines. These roots will often be referred to simply as *points* and *lines*. Let $\xi = e^{-\pi i/6}$ and $(\rho_1, \cdots, \rho_{26}) = (x_1, \cdots, x_{13}, \xi l_1, \cdots, \xi l_{13})$. We say ρ_i and ρ_j are connected if the corresponding roots are. For each i let Σ_i be the sum of the ρ_j's that are connected to ρ_i. For any line ρ_i the vector $\sqrt{3}\rho_i + \Sigma_i$ is an element of L that has norm 3 and is orthogonal to the points. But there is only one such vector. (This vector, called $w_{\mathcal{P}}$ in later sections, is fixed by the diagram automorphisms $PGL_3(\mathbb{F}_3)$). Thus for any two lines ρ_i and ρ_j we have

$$\sqrt{3}\rho_i + \Sigma_i = \sqrt{3}\rho_j + \Sigma_j \tag{2.7}$$

These relations generate all the relations among the roots of D. We shall see later that there is an automorphism σ of L interchanging ρ_i with $\xi\rho_{13+i}$. So equations (2.7) hold also for every pair of points ρ_i and ρ_j.

2.3.4 Remark. One can define the lattice L using the relations (2.7) as follows. Define a singular $\mathcal{E}$-lattice spanned by 26 linearly independent vectors of norm -3 corresponding to the vertices of D. Inner product of two vectors is zero if there is no edge between the corresponding vertices of D and $\sqrt{3}$ if there is an edge. Then L can be obtained by taking the quotient of this singular lattice by imposing the relations (2.7). The inner product is induced from the singular lattice.

2.4 Preliminary results about R(L)

In this section we give an elementary proof of the fact that the reflection group $R(L)$ act transitively on the root system Φ_L, and we describe how to write down generators for the reflection group of L.

2.4.1 Lemma. *Work in the co-ordinate system $\Lambda \oplus H$ for L. Look at the set of roots Ψ of the form $r = r_{\lambda,\beta} = (\lambda; 1, \theta\frac{(-3-|\lambda|^2)}{6} + \beta)$, where $\lambda \in \Lambda$ and $\beta \in \frac{1}{2}\mathbb{Z}$ is chosen satisfying $2\beta + 1 \equiv |\lambda|^2 \mod 2$, so that the last entry of r is in $\mathcal{E}$. Then Ψ is a connected set.*

Proof. Recall that $[a, b] = \frac{\operatorname{Im}\langle a,b\rangle}{\theta}$. By direct computation we have

$$\langle r, r'\rangle = (-3 - \frac{1}{2}|\lambda - \lambda'|^2) + \theta(\beta - \beta' + [\lambda, \lambda'])$$

We find that r and r' are adjacent if and only if $|\langle r, r'\rangle| = \sqrt{3}$. This can happen in two ways:

Case(I). $\langle r, r'\rangle = \pm\theta$, that is, $|\lambda - \lambda'|^2 = -6$ and $\beta' = \beta + [\lambda, \lambda'] \pm 1$

or,

Case(II). $\langle r, r'\rangle = \pm\theta\omega$ or $\pm\theta\omega^2$, that is, $|\lambda - \lambda'|^2 = -3$ or -9 and $\beta' = \beta + [\lambda, \lambda'] \pm 1/2$.

(note that the case $|\lambda - \lambda'|^2 = -3$ does not occur here)

Given a root $r_{\lambda,\beta} \in \Psi$ choose λ' satisfying $|\lambda - \lambda'|^2 = -9$, and choose $\beta' = \beta + [\lambda, \lambda'] + 1/2$. $r' = r_{\lambda',\beta'}$ belongs to Ψ. By case(II) above, both $r_{\lambda,\beta}$ and $r_{\lambda,\beta+1}$

are adjacent to r', hence are connected by a chain of length 2. It follows that any two roots $r_{\lambda,\beta}$ and $r_{\lambda,\alpha}$ in Ψ are connected by a chain, because $\beta - \alpha \in \mathbb{Z}$.

Given a root $r_{\lambda,\beta}$ in Ψ and λ' in Λ, with $|\lambda - \lambda'|^2 = -6$, it follows form case(I) above, that there is some β' such that $r_{\lambda',\beta'}$ is a root connected to $r_{\lambda,\beta}$. Hence two roots of the form $r_{\lambda,\alpha}$ and $r_{\lambda',\alpha'}$ are connected for all α and α' , whenever $|\lambda - \lambda'|^2 = -6$.

Now given any two roots $r_{\lambda,\beta}$ and $r_{\lambda',\beta'}$ in Ψ there is a connected chain $\lambda = \lambda_0, \lambda_1, \cdots, \lambda_n = \lambda'$ such that $|\lambda_i - \lambda_{i+1}|^2 = -6$, because the vectors of Λ of norm -6 generate Λ. Hence $r_{\lambda,\beta}$ and $r_{\lambda',\beta'}$ are connected. □

2.4.2 Proposition. *Let $L = \Lambda \oplus H$. Let Ψ be the roots of the form*

$$r = (\lambda; 1, \theta\frac{(-3 - |\lambda|^2)}{6} + \beta + n)$$

where β is chosen so that the last entry is in $\mathcal{E}$, $\lambda \in \Lambda$, n an integer. Then $\mathrm{rad}(\Psi) = \Phi_L$. *In particular Ψ generate the reflection group of L.*

Proof. This is exactly like Conway's calculation in [Con1] of the reflection group of $II_{1,25}$ as adapted by Allcock. Let $\rho = (0^{12}; 0, 1)$ and define $h(r) = |\langle r, \rho\rangle/\theta|$, so that Ψ is the set of roots r with $h(r) = 1$. (h is only used in this proof). We show that if $h(\mu) > 1$ then there exists a reflection $\phi_r^{\pm}$ in a root $r \in \Psi$ such that $h(\phi_r^{\pm}(\mu)) < h(\mu)$. This proof uses the covering radius of the Leech lattice being $\sqrt{2}$ and would not have worked if the radius were any bigger.

Suppose μ is a root such that $h(\mu) > 1$. Let $w = (l; 1, \alpha - \frac{\theta|l|^2}{6})$ be a scalar multiple of μ such that $h(w) = 1$. Let $\alpha = \alpha_1 + \theta\alpha_2$. We have $-3 < w^2 < 0$; it follows that $-\frac{1}{2} < \alpha_2 < 0$.

We try to find an ϵ-reflection in a root r of the form $(\lambda; 1, \theta\frac{(-3-|\lambda|^2)}{6} + \beta + n)$ so that $h(\phi_r^{\epsilon}(w)) < h(w)$. We have

$$\langle r, w\rangle = -3(a + b)$$

where a is real

$$a = \frac{1}{2} + \frac{|l - \lambda|^2}{6} - \alpha_2$$

and b is imaginary

$$b = n\theta^{-1} + \alpha_1\bar{\theta}^{-1} + \beta\theta^{-1} - \frac{\mathrm{Im}\langle\lambda, l\rangle}{3}$$

Then $h(\phi_r^\epsilon(w))$ is equal to $|1-(1-\epsilon)(a+b)|$.

By choice of n, can make $b \in [-\frac{\theta}{6}, \frac{\theta}{6}]$, and by choice of λ can make $0 \geq \frac{|l-\lambda|^2}{6} \geq -\frac{1}{2}$. (*Note:* covering norm of Leech lattice in our scaling is -3). Hence, using $-\frac{1}{2} < \alpha_2 < 0$, we get that $1 > a > 0$. If $b \in [-\frac{\theta}{6}, 0]$, choose $\epsilon = \omega^2$, and if $b \in [0, \frac{\theta}{6}]$ choose $\epsilon = \omega$. Then we get that $h(\phi_r^\epsilon(w))$ is strictly less than 1. □

2.4.3 Proposition. *Φ_L is connected. So $R(L)$ acts transitively on Φ_L.*

Proof. By 2.4.1 Ψ is connected and by 2.4.2 $\mathrm{rad}(\Psi) = \Phi_L$. So by 2.2.3 Φ_L is connected. Hence any two roots can be connected by a chain of adjacent roots. If a and b are adjacent roots then $\phi_a\phi_b a$ is an unit multiple of b, so a and b are in the same orbit under $R(L)$. □

2.4.4 **Generators for the reflection group of L**

Let $\mathbb{T}$ be the subgroup of $\mathrm{Aut}(L)$ consisting of the automorphisms of L that fix the vector $\rho = (0^{12}; 0, 1)$ and act trivially on $\rho^\perp/\rho$. $\mathbb{T}$ is called the group of *translations.*

$$\mathbb{T} = \{T_{\lambda,\theta\alpha/2} | \lambda \in \Lambda, \alpha \in \mathbb{Z}, \alpha = |\lambda|^2 \bmod 2\}$$

where $T_{\lambda,z}$ is given by

$$T_{\lambda,z} = \begin{pmatrix} I & \lambda & 0 \\ 0 & 1 & 0 \\ \theta^{-1}\lambda^* & \bar{\theta}^{-1}(z - |\lambda|^2/2) & 1 \end{pmatrix}$$

where I is the identity acting on Λ and λ^* is the linear map on Λ given by $x \mapsto \langle\lambda, x\rangle$. We call $T_{\lambda,z}$, a translation by λ.

2.4.5 Lemma. *Let $d = dim_{\mathcal{E}}(\Lambda)$. Let ϕ_{r_1} and ϕ_{r_2} be the ω-reflections in the roots $r_1 = (0^d; 1, \omega^2)$ and $r_2 = (0^d; 1, -\omega)$. Let $\lambda_1, \cdots, \lambda_{2d}$ be a basis of Λ over $\mathbb{Z}$. Choose z_j such that $T_{\lambda_j,z_j} \in \mathbb{T}$. Then reflections in the roots $T_{\lambda_j,z_j}(r_1)$ and $T_{\lambda_j,z_j}(r_2)$ for $j = 1, \cdots 2d$, together with r_1 and r_2 generate the reflection group of L.*

Proof. Here $d = 12$. The lemma is deduced from the work done in section 3 of [All1] as follows. From the equation $T_{\lambda_1,z_1} \circ T_{\lambda_2,z_2} = T_{\lambda_1+\lambda_2,z_1+z_2+\mathrm{Im}\langle\lambda_1,\lambda_2\rangle}$, we see that the group generated by the $2d$ translations of the form $T_{\lambda_i,*}$ contains a translation by every vector of the Leech lattice. It also contains all the central translations $T_{0,z}$ because if we take λ and λ' in Λ with $\langle\lambda,\lambda'\rangle = -\theta\omega$ then $T_{\lambda,z}^{-1}T_{\lambda',z'}^{-1}T_{\lambda,z}T_{\lambda',z'} = T_{0,\theta}$ Hence the $2d$ translations $T_{\lambda_i,*}$ generate whole of $\mathbb{T}$.
The orbit of the root $r_1 = (0^{12}; 1, \omega^2)$ under $\mathbb{T}$ is all the roots of the form $(\lambda; 1, *)$, and by 2.4.2 they generate $R(L)$. So $\mathbb{T}$ together with ϕ_{r_1} generate it too. Now from the equation $T_{\lambda,z}\phi_{r_1}\phi_{r_2}T_{\lambda,z}^{-1}(\phi_{r_1}\phi_{r_2})^{-1} = T_{-\omega\lambda,\theta|\lambda|^2/2}$ it follows that reflections in the roots $T_{\lambda_j,z_j}(r_1)$ and $T_{\lambda_j,z_j}(r_2)$ for $j = 1, \cdots 2d$, together with r_1 and r_2 generates $\mathbb{T}$. So these $4d+2$ roots generate the whole reflection group $R(L)$. □

2.5 The diagram automorphisms

2.5.1 The action of the automorphism group of the diagram D

The group $G \cong PGL_3(\mathbb{F}_3)$ acts on the 26 node graph D. We fix co-ordinates where $\mathcal{P} = \{a, c_i, e_i, a_i, g_i\}$ are the points and $\mathcal{L} = \{f, b_i, d_i, f_i, z_i\}$ are the lines of $P^2(\mathbb{F}_3)$. For computations on $P_2(\mathbb{F}_3)$, the points $x \in \mathcal{P}$ are represented as column vectors of length 3 and the lines $l \in \mathcal{L}$ as row vectors of length 3. The incidence pairing is given by $(x, l) \to lx$. (See the pari code **P2F3.gp** for an actual co-ordinates used). An element g of G can be represented by a 3×3 matrix which acts on the points by $x \mapsto gx$ and on the lines by $l \mapsto lg^{-1}$, so that it preserves the incidence pairing.

$\mathcal{P}$ and $\mathcal{L}$ will also denote the set of roots of L given in 2.3.1. An element of $\mathcal{P}$ will be simply referred to as a *point* and an element of $\mathcal{L}$ as a *line*.

2.5.2 Lemma. *The action of G on D induces a linear action of G on the lattice L.*

Proof. The lemma follows from the construction of the lattice L given in 2.3.4

since the relations (2.7) are invariant under G. It can also be proved directly by verifying that the generators of G acting on L satisfy the relations of the following presentation of $PGL_3(\mathbb{F}_3)$. $G = \langle x, y | x^2 = y^3 = (xy)^{13} = r = 1 \rangle$ where $r = ((xy)^4 xy^{-1})^2 xyxyxy^{-1}xy^{-1}xyxy^{-1}xy^{-1}xyxyxy^{-1}$. $\square$

2.5.3 The automorphism σ

Actually a group slightly larger that G acts naturally on mirrors orthogonal to the roots of D. To see this note that the automorphism $\bar{\sigma}$ that takes a point $x = [a : b : c]$ to the line $l = \{aX + bY + cZ = 0\}$ lifts to an automorphism σ of L of order 12. σ takes a point x to $-\omega l$ and a line l back to x and its square is equal to $-\omega$. Let M be the group generated by G and σ. M acts transitively and faithfully on the unit multiples of the roots of D. For $g \in G$, $\sigma g \sigma^{-1}$ preserves the set of points and hence is in G. So we have an exact sequence

$$1 \to PGL_3(\mathbb{F}_3) \to M \to \mathbb{Z}/12\mathbb{Z} \to 1 \tag{2.8}$$

Recall $(\rho_1, \cdots, \rho_{26}) = (x_1, \cdots, x_{13}, \xi l_1, \cdots, \xi l_{26})$. Then σ interchanges ρ_i with $\xi \rho_{13+i}$.

2.5.4 The fixed vectors of the action of G on L

The group G fixes a two dimensional sub-lattice F of signature $(1,1)$. F is spanned by two norm 3 vectors $w_{\mathcal{P}}$ and $w_{\mathcal{L}}$, where $w_{\mathcal{P}} = [(0,0,\theta,-2\theta)^3; 4\omega^2, 4]$ and $w_{\mathcal{L}} = [(\omega,\omega,2\omega,-3\omega)^3; -2-5\omega, -2\theta\omega]$. These two vectors can be expressed by

$$w_{\mathcal{P}} = \omega^2 \theta l + \sum_{x \in l} x, \qquad w_{\mathcal{L}} = -\omega\theta x + \sum_{x \in l} l \tag{2.9}$$

where x is any point and l is any line (also see (2.7)). $w_{\mathcal{P}}^{\perp}$ is the 13 dimensional negative definite subspaces containing the points $\mathcal{P}$, and $w_{\mathcal{L}}^{\perp}$ is the 13 dimensional negative definite subspaces containing the lines $\mathcal{L}$. We note $\langle w_{\mathcal{P}}, w_{\mathcal{L}} \rangle = -4\theta\omega$ and so discriminant of F is 39. From the explicit co-ordinates for $w_{\mathcal{P}}$ and $w_{\mathcal{L}}$ it can be checked that F is primitive, and thus is equal to the sub-lattice fixed by G.

The vectors corresponding to the sum of points,

$$\Sigma_{\mathcal{P}} = \sum_{x \in \mathcal{P}} x = [-\theta\omega^2(1,1,-2,5)^3; 1+9\omega, 10\omega^2]$$

and the vector corresponding to the sum of lines,

$$\Sigma_{\mathcal{L}} = \sum_{l \in \mathcal{L}} l = [(4,4,5,-6)^3; -8+4\omega, -4\theta]$$

are also fixed by the action of G. They span a finite index sub-lattice of F.

2.5.5 The "Weyl vector"

There is a unique fixed point $\bar{\rho}$ in $\mathbb{C}H^{13}$ under the action of the group M that we call the *Weyl vector*. Since $\sigma(\Sigma_{\mathcal{P}}) = -\omega\Sigma_{\mathcal{L}}$ and $\sigma(\Sigma_{\mathcal{L}}) = \Sigma_{\mathcal{P}}$, the lines containing the vectors

$$\bar{\rho}_{\pm} = (\Sigma_{\mathcal{P}} \pm \xi\Sigma_{\mathcal{L}})/26 \tag{2.10}$$

are fixed by M, where $\xi = e^{-\pi i/6}$. Of these two vectors $\bar{\rho} = \bar{\rho}_+$ has positive norm, and $\bar{\rho}_-$ have negative norm. So $\bar{\rho}$ given by

$$\bar{\rho} = (\Sigma_{\mathcal{P}} + \xi\Sigma_{\mathcal{L}})/26 = (w_{\mathcal{P}} + \xi w_{\mathcal{L}})/2(4+\sqrt{3}) \tag{2.11}$$

is the unique point of $\mathbb{C}H^{13}$ fixed by M. We note some inner products:

$$\langle \rho_i, \rho_j \rangle = \sqrt{3} \text{ or } 0 \tag{2.12}$$

$$\langle w_{\mathcal{P}}, \bar{\rho} \rangle = \sqrt{3}/2 \tag{2.13}$$

$$\langle \bar{\rho}, \rho_i \rangle = |\bar{\rho}|^2 = (4\sqrt{3}-3)/26 \tag{2.14}$$

$$|\bar{\rho}_-|^2 = (-3-4\sqrt{3})/26 \qquad \text{and} \qquad \langle \bar{\rho}_-, \rho_j \rangle = \pm|\bar{\rho}_-|^2 \tag{2.15}$$

In (2.15) the positive sign holds if ρ_j correspond to a point and negative sign holds otherwise.

The fixed vector $\bar{\rho}$ can be expressed more canonically as follows. Let χ be the character of the group $\mathbb{Z}/12\mathbb{Z}$ generated by σ, given by $\sigma \mapsto \xi^{-1}$ extended to a character of the group M. Define the sum

$$\tau = \frac{1}{m}\sum_{g\in M}\chi(g)g$$

and $\tau_1 = \sum_{g\in G} g$, where m is the order of the group M. Then

$$\begin{aligned} m\tau = \sum_{j=1}^{12}\sum_{g\in G}\chi(\sigma^j g)\sigma^j g &= \sum_{j=1}^{6}\sum_{g\in M}\xi^{-2j}\sigma^{2j}g + \xi^{-2j+1}\sigma^{2j-1}g \\ &= 6\sum_{g\in G} g + \xi^{-1}\sigma g = 6(1+\xi^{-1}\sigma)\tau_1 \end{aligned}$$

Let x be any point in $\mathcal{P}$. If $n_x = m/12.13$ is the size of the stabilizer of x in G we have $\tau_1 x = n_x \Sigma_{\mathcal{P}}$, $\sigma\tau_1 x = \xi n_x(\xi\Sigma_{\mathcal{L}})$, $\tau_1(\xi l) = n_x(\xi\Sigma_{\mathcal{L}})$ and $\sigma\tau_1(\xi l) = \xi n_x \Sigma_{\mathcal{P}}$. It follows that $\tau\rho_i = 6n_x.26\bar{\rho}/m = \bar{\rho}$. We also have $\langle \tau u, v\rangle = \langle u, \tau v\rangle$ for all u and v. Thus τ is a self adjoint rank one idempotent and its image gives the unique point in the hyperbolic space fixed by M.

2.5.6 **Height of a root**

Let $r = \sum \lambda_i\rho_i$. then

$$\langle\bar{\rho}, r\rangle = \langle\bar{\rho}, \sum\rho_i\lambda_i\rangle = |\bar{\rho}|^2\sum\lambda_i.$$

The vector $\bar{\rho}$ has some of the formal properties of a Weyl vector in a reflection group, while the 26 roots of D play the role of simple roots. Define the *height* of a root r by

$$\mathrm{ht}(r) = |\langle\bar{\rho}, r\rangle|/|\bar{\rho}|^2 \tag{2.16}$$

The 26 roots of D have height 1. We will see later that this is the minimum height of a root. This suggests the strategy for the proof the the following theorem.

2.5.7 Theorem. *The 26 ω-reflections in the roots of D generate the reflection group of L.*

Proof. Let R_1 be the group generated by the ω-reflections in the 26 roots of D. Take the 50 roots $g_1, \cdots, g_{50}$ as in Lemma 2.4.5 generating the reflection group of L. We convert them to co-ordinate system $3E_8 \oplus H$ using the explicit isomorphism found by the method described in 2.2.7. One can prove the theorem by showing that each ϕ_{g_i} is in R_1. The algorithm for this is the following.

Start with a root $y_0 = g_i$ and find a $r \in D$ and $\epsilon \in \{\omega, \bar{\omega}\}$ such that $y_1 = \phi_r^\epsilon(y_0)$ has height less than y_0. Then repeat the process with y_1 instead of y_0 to get an y_2 such that $\mathrm{ht}(y_2) < \mathrm{ht}(y_1)$, etc. Continue until some y_n is an unit multiple of an element of D.

In many cases, such as for $y_0 = g_3$, this algorithm works. That is, we find that some y_n is an unit multiple of D, showing that the reflection ϕ_{g_3} is in R_1.

But this does not work for all the generating roots g_i. For example, when the above algorithm is implemented starting with $y_0 = g_1$ it gets stuck at a root y_k whose height cannot be decreased by a single reflection in any root of D. In such a situation we perturb y_k by reflecting in one of the reflections ϕ_{g_j} that has already been shown in R_1. For example, we take $y_{k+1} = \phi_{g_3}(y_k)$ and then run the algorithm again on y_{k+1}. If the algorithm works now then too it follows that ϕ_{g_1} is in R_1. Or else one could repeat this process by perturbing again. It was verified that the algorithm works with at-most one perturbation for all the generating roots $g_1, \cdots, g_{50}$.

The details of the computer programs that are needed for the validation of the above claims are given in the last section of this chapter. □

2.5.8 Theorem. *The automorphism group of L is equal to the reflection group. So the 26 reflections of D generate the full automorphism group of L.*

Proof. The idea of the proof is due to D. Allcock. He showed in [All1] that the group $6 \cdot Suz$ of automorphisms of Λ maps onto the quotient $\mathrm{Aut}(L)/R(L)$. We find a permutation group S_3 in the intersection of $R(L)$ and $6 \cdot Suz$ showing that $6 \cdot Suz \subseteq R(L)$, because a normal subgroup of $6 \cdot Suz$ containing an S_3 has to be the whole group.

Working in the co-ordinates $\Lambda \oplus H$ we can find a M_{666} diagram inside L such that all the 12 roots of the three E_8 hands, that is, c_i, d_i, e_i, f_i have the form $(\lambda; 1, \eta)$ up-to units, where λ is in the first shell of the Leech lattice. Such a set of vectors is written down in 2.8.2. See the matrix E_1' and the vectors $a', b_i', \cdots, f_i'$.

Consider the automorphisms φ_{12} and φ_{23} of order two that fixes respectively the third and the first hand of the M_{666} diagram and flips the other two. This corresponds to interchanging two E_8 components fixing the third E_8. They generate a subgroup S_3 of diagram automorphisms.

The automorphism φ_{12} flips the A_{11} braid diagram of the first and the second hand formed by $f_1, e_1, d_1, c_1, b_1, a, b_2, c_2, d_2, e_2, f_2$ and fixes the A_4 disjoint from it in the third hand formed by f_3, e_3, d_3, c_3. This determines φ_{12} because these 15 roots contain a basis of L.

The automorphism of flipping the braid diagram in 12 strand braid group is inner, so φ_{12} and φ_{23} are in the reflection group. But they also fix the vector $\rho = (0^{12}; 0, 1) \in \Lambda \oplus H$ which is a norm zero vector of Leech type. In-fact one can directly check that the automorphisms φ_{12} and φ_{23} fix the hyperbolic cell spanned by $(0^{12}; 0, 1)$ and $(-2, \omega^2, \omega^2, 1, \omega, -2\omega, 1, \omega^2, 1, 1, 1, \omega, 1, \theta)$ in $\Lambda \oplus H$. So they are in $6 \cdot Suz$ too. □

2.6 The minimum height of a root

Let $\xi = e^{-\pi i/6}$ and $(\rho_1, \cdots \rho_{26}) = (x_1, \cdots, x_{13}, \xi l_1, \cdots, \xi l_{13})$, as before. If y is equal to a multiple of a root r then $\phi_y = \phi_r$. We see that $\phi^{\pm}_{\rho_j}(\rho_i)$ is equal to $\rho_i + \xi^{\pm}\rho_j$ or ρ_i according to whether the two nodes ρ_i and ρ_j are connected in D or not. Let $r = \sum \alpha_j \rho_j$. Then

$$\begin{aligned}\phi^{\pm}_{\rho_i}(r) &= \omega^{\pm}\alpha_i\rho_i + \sum_{j:\langle\rho_i,\rho_j\rangle\neq 0} \alpha_j(\rho_j + \xi^{\pm}\rho_i) + \sum_{j:\langle\rho_i,\rho_j\rangle = 0} \alpha_j\rho_j \\ &= \Big(\omega^{\pm}\alpha_i + \xi^{\pm} \sum_{j:\langle\rho_i,\rho_j\rangle\neq 0} \alpha_j\Big)\rho_i + \sum_{j\neq i} \alpha_j\rho_j\end{aligned}$$

By induction we can prove that all elements r in $\mathrm{rad}(D)$ can be written as $\sum \alpha_i \rho_i$ with $\alpha_i \in \mathbb{Z}[\xi]$. Thus $\langle r, \bar{\rho}\rangle / |\bar{\rho}|^2 = \sum \alpha_i$ is an algebraic integer. Note that the norm of the the algebraic integer $\langle r, \bar{\rho}\rangle / |\bar{\rho}|^2$ is equal to $\mathrm{Nm}(r) = \frac{|\langle r,\bar{\rho}\rangle|^2}{|\bar{\rho}|^4} \frac{|\langle r,\bar{\rho}_-\rangle|^2}{|\bar{\rho}_-|^4} \in \mathbb{Z}$. Experimentally we find that a lot of roots have $\mathrm{Nm}(r) = 1$. We tried defining the height of r by $\mathrm{Nm}(r)$ in the proof of 2.5.7 but failed to make the height reduction algorithm converge even with a lot of computer time. We also tried other definitions of height, for example, absolute value of inner product with some other norm zero or norm 3 vectors of the lattice L, with the same effect.

The following proposition also seem to justify the definition of height we use.

2.6.1 Proposition. *All the roots of L have height greater than or equal to one and the unit multiples of the 26 roots of D are the only ones with minimal height.*

We use the triangle inequality for distances in the complex hyperbolic space to bound the possibilities for a root r with $\mathrm{ht}(r) \leq 1$. These calculations (that we are going to need more than once) are contained in the following lemma.

2.6.2 Lemma. *Suppose L is a $\mathcal{E}$ lattice, $\rho_i \in L \otimes \mathbb{C}, i = 1, \cdots, k$ is a finite set of vectors with $|\rho_i|^2 = -3$ and $\langle \bar{\rho}, \rho_i\rangle = |\bar{\rho}|^2$ where $\bar{\rho} = \sum_1^k \rho_i / k$. Suppose if possible r be a root of L with $\mathrm{ht}(r) = |\langle \bar{\rho}, r\rangle| / |\bar{\rho}|^2 \leq 1$. Then, for each i we have*

$$|\langle r, \rho_i\rangle| \leq \min\{3, (2|\bar{\rho}|^2 + 3)\} \tag{2.17}$$

If w is a vector of positive norm in L then

$$|\langle r, w\rangle| \leq 3 \sinh\Big(\sinh^{-1}\Big(\frac{|\bar{\rho}|}{\sqrt{3}}\Big) + \cosh^{-1}\Big(\frac{|\langle w, \bar{\rho}\rangle|}{\sqrt{3}|\bar{\rho}|}\Big)\Big) \tag{2.18}$$

Proof. Recall the few elementary fact about the complex hyperbolic space from 2.2.8. From (2.3) and (2.16) We have

$$c(\bar{\rho}, r)^2 = -\,\mathrm{ht}(r)^2 |\bar{\rho}|^2 / 3 \tag{2.19}$$

From the triangle inequality we get $d(r^\perp, y^\perp) \leq d(r^\perp, \bar{\rho}) + d(y^\perp, \bar{\rho})$ for any other root y. Using the distance formulae (2.5) and (2.6) the inequality takes the form

$$c(r,y)^2 = \cosh^2(d(r^\perp, y^\perp)) \leq \cosh^2(\sinh^{-1}(|c(\bar{\rho}, r)|) + \sinh^{-1}(|c(\bar{\rho}, y)|) \quad (2.20)$$

$$\leq \cosh^2\Big(\sinh^{-1}\Big(\frac{|\bar{\rho}|}{\sqrt{3}}\Big) + \sinh^{-1}\Big(\frac{\mathrm{ht}(y)|\bar{\rho}|}{\sqrt{3}}\Big)\Big) \quad (2.21)$$

if $c(y,r)^2 > 1$. From this we get the following bound: either $|\langle r, y\rangle| \leq 3$ or

$$|\langle r, y\rangle| \leq 3\cosh\Big(\sinh^{-1}\Big(\frac{|\bar{\rho}|}{\sqrt{3}}\Big) + \sinh^{-1}\Big(\frac{\mathrm{ht}(y)|\bar{\rho}|}{\sqrt{3}}\Big)\Big) \quad (2.22)$$

Using the inequality (2.22) with $y = \rho_i$ gives

$$|\langle r, \rho_i\rangle| \leq 3\cosh(2\sinh^{-1}(|\bar{\rho}|/\sqrt{3})) = (2|\bar{\rho}|^2 + 3) \quad (2.23)$$

Similarly using (2.4), (2.5), (2.13), (2.19) and the triangle inequality $d(w_{\mathcal{P}}, r^\perp) \leq d(w_{\mathcal{P}}, \bar{\rho}) + d(\bar{\rho}, r^\perp)$ we get the bound

$$|\langle w_{\mathcal{P}}, r\rangle|^2 \leq 9\sinh^2(\sinh^{-1}(|c(\bar{\rho}, r)|) + \cosh^{-1}(|c(\bar{\rho}, w_{\mathcal{P}})|)) \quad (2.24)$$

This proves (2.18) □

Now we come to the proof of the proposition 2.6.1. The proof does not use the fact that the reflection group is generated by the 26 nodes of D.

Proof. Suppose if possible r be a root of L with $\mathrm{ht}(r) = |\langle \bar{\rho}, r\rangle|/|\bar{\rho}|^2 \leq 1$. We will show that r must be an unit multiple of an element of D. Using the inequality (2.17) we get, either $|\langle r, \rho_i\rangle| \leq 3$ or

$$|\langle r, \rho_i\rangle|^2 \leq (2|\bar{\rho}|^2 + 3)^2 \approx 10.9$$

But the inner product between lattice vectors are in $\theta\mathcal{E}$, hence we have

$$|\langle r, \rho_i\rangle| \leq 3 \quad (2.25)$$

Similarly using (2.18) we get the bound

$$|\langle w_{\mathcal{P}}, r\rangle|^2 \leq 9\sinh^2(\sinh^{-1}(|c(\bar{\rho}, r)|) + \cosh^{-1}(|c(\bar{\rho}, w_{\mathcal{P}})|)) \approx 11.2$$

And thus

$$|\langle r, w_{\mathcal{P}}\rangle| \leq 3 \tag{2.26}$$

Write r in terms of the basis $x_1, \cdots, x_{13}, w_{\mathcal{P}}$ as

$$r = \sum -\langle x_i, r\rangle x_i/3 + \langle w_{\mathcal{P}}, r\rangle w_{\mathcal{P}}/3 \tag{2.27}$$

Taking norm we get

$$-3 = -\sum |\langle x_i, r\rangle|^2/3 + |\langle w_{\mathcal{P}}, r\rangle|^2/3 \tag{2.28}$$

Changing r up-to units we may assume that $\langle r, w_{\mathcal{P}}\rangle$ is either 0 or θ or 3 and it follows from (2.28) that $\sum|\langle x_i, r\rangle|^2$ is equal to 9, 12 and 18 respectively. In each of these three cases we will check that there is no root of height less than one thus finishing the proof. In the following let u_1, u_2 etc. denote units in $\mathcal{E}^*$.

If $\langle r, w_{\mathcal{P}}\rangle = 0$ and $\sum|\langle x_i, r\rangle|^2 = 9$, the unordered tuple $(\langle x_1, r\rangle, \cdots, \langle x_{13}, r\rangle)$ is equal to $(3u_1, 0^{12})$ or $(\theta u_1, \theta u_2, \theta u_3, 0^{10})$. So either r is an unit multiple of x_i (in which case it has height equal to one) or $r = (\theta u_1 x_1 + \theta u_2 x_2 + \theta u_3 x_3)/(-3)$. Using diagram automorphisms we can assume that $x_1 = a$ and $x_2 = c_1$ and then check that there is no such root r.

If $\langle r, w_{\mathcal{P}}\rangle = \theta$ and $\sum|\langle x_i, r\rangle|^2 = 12$, the unordered tuple $(\langle x_1, r\rangle, \cdots, \langle x_{13}, r\rangle)$ is equal to $(3u_1, \theta u_2, 0^{11})$ (Case I) or $(\theta u_1, \theta u_2, \theta u_3, \theta u_4, 0^9)$ (Case II). In (case I) we get $r = \theta u_1 x_1/(-3) + 3u_2 x_2/(-3) + \theta w_{\mathcal{P}}/3$. Taking inner product with $\bar{\rho}$ and using $\langle \bar{\rho}, w_{\mathcal{P}}\rangle/|\bar{\rho}|^2 = 4 + \sqrt{3}$ we get $\langle \bar{\rho}, r\rangle/|\bar{\rho}|^2 = (u_1 - u_2\theta - 4 - \sqrt{3})/\theta$ which clearly has norm greater than one. In (case II) we get $r = \theta w_{\mathcal{P}}/3 + \sum_{i=1}^4 \theta u_i x_i/(-3)$ which give $\langle \bar{\rho}, r\rangle/|\bar{\rho}|^2 = (-4 - \sqrt{3} + \sum_{i=1}^4 u_i)/\theta$, again this quantity has norm at least one. Note that the lines $l_1, \cdots, l_{13}$ fall in this case and they have height one. Now we prove that in this case they are the only ones.

In the above paragraph we can have $\mathrm{ht}(r) = 1$ only if r has inner product θ with four of the points $x_1, \cdots, x_4$ and orthogonal to others. If $x_1, \cdots, x_4$ do not all lie on a line then there is a line l that avoids all these four points. (To see this, take a point x_5 not among $x_1, \cdots, x_4$ on the line joining x_1 and x_2. There are four lines though x_5 and one of them already contain x_1 and x_2. So there is one that does not contain any of $x_1, \cdots, x_4$.) Taking inner products with r in the equation $w_{\mathcal{P}} = \omega^2\theta l + \sum_{x\in l} x$ gives $\theta = \omega^2\theta\langle l, r\rangle$ contradicting $\theta L' = L$. So $x_1, \cdots, x_4$ are points on a line l_1. So r and ωl_1 has the same inner product with $w_{\mathcal{P}}$ and the elements of $\mathcal{P}$. So $r = \omega l_1$.

If $\langle r, w_{\mathcal{P}}\rangle = 3$, and $\sum |\langle x_i, r\rangle|^2 = 18$, the unordered tuple $(\langle x_1, r\rangle, \cdots, \langle x_{13}, r\rangle)$ is equal to $(3u_1, 3u_2, 0^{11})$ or $(3u_1, \theta u_2, \theta u_3, \theta u_4, 0^9)$ or $(\theta u_1, \cdots, \theta u_6, 0^7)$. Using similar calculation as above, we get $\langle r, \bar{\rho}\rangle/|\bar{\rho}|^2$ is equal to $(-u_1 - u_2 + 4 + \sqrt{3})$ or $(-u_1 + (u_2 + u_3 + u_4)/\theta + 4 + \sqrt{3})$ or $((u_1 + \cdots + u_6)/\theta + 4 + \sqrt{3})$ respectively. Again each of these quantities are clearly seen to have norm strictly bigger than one. ☐

Let $\mathcal{M}$ be the set of mirrors of reflections in $R(L)$. Let $d_{\mathcal{M}}(x)$ be the distance of a point x in $\mathbb{C}H^{13}$ from $\mathcal{M}$. Then we have

2.6.3 Proposition. *The function $d_{\mathcal{M}}$ attains a local maximum at the point $\bar{\rho}$.*

Proof. Let $N(x)$ be the set of mirrors that are at minimum distance from a point x in $\mathbb{C}H^{13}$. By 2.6.1 we know $N(\bar{\rho})$ consists of the 26 mirrors $\rho_j^{\perp}$ corresponding to the vertices of D. If x is close enough to $\bar{\rho}$ then $N(x) \subseteq N(\bar{\rho})$. Thus it suffices to show that for any vector v, moving the point $\bar{\rho}$ a little in v direction decreases its distance from at-least one of $\rho_1^{\perp}, \cdots, \rho_{26}^{\perp}$. Let ϵ be a small positive real number. Using the formula (2.5) for distance between a point and hyperplane and ignoring the terms of order ϵ^2 we have,

$$\begin{aligned} d(\bar\rho+\epsilon v,\rho_j^\perp)<d(\bar\rho,\rho_j^\perp) &\iff -\frac{|\langle\bar\rho+\epsilon v,\rho_j\rangle|^2}{|\bar\rho+\epsilon v|^2|\rho_j|^2}<-\frac{|\langle\bar\rho,\rho_j\rangle|^2}{|\bar\rho|^2|\rho_j|^2}\\ &\iff \frac{|(|\bar\rho|^2+\epsilon\langle v,\rho_j\rangle)|^2}{|\bar\rho|^2+2\,\mathrm{Re}\langle v,\bar\rho\rangle\epsilon}<|\bar\rho|^2\\ &\iff \frac{|\bar\rho|^4+2|\bar\rho|^2\,\mathrm{Re}(\langle v,\rho_j\rangle\epsilon)}{|\bar\rho|^2+2\,\mathrm{Re}\langle v,\bar\rho\rangle\epsilon}<|\bar\rho|^2\\ &\iff \frac{|\bar\rho|^2+2\,\mathrm{Re}(\langle v,\rho_j\rangle\epsilon)}{(|\bar\rho|^2+2\,\mathrm{Re}\langle v,\bar\rho\rangle\epsilon)}<1\\ &\iff \mathrm{Re}(\langle v,\rho_j\rangle\epsilon)<\mathrm{Re}(\langle v,\bar\rho\rangle\epsilon) \end{aligned}$$

For any v such that $\mathrm{Re}\langle v,\rho_j\rangle, j=1,\cdots,26$ are not all equal, there is an j_0 such that $\mathrm{Re}\langle v,\rho_{j_0}\rangle<\mathrm{Re}\langle v,\bar\rho\rangle$ since $\mathrm{Re}\langle v,\bar\rho\rangle=\frac{1}{26}\sum_{j=1}^{26}\mathrm{Re}\langle v,\rho_j\rangle$. So moving the point $\bar\rho$ a little in v direction will decrease its distance from $\rho_{j_0}^\perp$ and hence decrease the value of $d_{\mathcal{M}}$.

It remains to check the directions v for which $\mathrm{Re}\langle v,\rho_j\rangle=\mathrm{Re}\langle v,\rho_k\rangle$ for $j,k=1,\cdots,26$. Such vectors form a real vector space of dimension three spanned by $\{\bar\rho_+,i\bar\rho_+,i\bar\rho_-\}$. So we only need to check for $v=i\bar\rho_-$. To calculate $d(\bar\rho_++i\epsilon\bar\rho_-,\rho_j^\perp)$ we recall the formulae (2.10), (2.14) and (2.15). Let $\alpha=-|\bar\rho_-|^2/|\bar\rho_+|^2$. Then $\alpha>1$, and

$$\begin{aligned} \sinh^2(d(\bar\rho_++i\epsilon\bar\rho_-,\rho_j^\perp)) &= -\frac{|\langle\bar\rho_++i\epsilon\bar\rho_-,\rho_j\rangle|^2}{|\bar\rho+i\epsilon\bar\rho_-|^2|\rho_j|^2}=\frac{|(|\bar\rho_+|^2\pm i\epsilon|\bar\rho_-|^2)|^2}{3(|\bar\rho_+|^2+|i\epsilon\bar\rho_-|^2)}\\ =\frac{|\bar\rho_+|^4-\epsilon^2|\bar\rho_-|^4}{3(|\bar\rho_+|^2+\epsilon^2|\bar\rho_-|^2)} &= \frac{|\bar\rho_+|^2}{3}\Big(\frac{1-\epsilon^2|\bar\rho_-|^4/|\bar\rho_+|^4}{1+\epsilon^2|\bar\rho_-|^2/|\bar\rho_+|^2}\Big)=\frac{|\bar\rho_+|^2}{3}\Big(\frac{1-\epsilon^2\alpha^2}{1-\epsilon^2\alpha}\Big)<\frac{|\bar\rho_+|^2}{3} \end{aligned}$$

So $d(\bar\rho_++i\epsilon\bar\rho_-,\rho_j^\perp)<d(\bar\rho_+,\rho_j^\perp)$. □

We shall end this section with a lemma that might be useful in calculating the fundamental group of the complement of mirror arrangement in $\mathbb{C}H^{13}$. Let F be the complex line joining $w_{\mathcal{P}}$ and $w_{\mathcal{L}}$. The simple mirrors corresponding to the points meet F at $w_{\mathcal{P}}$ and the simple mirrors corresponding to the lines meet F at

$w_{\mathcal{L}}$. The lemma gives conditions for a mirror to intersect F inside $\mathbb{C}H^{13}$. Though the conditions are quite restrictive we find that there are mirrors other than the simple ones that do meet F.

2.6.4 Lemma. *Let $u \in L$, $(c_1, \cdots, c_{26}) = (\langle u, \rho_1\rangle, \cdots, \langle u, \rho_{26}\rangle)$, $c_{\mathcal{P}} = \langle u, w_{\mathcal{P}}\rangle$ and $c_{\mathcal{L}} = \langle u, w_{\mathcal{L}}\xi\rangle$. Let $S_{\mathcal{P}}(u) = \sum_{1\le i<j\le 13}|c_i - c_j|^2$ and $S_{\mathcal{L}}(u) = \sum_{14\le i<j\le 26}|c_i - c_j|^2$. Let $I(u) = c_{\mathcal{P}}w_{\mathcal{L}}\xi - c_{\mathcal{L}}w_{\mathcal{P}}$ be the intersection of L with $u^{\perp}$. Then $|I(u)|^2 \equiv 0 \bmod 9$ and*

$$-|I(u)|^2 - 39|u|^2 = S_{\mathcal{P}}(u) = S_{\mathcal{L}}(u) \tag{2.29}$$

Proof. $\langle w_{\mathcal{P}}, w_{\mathcal{L}}\xi\rangle = 4\sqrt{3}$ implies

$$|I(u)|^2 = 3|c_{\mathcal{P}}|^2 + 3|c_{\mathcal{P}}|^2 - 8\sqrt{3}\,\mathrm{Re}(\bar{c}_{\mathcal{P}}c_{\mathcal{L}}) \tag{2.30}$$

We write

$$u = \bar{c}_{\mathcal{P}}w_{\mathcal{P}}/3 + \sum_{i=1}^{13}\bar{c}_i\rho_i/(-3) \tag{2.31}$$

Taking norm on both sides of (2.31) we get

$$|c_{\mathcal{P}}|^2 - 3|u|^2 = \sum |c_i|^2 \tag{2.32}$$

Taking inner product with $w_{\mathcal{L}}\xi$ in (2.31) we get $\langle u, w_{\mathcal{L}}\xi\rangle = c_{\mathcal{P}}4\sqrt{3}/3 + \sum_{i=1}^{13} c_i\sqrt{3}/(-3)$ or $4c_{\mathcal{P}} - \sqrt{3}c_{\mathcal{L}} = \sum c_i$. Taking norm we get

$$\begin{aligned} 16|c_{\mathcal{P}}|^2 + 3|c_{\mathcal{L}}|^2 - 8\sqrt{3}\,\mathrm{Re}(\bar{c}_{\mathcal{P}}c_{\mathcal{L}}) = |\sum c_i|^2 &= 13\sum_i |c_i|^2 - S_{\mathcal{P}}(u) \\ &= 13(|c_{\mathcal{P}}|^2 - 3|u|^2) - S_{\mathcal{P}}(u) \end{aligned}$$

The last equality is from (2.32). (2.29) now follows from (2.30). □

When u is a root we get $|I(u)|^2 = 117 - S_{\mathcal{P}}(u)$. So if the line $F_{\mathbb{C}}$ meets a mirror $u^{\perp}$ in the hyperbolic space then $S_{\mathcal{P}}(u) = S_{\mathcal{L}}(u) \le 117$. This, together with $|I(u)|^2 \equiv 0 \bmod 9$ puts lots of restriction on u.

2.7 Relations in the reflection group of L

2.7.1 The presentation of the bimonster

Let $M \wr 2$ be the wreath product of the monster simple group M with the group of order two. It is defined as the semi-direct product of $M \times M$ with the group or order 2 acting on it by interchanging the two copies of M. This group, called the bimonster, has two simple presentations: see [CNS], [Iva]. The first presentation is on 16 generators of order 2 corresponding to the vertices of the M_{666} diagram, satisfying the Coxeter relations of this diagram and the one extra relation called the "spider relation" given by $S^{10} = 1$, where $S = ab_1c_1ab_2c_2ab_3c_3$.

The other presentation of $M \wr 2$ is on the 26 node diagram D. Each vertex is of order 2, they satisfy the Coxeter relations of the diagram D and we need the extra relations that every free 12-gon in D should generate the symmetric group S_{12} (see [CS]).

There are some other relators for $M \wr 2$ that are particularly nice. Let Δ be any spherical Dynkin Diagram sitting inside M_{666} or D. A *Coxeter element* $w(\Delta)$ of Δ is the product (in any order) of the generators corresponding to the vertices of the diagram Δ. The order of $w(\Delta)$ in the Euclidean reflection group determined by Δ is called the *Coxeter number* denoted by $h(\Delta)$. The relation $w(\Delta)^{h(\Delta)/2} = 1$, is satisfied in $M \wr 2$. We find analogs of all the above relations in the reflection group of L.

2.7.2 Conjectured relation of $\mathrm{Aut}(L)$ with the bimonster

D. Allcock conjectured the following connection between the automorphism group of L and the bimonster, in the late 90's on the basis of the appearance of the M_{666} diagram as the "hole diagram" for E_8^3 and the idea that the orbifold construction is the natural way to turn triflections into biflections.

Conjecture [D. Allcock, personal communication]: Let L be the unique Eisenstein lattice of signature $(1, 13)$ with $\theta L^* = L$. Let X be the quotient $\mathbb{C}H^{13}/\,\mathrm{Aut}(L)$

and $\mathfrak{D}$ be the image therein of the mirrors for the triflections in Aut(L). Then the bimonster is a quotient of the orbifold π_1 of $X \setminus \mathfrak{D}$, namely the quotient by the normal subgroup generated by the squares of the meridians around $\mathfrak{D}$.

This conjecture were made long before the 26 generators of Aut(L) were found. The results in this chapter can thus be taken as supporting evidence for the suggested connection. Allcock also pointed out to me the possible connection with the "Monster Manifold" conjectured by Hirzebruch et.al. (see [HBJ]). Recently Allcock has written a preprint [All4] where he discusses the conjecture and the evidence for it in more detail.

2.7.3 **The relations in the reflection group of L**

All the Coxeter relations of the diagram D hold in $R(L)$, except that now the nodes have order three instead of two. In addition the following relations hold:

1. the spider: $S^{20} = 1$, where $S = ab_1c_1ab_2c_2ab_3c_3$. (Here a means ω reflection in a, etc.)

2. The relation deflate(y) present in the Conway-Simons presentation of the bimonster holds in Aut(L) too. The element $A = ab_2c_2d_2e_2f_2a_3f_1e_1d_1c_1b_1$ has order 11 in Aut(L).

3. Let $w = w(\Delta)$ be the Coxeter element of a free spherical Dynkin diagram Δ inside the sixteen node diagram, then in all the cases except $\Delta = D_4$ and A_5, the element w has finite order in $R(L)$ and the order is a simple multiple of half-Coxeter number in all these cases. The two exceptional cases are the "affine diagrams" for the Eisenstein version of E_6 and E_8. and in these two cases the corresponding Coxeter elements in $R(L)$ have infinite order.

Δ:	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}
order $w(\Delta)$:	3	6	12	30	∞	42	24	18	30	66	12

Δ:	D_4	D_5	D_6	D_7	D_8	E_6	E_7	E_8
order $w(\Delta)$:	∞	24	15	12	21	12	9	15

4. More relators can be found using the action of M on $R(L)$, induced from its

action on L. If $r = 1$ is a relation valid in $R(L)$ and σ is in M then $\sigma(r) = 1$ is also a relator.

2.8 Co-ordinates for the codes and lattices

2.8.1 The codes

A *code* is, for us, a linear subspace of a vector space over a finite field. A *generator matrix* for a code is a matrix whose rows give a basis for the subspace. We give generator matrices for the codes that are used in the computer calculation with the lattice L. The *ternary tetracode* $\mathcal{C}_4$ is a two dimensional subspace of $\mathbb{F}_3^4$ with generator matrix

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

The *ternary Golay code* $\mathcal{C}_{12}$ is a six dimensional subspace of $\mathbb{F}_3^{12}$ with generator matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & -1 & 1 & 0 \end{bmatrix}$$

The binary Golay code $\mathcal{C}_{24}$ (which is a 12 dimensional subspace of $\mathbb{F}_2^{24}$) and the ternary Golay code $\mathcal{C}_{12}$ are obtained from the Golay codes $\mathcal{C}_{23}$ and $\mathcal{C}_{11}$ by appending "zero sum check digit". $\mathcal{C}_{11}$ and $\mathcal{C}_{23}$ are both "cyclic quadratic residue codes". They can be generated as follows:

Let q be equal to 11 or 23. Label the the entries of a vector of length $q+1$ by elements of $\Omega = \mathbb{F}_q \cup \{\infty\}$. Let $N = \Omega \setminus \{x^2 : x \in \mathbb{F}_q\}$. Let v be a vector of length 12 with -1 in the places of N and the 1 in the places of $\Omega \setminus N$, or vectors of length 24 with 1 in the places of N and 0 in the places of $\Omega \setminus N$. Then the vectors obtained by cyclic permutation of the co-ordinates of v span the code $\mathcal{C}_{12}$ and $\mathcal{C}_{24}$ respectively. The above are all taken from chapter 10 of the book [SPLAG], where much more can be found. The co-ordinates for the ternary Golay code $\mathcal{C}_{12}$ used to list the vectors in first shell of the Leech lattice is taken from page 85 of the above book. Note that this gives different co-ordinates on the Leech lattice than those used in [Wil2] or in chapter 11 of the book [SPLAG].

2.8.2 Change of basis between two preferred co-ordinates

Let E_2 be the matrix whose rows are given by the vectors $f_i, \omega e_i, d_i, \omega c_i, i = 1, 2, 3$, $(0^{12}, 0, 1)$ and $(0^{12}, 1, 0)$ respectively (see 2.3.1). Clearly E_2 forms a basis for the $3E_8 \oplus H$. The calculation procedure described in 2.2.7 produces the following matrix E_1. (*notation:* the matrix entry a, b stands for the number $a + \omega b$ and $\overline{x}$ stands for $-x$.)

$$E_1 = \begin{pmatrix}
\overline{3},0 & 0,0 & 3,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 1,0 & 0,1 \\
2,\overline{1} & 0,1 & \overline{1,1} & 0,1 & \overline{1,1} & 0,1 & \overline{1,1} & \overline{1,1} & 0,1 & 0,1 & 0,1 & \overline{1,1} & \overline{1},0 & \overline{1,1} \\
\overline{3},0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 3,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 1,0 & 0,1 \\
3,1 & 0,1 & 0,1 & 0,1 & \overline{1,1} & \overline{1,1} & \overline{1,1} & 0,1 & \overline{1,1} & 0,1 & \overline{1,1} & \overline{1,1} & \overline{1},0 & 0,\overline{1} \\
\overline{3},0 & 3,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 1,0 & 0,1 \\
3,1 & \overline{1,1} & 0,1 & \overline{1,1} & \overline{1,1} & 0,1 & 0,1 & \overline{1,1} & \overline{1,1} & 0,1 & 0,1 & \overline{1,1} & \overline{1},0 & 0,\overline{1} \\
\overline{3},0 & 0,0 & 0,0 & 3,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 1,0 & 0,1 \\
2,\overline{1} & 0,1 & 0,1 & \overline{1,1} & \overline{1,1} & \overline{1,1} & 0,1 & 0,1 & 0,1 & \overline{1,1} & 0,1 & \overline{1,1} & \overline{1},0 & \overline{1,1} \\
\overline{3},0 & 0,0 & 0,0 & 0,0 & 0,3 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 1,0 & 0,1 \\
2,0 & \overline{1},0 & \overline{1},0 & \overline{1},0 & 2,0 & \overline{1},0 & \overline{1},0 & \overline{1},0 & \overline{1},0 & \overline{1},0 & \overline{1},0 & \overline{1},0 & \overline{1},0 & 0,\overline{1} \\
\overline{3,2} & 1,0 & 1,0 & 1,0 & 0,1 & \overline{1,1} & 1,0 & \overline{1,1} & 0,1 & 1,0 & 1,0 & \overline{1,1} & 1,1 & 0,1 \\
\overline{2,2} & 1,1 & 1,1 & 1,1 & \overline{1},0 & 0,2 & 1,1 & \overline{1},0 & 1,1 & 0,\overline{1} & 0,\overline{1} & \overline{1},0 & 1,1 & \overline{1},0 \\
12,\overline{2} & \overline{3},1 & \overline{3},1 & \overline{3},1 & \overline{1,4} & \overline{2},0 & \overline{3},1 & \overline{1,1} & \overline{3,2} & \overline{2},0 & \overline{2},0 & \overline{1,4} & \overline{5},0 & \overline{2,6} \\
12,16 & \overline{4,4} & \overline{4,4} & \overline{4,4} & 4,0 & 0,\overline{2} & \overline{4,4} & 2,\overline{1} & 0,\overline{2} & \overline{2,3} & \overline{2,3} & 2,\overline{1} & \overline{4,6} & 6,0
\end{pmatrix}$$

The rows of E_1 belong to $\Lambda \oplus H$. The rows of E_1 and the rows of E_2 have the same Gram matrix. Hence E_1 is a basis for $\Lambda \oplus H$ and so left multiplication of a column vector by $C = E_2 E_1^{-1}$ gives an explicit isomorphism from $\Lambda \oplus H$ to $3E_8 \oplus H$. This is the isomorphism we used to transfer the generators for the reflection group found in 2.4.5 to the co-ordinate system $3E_8 \oplus H$ because the 26 nodes of the diagram D are naturally given in co-ordinate system $3E_8 \oplus H$.

The proof of equality of the automorphism group and the reflection group of L in 2.5.8 depends on the existence of a M_{666} diagram in $\Lambda \oplus H$ such the 12 roots in the three hands of the diagram spanning $3E_8$ all have the form $(*; 1, *)$ up to units; so that the diagram automorphisms permuting the hands of the M_{666} diagram will fix the vector $(0^{12}; 0, 1)$ in $\Lambda \oplus H$. For this purpose the matrix E'_1 given below was found, by basically the same calculation scheme as described in 2.2.7.

$$E'_1 = \begin{pmatrix}
\bar{3},0 & 0,0 & 3,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 1,0 & 0,1 \\
2,\bar{1} & 0,1 & \overline{1,1} & 0,1 & \overline{1,1} & 0,1 & \overline{1,1} & \overline{1,1} & 0,1 & 0,1 & 0,1 & \overline{1,1} & \bar{1},0 & \overline{1,1} \\
\bar{3},0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 3,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 1,0 & 0,1 \\
3,1 & 0,1 & 0,1 & 0,1 & \overline{1,1} & \overline{1,1} & \overline{1,1} & 0,1 & \overline{1,1} & 0,1 & \overline{1,1} & \overline{1,1} & \bar{1},0 & 0,\bar{1} \\
\bar{3},0 & 3,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 1,0 & 0,1 \\
3,1 & \overline{1,1} & 0,1 & \overline{1,1} & \overline{1,1} & 0,1 & 0,1 & \overline{1,1} & \overline{1,1} & 0,1 & 0,1 & \overline{1,1} & \bar{1},0 & 0,\bar{1} \\
\bar{3},0 & 0,0 & 0,0 & 3,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 1,0 & 0,1 \\
2,\bar{1} & 0,1 & 0,1 & \overline{1,1} & \overline{1,1} & \overline{1,1} & 0,1 & 0,1 & 0,1 & \overline{1,1} & 0,1 & \overline{1,1} & \bar{1},0 & \overline{1,1} \\
\bar{2},0 & 1,0 & 1,0 & 1,0 & 0,1 & 2,2 & 1,0 & 0,1 & 1,0 & \overline{1,1} & \overline{1,1} & 0,1 & 1,0 & 0,1 \\
2,\bar{1} & 0,1 & 0,1 & 0,1 & \overline{1,1} & 1,0 & 0,1 & 1,0 & \overline{1,1} & 0,1 & 0,1 & 1,0 & \bar{1},0 & \overline{1,1} \\
\bar{3},0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,3 & 1,0 & 0,1 \\
2,0 & \bar{1},0 & \bar{1},0 & \bar{1},0 & \bar{1},0 & \bar{1},0 & \bar{1},0 & \bar{1},0 & \bar{1},0 & \bar{1},0 & \bar{1},0 & 2,0 & \bar{1},0 & 0,\bar{1} \\
\overline{5,18} & 2,5 & 2,5 & 2,5 & \overline{4,1} & \bar{1},2 & 2,5 & \overline{2,0} & \overline{2,0} & 1,3 & 1,3 & \overline{5,3} & 1,6 & \overline{8,4} \\
12,\bar{2} & \bar{3},1 & \bar{3},1 & \bar{3},1 & \overline{1,4} & \bar{2},0 & \bar{3},1 & \overline{1,1} & \overline{3,2} & \bar{2},0 & \bar{2},0 & \overline{1,4} & \bar{5},0 & \overline{2,6}
\end{pmatrix}$$

Let the rows of E'_1 be $f'_i, \omega e'_i, d'_i, \omega c'_i, i = 1, 2, 3, n'_1, n'_2$ respectively. Then the fourteen vectors $f'_i, e'_i, d'_i, c'_i, n'_1, n'_2$ have the same inner product matrix as that of the vectors $f_i, e_i, d_i, c_i, n_1, n_2$ of $3E_8 \oplus H$, where $n_1 = (0^{12}, 0, 1)$ and $n_2 = (0^{12}, 1, 0)$. Let $a' = n'_2 + \omega^2 n'_1$ and $b'_i = -n'_2 - (f'_i + (2+\omega)e'_i + 2d'_i + (2+\omega)c'_i)$. Then $a', b'_i, c'_i, d'_i, e'_i, f'_i$ give 16 roots of $\Lambda \oplus H$ that have the same inner products as the vectors $a, b_i, c_i, d_i, e_i, f_i$ in $3E_8 \oplus H$. These 16 vectors form the M_{666} diagram as required in the proof of Theorem 2.5.8. In-fact one can directly check that the diagram automorphism φ_{12} and φ_{23} fix the hyperbolic cell spanned by $(0^{12}; 0, 1)$ and $(-2, \omega^2, \omega^2, 1, \omega, -2\omega, 1, \omega^2, 1, 1, 1, \omega, 1, \theta)$ in $\Lambda \oplus H$.

2.9 Some computer programs

We describe some of the computer programs required for the validation of the proof of Theorem 2.5.7. They are available at http://www.math.berkeley.edu/˜tathagat. The calculations were done using the **GP** calculator. All the calculations that are needed for verification of 2.5.7 are exact. We used some floating point calculation to find the proof, because ht(r) was defined as a floating point function, but once we had the answer, its validity was checked by exact arithmetic.

We give the names of some of the files that contains the relevant codes, and indicate how to execute them to verify our result. Often there are brief comments above the codes explaining their function.

We represent $a + \omega b \in \mathcal{E}$ as [**a,b**]. So elements of $\mathcal{E}^{12}$ are represented as 12×2 matrices. In the gp calculator, we first need to read the file **inp.gp**. The command for this is

```
\r inp.gp
```

This file contains the codes for the basic linear algebra operations for $\mathcal{E}$-lattices. The other codes used and vectors mentioned (and many other things) are contained in the files **isom.gp**, **c12.gp** and **ht.gp** that we also want to read.

The code **C12** generates the codewords of $\mathcal{C}_{12}$ Golay code. The output is saved in the file **c12.gp**. The code **checkleech(v)** returns one if the vector **v** is in Λ and zero otherwise. The vectors **bb**[**1**],**bb**[**2**],...,**bb**[**12**] given in **isom.gp** form a basis for Λ. Using **checkleech** one can check that they are in Λ and one can calculate the discriminant to see that they indeed form a basis. The commands are

```
for( i = 1, 12, print(checkleech(bb[i])))
mdet(checkipd(bb))
```

The code **mdet** calculates determinant and the code **checkipd** calculates the inner product matrix using the inner product **ipd**. Note that the inner product of Λ is one third of **ipd**.

Next, the generators $g_1, \cdots, g_{50}$ are generated as described in Lemma 2.4.5 using the basis **bb**. The code **Trans(l,z)** generates the matrix of the translation $T_{l,z}$. The generators, $g_1, \cdots, g_{50}$ are obtained in the co-ordinates $\Lambda \oplus H$ using the code **genge**. The Matrix $C = E_2 E_1^{-1}$ given in appendix 2.8.2 gives the isomorphism from $\Lambda \oplus H$ to $3E_8 \oplus H$. Multiplying by C we get the generators in the $3E_8 \oplus H$-co-ordinate system. They are called **ge[1],...,ge[50]**. **ge[i]** and were generated by:

```
for(i=1,50,ge[i]=mv(C,genge[i]))
```

They are already stored in the file **ht.gp**.

Now we can use the code **decreasehtnew(x,m)** to run the height reduction algorithm described in 2.5.7, with the arguments **x=ge[i], m = i** where $i = 1, \cdots, 50$. (First check this for $i = 3, 4, 6$; in these cases the algorithm leads to one of the simple roots showing these generators are in rad(D)). For some of the vectors **ge[i]** this algorithm gets stuck and then we perturbed it by reflecting in either **ge[3]**,**ge[4]** or **ge[6]**. Which vector to perturb by is given in the list **perturb**. These are also contained in **ht.gp**.

In-fact the file **path.gp** contains the output obtained by running **decreaseht-new(x,m)** with the given values of **perturb**. Read in this file **path.gp**. To validate the proof of 2.5.7 one only needs to run the code **checktrack(i)** for $i = 1, \cdots, 50$. All that the program does is to start with the vector **ge[i]** and reflect by the simple roots in the order given by the vectors **trac1(i)** and **trac2(i)** that are contained in **path.gp** (or one of **ge[3], ge[4]** or **ge[6]** when perturbation is needed). We use the command

```
for(i=1,50, checktrack(i))
```

In every case one arrives at a unit multiple of one of the simple roots. This verifies the proof.

Chapter 3

Quaternionic Lorentzian Leech lattice

3.1 Introduction

Let $\mathcal{H}$ denote the ring of Hurwitz integers, consisting of the quaternions $(a + bi + cj + dk)/2$ where a, b, c and d are integers, all congruent modulo 2. Let Λ and E_8 be the Leech lattice and E_8 root lattice respectively considered as Hermitian lattices over $\mathcal{H}$. Let H be the 2 dimensional lattice $\mathcal{H} \oplus \mathcal{H}$ with Gram matrix $\left(\begin{smallmatrix} 0 & 1+i \\ 1-i & 0 \end{smallmatrix}\right)$ and let $L = \Lambda \oplus H$.

In this chapter we investigate a second example of the phenomenon studied in the previous chapter on complex Lorentzian Leech lattice and bimonster. Here the base ring $\mathbb{Z}[e^{2\pi i/3}]$ is replaced by the non-commutative ring $\mathcal{H}$, the incidence graph of $P^2(\mathbb{F}_3)$ is replaced by $P^2(\mathbb{F}_2)$ and then there is a parallel story.

We find 14 quaternionic reflections of order 4 in the reflection group of the Lorentzian quaternionic Leech lattice L that form the Coxeter diagram given by the incidence graph of the projective plane over $\mathbb{F}_2$. (When I told D. Allcock about it, I got to know that he had found this diagram too). This 14 node diagram D (see Fig. 3.1) is obtained by extending the M_{444} diagram which comes up naturally, as the lattice L is isomorphic to $3E_8 \oplus H$. (We shall describe an explicit isomorphism from $\Lambda \oplus H$ to $3E_8 \oplus H$ over $\mathcal{H}$ later in 3.2.5, because it is needed for our computations.) The three hands of the M_{444} diagram correspond to the three copies of E_8.

The main results are the following. We see that there is a "Weyl vector" for D in the quaternionic hyperbolic space that is fixed by the diagram automorphisms and that the 14 reflections of D generate the reflection group of L. Allcock showed that the reflection group is finite index in the automorphism group of L. We also see that the mirrors of the 14 reflections of D are the mirrors closest to the Weyl vector, and in that sense, the roots of D are the analog of the simple roots.

Almost all the definitions and proofs here are similar to those in the previous chapter, so we shall not bother to mention them at each step. In view of the two examples, one would surely like to know whether there are other examples of similar phenomenon for other Lorentzian lattices; and if a more conceptual meaning can be given to these diagrams in a suitably general context that would explain all the numerical coincidences seen in this and the previous chapter.

3.2 Preliminaries

3.2.1 Notation

All the notations are borrowed from the previous chapter on complex Lorentzian Leech lattice and are mostly consistent with the ones used in [All1]. As the ring of Eisenstein integers $\mathcal{E}$ is replaced by the ring $\mathcal{H}$ of Hurwitz integers, the 14 node diagram D replaces the 26 node diagram, p replaces $\theta = \sqrt{-3}$, $\mathbb{F}_2$ replaces $\mathbb{F}_3$, et cetera.

D	The incidence graph of $P^2(\mathbb{F}_2)$
H	the hyperbolic cell over $\mathcal{H}$ with Gram matrix $\begin{pmatrix} 0 & \bar{p} \\ p & 0 \end{pmatrix}$
$\mathcal{H}$	The ring of Hurwitz integers, a copy of D_4 root lattice sitting inside the quaternions.

$\mathbb{H}$	The skew field of real quaternions.
Λ	The Leech lattice as a 6 dimensional negative definite $\mathcal{H}$-lattice
p	$1-i$
ϕ_r^α	the α-reflection in the vector r
ξ	$(1+i)/\sqrt{2}$

3.2.2 The ring of quaternions

let $\mathcal{H}$ be the ring of *Hurwitz integers* generated over $\mathbb{Z}$ by the 24 unit quaternions $\pm 1, \pm i, \pm j, \pm k$, and $(\pm 1 \pm i \pm j \pm k)/2$. $\mathcal{H}$ consists of the elements $(a+bi+cj+dk)/2$, where a, b, c, d are integers all congruent modulo 2, with the standard multiplication rules $i^2 = j^2 = k^2 = ijk = -1$. When tensored with $\mathbb{R}$ we get the skew field of quaternions called $\mathbb{H}$. The conjugate of $q = a + bi + cj + dk$ is $\bar{q} = a - bi - cj - dk$. The *real part* of q is $\mathrm{Re}(q) = a$ and the *imaginary part* is $\mathrm{Im}(q) = q - \mathrm{Re}(q)$. The *norm* of q is $|q|^2 = \bar{q}.q$.

We let $\alpha = (1 + i + j + k)/2$ and $p = (1 - i)$. In the constructions of the lattices given below, the number $p = 1 - i$ plays the role of $\sqrt{-3}$ in chapter 2. The quaternion p generates a two sided ideal $\mathfrak{p}$ in $\mathcal{H}$. We have $i \equiv j \equiv k \equiv 1 \bmod \mathfrak{p}$. (observe that $(1 - i)\alpha = 1 + j$, so $j \equiv 1 \bmod \mathfrak{p}$). The group $\mathcal{H}/\mathfrak{p}\mathcal{H}$ is $\mathbb{F}_4$ generated by $0, 1, \alpha, \bar{\alpha}$.

The multiplicative group of units $\mathcal{H}^*$ is $2 \cdot A_4$. The quotient $\mathcal{H}^*/\{\pm 1\}$ has four Sylow 3 subgroups generated by $\alpha, i\alpha, j\alpha, k\alpha$. The permutation representation of $\mathcal{H}^*/\{\pm 1\}$ on the Sylow 3-subgroups identifies it with A_4.

We also identify the quaternions $a + bi$ with the complex numbers. So any quaternion can be written as $z_1 + z_2 j$ for complex numbers z_1 and z_2. The multiplication is defined by $j^2 = -1$ and $jz = \bar{z}j$. The complex conjugation becomes conjugation by the element j.

3.2.3 Lattices over Hurwitz integers

General reference for lattices is [SPLAG]. An $\mathcal{H}$*-lattice* is a free finitely generated right $\mathcal{H}$-module with an $\mathcal{H}$-valued bilinear form $\langle\ ,\ \rangle$ satisfying $\overline{\langle x,y\rangle} = \langle y,x\rangle$, $\langle x, y\alpha\rangle = \langle x,y\rangle\alpha$ and $\langle x\alpha, y\rangle = \bar{\alpha}\langle x,y\rangle$, for all x, y in the lattice and α in $\mathcal{H}$. In this article, by a lattice we shall mean an $\mathcal{H}$-lattice, unless otherwise stated. Definite lattices will usually be negative definite. The standard negative definite lattice $\mathcal{H}^n$ has the inner product $\langle x,y\rangle = -\bar{x}_1y_1 - \cdots - \bar{x}_ny_n$. The indefinite lattice given by $\mathcal{H}\oplus\mathcal{H}$ with inner product $\langle (x,y),(x',y')\rangle = (\bar{x},\bar{y})\left(\begin{smallmatrix} 0 & \bar{p} \\ p & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} x' \\ y'\end{smallmatrix}\right)$ is denoted by H. We call H the *hyperbolic cell*.

The E_8 root lattice can be defined as a sub-lattice of $\mathcal{H}^2$ as $E_8 = \{(x_1,x_2) | x_1 \equiv x_2 \bmod \mathfrak{p}\}$. It has minimal norm -2 and the underlying $\mathbb{Z}$-lattice is the usual E_8 root lattice.

The Leech lattice Λ can be defined as a 6 dimensional negative definite $\mathcal{H}$-lattice with minimal norm -4 whose real form is the usual real Leech lattice. The automorphism group of this lattice was studied by Wilson in [Wil1]. We quote the facts we need from there. Let $\omega = (-1+i+j+k)/2$. The lattice Λ consists of all vectors $(v_\infty, v_0, v_1, v_2, v_3, v_4)$ in $\mathcal{H}^6$ such that $v_2 \equiv v_3 \equiv v_4 \bmod \mathfrak{p}$, $(v_1+v_4)\bar{\omega} + (v_2+v_3)\omega \equiv (v_0+v_1)\omega + (v_2+v_4)\bar{\omega} \equiv 0 \bmod 2$, and $-v_\infty(i+j+k) + v_0 + v_1 + v_2 + v_3 + v_4 \equiv 0 \bmod 2+2i$. The inner product we use is $-1/2$ of the one used in [Wil1], so that the following basis vectors have norm -4. We use the following $\mathcal{H}$-basis for the lattice Λ given in [Wil1] for some computations.

$$bb[1] = [2+2i, 0, 0, 0, 0, 0]$$
$$bb[2] = [2, 2, 0, 0, 0, 0]$$
$$bb[3] = [0, 2, 2, 0, 0, 0]$$
$$bb[4] = [i+j+k, 1, 1, 1, 1, 1]$$
$$bb[5] = [0, 0, 1+k, 1+j, 1+j, 1+k]$$
$$bb[6] = [0, 1+j, 1+j, 1+k, 0, 1+k]$$

Let L be the Lorentzian lattice $L = \Lambda\oplus H \cong 3E_8 \oplus H$. The real form of this lattice is $II_{4,28}$. E_8, Λ, H and L each satisfy $L'p = L$, where L' is the *dual lattice* of L defined by $L' = \{x \in L \otimes \mathbb{H} : \langle x, y\rangle \in \mathcal{H} \forall y \in L\}$.

3.2.4 Quaternionic reflections

A *μ-reflection* in a vector r of a lattice is given by

$$\phi_r^\mu(v) = v - r(1-\mu)\langle r, v\rangle/|r|^2 \tag{3.1}$$

where $\mu \neq 1$ is a unit in $\mathcal{H}$. Note that

$$\phi_{r\alpha}^\mu = \phi_r^{\alpha\mu\alpha^{-1}} \tag{3.2}$$

for any unit $\alpha \in \mathcal{H}$. Note also that ϕ_r^μ can be characterized as the automorphism of the lattice that fixes the orthogonal complement of r and multiplies r by the root of unity μ. It follows that for every automorphism γ of the lattice we have $\gamma\phi_r^\mu\gamma^{-1} = \phi_{\gamma r}^\mu$. We say the two reflections ϕ_1 and ϕ_2 *braid* if $\phi_1\phi_2\phi_1 = \phi_2\phi_1\phi_2$. A *root* of a negative definite or a Lorentzian lattice is a lattice vector of negative norm such that there is a nontrivial reflection in it that is an automorphism of the lattice. If $L'p = L$, the roots of L are all the vectors of norm -2. For such a root r, $R(L)$ contains six reflections of order four given by $\phi_r^{\pm i}, \phi_r^{\pm j}$ and $\phi_r^{\pm k}$. The square of each of them is the order 2 reflection ϕ_r^-. Note that the six units $\pm i, \pm j$, and $\pm k$ form a conjugacy class in $\mathcal{H}^*$. For a set of reflections (or roots) that either braid or commute, we form the *Coxeter diagram* by taking one vertex for each reflection and joining them only if the reflections braid.

For computational purpose we note the following (one needs to be careful because $\mathcal{H}$ is not commutative). Let $e_1, \cdots, e_n$ be a basis for a lattice. For a linear transformation ϕ, if $\phi(e_k) = \sum_j e_j\phi_{jk}$ then let $mat(\phi) = ((\phi_{jk}))$; if a vector $x = \sum e_i x_i$ is written in co-ordinates as a column vector then $\phi(x) = mat(\phi)(x_1, \cdots, x_n)'$.

3.2.5 Computation for explicit isomorphism between $3E_8 \oplus H$ and $\Lambda \oplus H$ over $\mathcal{H}$

The reflection group of the $\mathcal{H}$-lattice E_8 can be generated by two reflections of order 4 that braid with each other. So an E_8-diagram for us, looks like an A_2 Dynkin diagram. We say that two diagrams are orthogonal if the corresponding lattices are orthogonal. We need to find three orthogonal E_8-diagrams in the lattice $\Lambda \oplus H$ and a hyperbolic cell orthogonal to this $3E_8$, to get an explicit change of basis matrix. We find the 6 vectors of the form $r = (l; 1, \bar{p}^{-1}(1+\beta))$ (with $\beta \in \operatorname{Im}\mathbb{H}$ and l in the first shell of the leech lattice) in $\Lambda \oplus H$ forming three copies of E_8 by a computer search. The steps of the computation are given below.

First, note that i-reflections in two roots $r = (l; 1, \bar{p}^{-1}(1+\beta))$ and $r' = (l'; 1, \bar{p}^{-1}(1+\beta'))$ will commute if $|l - l'|^2 = -4$ and $\beta - \beta' = [l, l']$, where $[l, l'] = \operatorname{Im}\langle l, l' \rangle$. The i-reflections in r and r' will braid if $|l - l'|^2 = 6$ and $\beta - \beta' = [l, l'] \pm i$.

It is easy to find one copy of E_8. One just have to find two vectors l_1, l_2 in first shell of Leech lattice that are at a distance -6 and find β_1, β_2 accordingly.

Now we generate a large list of vectors in the first shell of Leech lattice using the basis *bb* given in 3.2.3. We find all the vectors l in the first shell (actually from the almost complete list that we had), that might give a root $(l; 1, *)$ of an orthogonal E_8. The conditions that l has to satisfy are $|l - l_1|^2 = |l - l_2|^2 = -4$ and $[l_1, l_2] - [l_1, l] + [l_2, l] = i$. This gives a small list of vectors l. (It is amusing to note that the equation looks like a "co-cycle condition").

Next, find pairs of vectors l_3, l_4 from the previous list that can actually give an E_8 orthogonal to the first one. The conditions to be satisfied are $[l_1, l_2] - [l_3, l_4] + [l_2, l_4] - [l_1, l_3] = 0$ and $|l_3 - l_4|^2 = -6$. This way one can find two E_8 diagrams orthogonal to the first one, that happen to be orthogonal to each other too. Thus we get the 6 vectors forming a $3E_8$ diagram.

Now we take the orthogonal complement and find two norm zero vectors in the complement forming an hyperbolic cell.

The actual co-ordinates for the 8 vectors found by above calculation are given below. A quaternion $a + bi + cj + dk$ is represented as $a + bi, c + di$.

$$\begin{pmatrix} 2 & 2 & 0 & 0 & 0 & 0 & 1 & 1 \\ i+j+k & 1 & 1 & 1 & 1 & 1 & 1 & 2+i+j \\ \frac{1+i+j+k}{2} & \frac{3+i-j+k}{2} & \frac{1-i+j-k}{2} & \frac{1-i-j+k}{2} & \frac{-1-i-j-k}{2} & \frac{1-i-j+k}{2} & 1 & \frac{3+i+j+k}{2} \\ 1+i+k & 1 & k & -k & 1 & -j & 1 & \frac{3+i+j+k}{2} \\ 1+i+k & 1 & i & -j & -i & 1 & 1 & \frac{3+i+j+k}{2} \\ 1+i & 1+k & 1-i & 1-k & 0 & 0 & 1 & \frac{3+i+j+k}{2} \\ \frac{-7+i-3j-5k}{2} & \frac{-7+3i-3j+k}{2} & \frac{-1+3i-j+k}{2} & \frac{-1+i-j+3k}{2} & \frac{-1+3i+j+3k}{2} & \frac{-3+i-j+k}{2} & \frac{-5+3i-3j+k}{2} & \frac{-4+2i-3j-k}{1} \\ \frac{-5-3i-j-5k}{2} & \frac{-7+i-j-3k}{2}, & \frac{-1+i-j-k}{2} & \frac{-3+i-j+k}{2} & \frac{-1+3i-j+k}{2} & \frac{-3+i+j-k}{2} & \frac{-5+i-j-k}{2} & -4-2j-2k \end{pmatrix}$$

3.3 The 14 node diagram

3.3.1 The diagram of 14 roots

The i-reflection of order 4 in a root r braids with the i-reflection in a root r' if $\langle r, r' \rangle = p$. In this section we work in the co-ordinates $3E_8 \oplus H$ for L. We can find 10 roots a, b_i, c_i, d_i for $i = 1, 2, 3$ forming an M_{444} diagram inside the reflection group $R(L)$. See [Iva] for more on these groups (called Y-groups there). The hands of the M_{444}-diagram correspond to the three copies of E_8 in L, in the sense that c_i, d_i generate the reflection group of the i-th E_8, the b_i are the affinizing node and a is the hyperbolizing node. These 10 roots can be extended to a set of 14 roots forming the incidence graph of $\mathbb{P}^2(\mathbb{F}_2)$. The roots a, c_i, e_i correspond to the points of $\mathbb{P}^2(\mathbb{F}_2)$ (See Fig. 3.1) and f, b_i, d_i correspond to the lines. There is an edge between them if the point lies on the line. This 14 vertex diagram is called D. the seven roots a, c_i, e_i (or their unit multiples) are called "points" and the seven roots f, b_i, d_i (or their unit multiples) are called "lines". We use the same shorthand notation for the diagram D that was used in in the figure 2.3. One choice of the

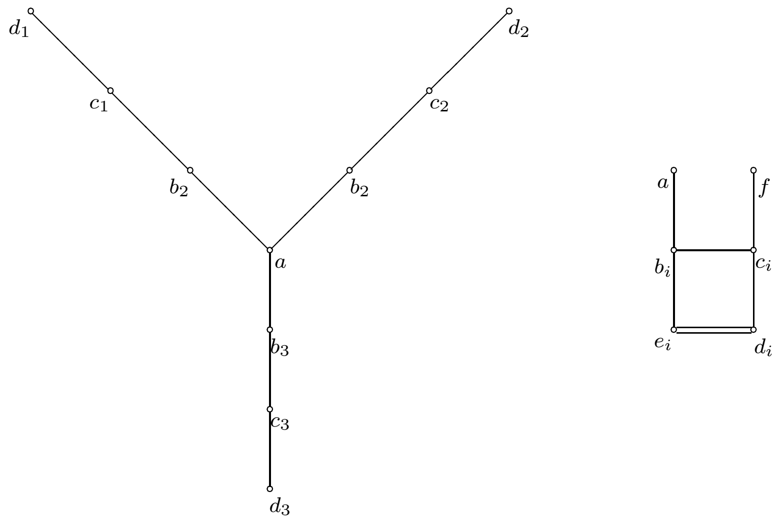


Figure 3.1: The diagrams M_{444} and $D \cong Inc(\mathbb{P}^2(\mathbb{F}_3))$

explicit co-ordinates of these roots are given by

$$
\begin{array}{rrlcccccccl}
r[1] = & a = [& & , & & ; & 1 & , & -1 &] \\
r[1+i] = & c_i = [& (1,-1)_i & , & & ; & & , & &] \\
r[4+i] = & e_i = [& & , & (1,1)_{jk} & ; & -i & , & -1 &] \\
r[8] = & f = [& (\ ,p)^3 & & & ; & -ip & , & -p &] \\
r[8+i] = & b_i = [& (\ ,p)_i & , & & ; & -1 & , & &] \\
r[11+i] = & d_i = [& (-p,\ )_i & , & & ; & & , & &]
\end{array}
$$

An E_8 vector with subscript i means we put it in place of the i-th E_8, while the subscript jk means that we put it at the j-th and k-th place. The indices i, j and k are in cyclic permutation of $(1, 2, 3)$. Blank spaces are to be filled with zero.

3.3.2 **Linear relations among the roots of D**

Let x be a point and l be a line of D. $\langle x, l\rangle$ is equal to p or 0 depending on whether there is a edge between x and l or not. Using this we find that the vector

$$w_{\mathcal{P}} = l\bar{p} + \sum_{x \in l} x \tag{3.3}$$

is orthogonal to the points and has norm 2. But there is only one such vector. Hence we get the relations $l\bar{p} + \sum_{x\in l} x = l'\bar{p} + \sum_{x\in l'} x$. for any two lines l and l'. These relations generate all linear relations in the 14 roots. Using the automorphism σ that takes l to x and x to li we see that the element

$$w_{\mathcal{L}} = xp + \sum_{x\in l} l \tag{3.4}$$

is constant for all point x. We shall see that the elements $w_{\mathcal{P}}$ and $w_{\mathcal{L}}$ determine points in the quaternionic hyperbolic space fixed by the diagram automorphisms $PGL_2(\mathbb{F}_2)$.

3.4 Generators for the reflection group of L

3.4.1 The Heisenberg group

We follow the definitions and notations of section 6 of the article [All1]. Let $\mathbb{T}$ be the *Heisenberg group* generated by the translations $T_{\lambda,z}$ for every λ in $\Lambda = \Lambda \cap \Lambda' p$. In section 6 Allcock mentions that formulae (3.2) - (3.5) of [All1] holds in the quaternionic case too. We need to do a little more calculation to get a little stronger version of Theorem 6.1 of [All1]. We use the three roots $r_1 = (0^6; 1, i)$, $r_2 = (0^6; 1, -1)$ and $r_3 = (0^6; 1, -\epsilon)$ where $\epsilon = (1 - i - j + k)/2$. (Actually there are six roots of the form $(0^6; 1, \mu)$ with $\mu \in \mathcal{H}^*$).

3.4.2 Lemma. *Let $\epsilon = (1 - i - j + k)/2$, r_3 and r_2 be the roots $(0^6; 1, -\epsilon)$ and $(0^6; 1, -1)$ and let $R = \phi^i_{r_3}\phi^j_{r_2}$. Then*

$$T^{-1}_{\lambda,z} R T_{\lambda,z} R^{-1} = T_{\lambda(\bar{\epsilon}-1),\epsilon z\bar{\epsilon}-z+\mathrm{Im}\langle\lambda\bar{\epsilon},-\lambda\rangle} \tag{3.5}$$

Since $(\bar{\epsilon}-1)$ is an unit of $\mathcal{H}$ we see that for every $\lambda \in \Lambda$, $R(L)$ contains a reflection in λ.

Proof. R is of the form $\left(\begin{smallmatrix} I & 0 \\ 0 & R_H \end{smallmatrix}\right)$ where $R_H = \left(\begin{smallmatrix} \epsilon & 0 \\ u & \delta \end{smallmatrix}\right)$ is the matrix acting on H with $u = (3 - i + j - k)/2$ and $\delta\bar{p}^{-1} = \bar{p}^{-1}\epsilon$. Now $R_H^{-1} = \left(\begin{smallmatrix} \epsilon^{-1} & 0 \\ v & \delta^{-1} \end{smallmatrix}\right)$ where $v = -\delta^{-1}u\epsilon^{-1}$.

Now just multiply the matrices to see that

$$RT_{\lambda,z}R^{-1} = \begin{pmatrix} I & & \\ & \epsilon & \\ & u & \delta \end{pmatrix} \begin{pmatrix} I & \lambda & \\ & 1 & \\ -\bar{p}^{-1}\lambda^* & \bar{p}^{-1}(z-\lambda^2/2) & 1 \end{pmatrix} \begin{pmatrix} I & & \\ & \bar{\epsilon} & \\ & v & \bar{\delta} \end{pmatrix}$$

$$= \begin{pmatrix} I & \lambda & \\ & \epsilon & \\ -\delta\bar{p}^{-1}\lambda^* & u+\delta\bar{p}^{-1}(z-\lambda^2/2) & \delta \end{pmatrix} \begin{pmatrix} I & & \\ & \bar{\epsilon} & \\ & v & \bar{\delta} \end{pmatrix}$$

$$= \begin{pmatrix} I & \lambda\epsilon^{-1} & \\ & 1 & \\ -\delta\bar{p}^{-1}\lambda^* & x & 1 \end{pmatrix}$$

where

$$x = (u+\delta\bar{p}^{-1}(z-\lambda^2/2))\bar{\epsilon}+\delta v = \delta\bar{p}^{-1}(z-\lambda^2/2)\bar{\epsilon} = \bar{p}^{-1}\epsilon(z-\lambda^2/2)\bar{\epsilon} = \bar{p}^{-1}(\epsilon z\bar{\epsilon}-\lambda^2/2)$$

In the second equality we use $u\bar{\epsilon} = -\delta v$ and in the third $\delta\bar{p}^{-1} = \bar{p}^{-1}\epsilon$. Also note that $-\delta\bar{p}^{-1}\lambda^* = -\bar{p}^{-1}\epsilon\lambda^* = -\bar{p}^{-1}(\lambda\bar{\epsilon})^*$. Thus we have

$$RT_{\lambda,z}R^{-1} = T_{\lambda\bar{\epsilon},\epsilon z\bar{\epsilon}} \tag{3.6}$$

Now (3.5) follows from (3.6) and $T_{\lambda,z}T_{\lambda',z'} = T_{\lambda+\lambda',z+z'+\mathrm{Im}\langle\lambda',\lambda\rangle}$ (equation (3.2) in [All1]). □

3.4.3 Lemma. *Let Ψ be the set of roots that are unit multiples of the roots of the form $(\lambda; 1, *)$. The reflections in Ψ act transitively on all root. All reflections in all the roots of the form $(\lambda, 1, *)$ generate the reflection group of L.*

Proof. The calculation here is almost identical to Theorem 6.2 in [All1] which uses the idea in [Con1]. For this lemma only, let $h(\lambda; \mu, \eta) = |\mu|$. (Later we are going to use a different definition of height). The roots r in Ψ are the ones with $h(r) = 1$. We show that if we have a root r with $h(r) > 1$, then we can i-reflect it in a root of Ψ to decrease its height. The covering radius of the Leech lattice is used in this proof and it has just the right value to make things work.

Let $y = (l; 1, \bar{p}^{-1}(\alpha - l^2/2))$ be a multiple of a root r with $h(r) > 1$. We have $|y|^2 \in (-2, 0)$, which amount to $\mathrm{Re}(\alpha) \in (-1, 0)$ because $|y|^2 = 2\,\mathrm{Re}(\alpha)$. Consider the i-reflection in the root $r = (\lambda; 1, \bar{p}^{-1}(-1 - \lambda^2/2 + \beta + n)$; where $\beta \in \mathrm{Im}\,\mathbb{H}$ is fixed so that $\bar{p}^{-1}(-1 - \lambda^2/2 + \beta)$ is in $\mathcal{H}$ and n (to be chosen later) can be any element of $\mathrm{Im}\,\mathfrak{p} = \mathfrak{p} \cap \mathrm{Im}\,\mathbb{H}$. Calculation yields $\langle r, y \rangle = -2(a + b)$, where

$$-2a = -\frac{1}{2}|l - \lambda|^2 - 1 + Re(\alpha) \in \mathbb{R}$$

and

$$-2b = \mathrm{Im}(\alpha) + \mathrm{Im}\langle \lambda, l \rangle - \beta - n \in \mathrm{Im}\,\mathbb{H}$$

So $h(\phi_r^i(y)) = \bar{p}(1 - (1 - i)(a + b))$. Thus we want to make $|1 - (1 - i)(a + b)|^2 < 1$, which amount to

$$|1/2 - a|^2 + |i/2 - b|^2 < 1/2 \tag{3.7}$$

Because the covering norm of Leech lattice is 2 we can make $|l - \lambda|^2 \in [-2, 0]$. This, together with $Re(\alpha) \in (-1, 0)$ gives $a \in (0, 1)$. So $|1/2 - a|^2 < 1/4$. As for the second term of (3.7), $(i - 2b)$ is in $\mathrm{Im}\,\mathbb{H}$, and in the expression for $-2b$ we are free to choose $n \in \mathrm{Im}\,\mathfrak{p}$ which forms a copy of D_3 root lattice: $\{(ai + bj + ck) : a + b + c \equiv 0 \bmod 2\}$ in $\mathrm{Im}\,\mathbb{H}$. D_3 has covering Radius 1. So we can make the norm of $(i - 2b)$ less than 1 by choice of n and thus make $|i/2 - b|^2 \leq 1/4$.

So if ϕ_r^μ is a reflection in any root r, after conjugating finitely many times by i-reflections in roots of the form $(\lambda; 1, *)$ we get a reflection $\phi_{r'}^\zeta$ in a root $r' = r''u$ where $r'' = (\lambda''; 1, *)$ and u is an unit. But then $\phi_{r'}^\zeta = \phi_{r''}^{u\zeta u^{-1}}$. Thus ϕ_r^μ can be obtained as a product of reflections in the roots of the form $(\lambda; 1, *)$. ☐

3.4.4 Lemma. *Let $\lambda_1, \lambda_2, \cdots, \lambda_{24}$ be elements of Λ that make a $\mathbb{Z}$-basis. Let $r_1 = (0^6; 1, i)$ and r_2, r_3 be as given in Lemma 3.4.2. Fix z_s such that $T_{\lambda_s, z_s} \in \mathbb{T}$, for $s = 1, \cdots, 24$. Let R_1 temporarily denote the group generated by all the reflections in the 81 roots $T_{\lambda_s, z_s}(r_t)$, $T_{0,i+j}(r_t)$, $T_{0,i+k}(r_t)$ and r_t where $s = 1, \cdots, 24$ and $t = 1, 2, 3$. Then R_1 contains the Heisenberg group $\mathbb{T}$. In-fact R_1 is equal to the reflection group of L.*

Proof. From (3.5) in lemma 3.4.2 we get that R_1 contains a translation in the vectors $\lambda_s(\bar{\epsilon}-1)$. These vectors form a $\mathbb{Z}$-basis of L as $\bar{\epsilon}-1$ is an unit. Using $T_{\lambda,z} \circ T_{\lambda',z'} = T_{\lambda+\lambda',z+z'+\mathrm{Im}\langle\lambda',\lambda\rangle}$ and $T^{-1}_{\lambda,z} = T_{-\lambda,-z}$ (equations (3.2) and (3.3) in [All1]) we see that R_1 contains translation in every vector of Λ.

Now we argue that all the central translations of the form $T_{0,z}$ are in R_1. Choosing λ and λ' such that $\langle\lambda',\lambda\rangle = p$ and using the identity $T^{-1}_{\lambda,z}T^{-1}_{\lambda',z'}T_{\lambda,z}T_{\lambda',z'} = T_{0,2\,\mathrm{Im}\langle\lambda',\lambda\rangle}$ (equation (3.4) in [All1]) conclude that the central translation $T_{0,2i}$ is in R_1. Similarly taking $\langle\lambda',\lambda\rangle$ to equal $p\alpha = (1+j)$ and $p(1+i-j+k)/2 = (1+k)$ respectively, it follows that the central translation $T_{0,2j}$ and $T_{0,2k}$ are also in R_1. From (3.6) in 3.4.2, it follows that $T^{-1}_{0,z}RT_{0,z}R^{-1} = T_{0,\epsilon z\bar{\epsilon}-z}$ which equals $T_{0,-i-2j-k}$ for $z = i+j$ and $T_{0,-j-k}$ for $z = i+k$. So these central translations are in R_1 too. We found that $T_{0,i+k}$ and $T_{0,j+k}$ are in R_1, as are $T_{0,2i}, T_{0,2j}$ and $T_{0,2k}$. These central translations clearly generate all the central translations of the form $T_{0,z}$, $z \in \mathfrak{p}\mathcal{H}$. So R_1 contains $\mathbb{T}$.

The orbit of $(0^6; 1, -1)$ under $\mathbb{T}$ is all roots of the form $(\lambda, 1, \bar{p}^{-1}(\beta - 1 - \lambda^2/2))$. So all these roots are in R_1 and we have already seen that these generate the whole reflection group of L.

For the actual computation multiply the basis vectors *bb* given in 3.2.3 by $1, i, j, (-1+i+j+k)/2$ to get a basis of Λ over $\mathbb{Z}$. □

3.5 The fixed points of diagram automorphisms and height of a root

3.5.1 The fixed points under diagram automorphism

The group $PGL_2(\mathbb{F}_2)$ acts on the diagram D and this induces a linear action of $PGL_2(\mathbb{F}_2)$ on L. The graph automorphism switching points with lines in $Inc(\mathbb{P}^2(\mathbb{F}_2))$ lifts to a automorphism σ taking a line l to a point x and x to li (note: $\langle x, l\rangle = p$ if and only if $\langle li, x\rangle = p$). This gives action of the extended diagram automorphism group $M = 8 \cdot PGL_2(\mathbb{F}_2)$ on L and hence on the *quaternionic hyperbolic space* $\mathbb{H}H^7$, which consists of the positive norm lines in $L \otimes \mathbb{R}$. The vectors $w_{\mathcal{P}}$ and $w_{\mathcal{L}}$ defined in (3.3) and (3.4) of section 3.3.2 span the 2 dimensional space of fixed vectors of the action of $PGL_2(\mathbb{F}_2)$. From (3.3) and (3.4) it follows that $\sigma(w_{\mathfrak{p}}) = w_{\mathcal{L}} i$ and $\sigma(w_{\mathcal{L}}) = w_{\mathcal{P}}$.

Let $\Sigma_{\mathcal{P}}$ and $\Sigma_{\mathcal{L}}$ be the sum of the points and the sum of the lines respectively. These too are fixed by the $PGL_2(\mathbb{F}_2)$ action and σ takes $\Sigma_{\mathcal{L}}$ to $\Sigma_{\mathcal{P}}$ and $\Sigma_{\mathcal{P}}$ to $\Sigma_{\mathcal{L}} i$. So, there is a unique fixed point in $\mathbb{H}H^7$ under the action of this extended group of diagram automorphisms M, given by the image of the vector $\Sigma_{\mathcal{P}} + \Sigma_{\mathcal{L}}\xi$ or $w_{\mathcal{P}} + w_{\mathcal{L}}\xi \in L \otimes \mathbb{R}$ where $\xi = (1+i)/\sqrt{2}$ is an eighth root of unity: $\xi^2 = i$. We call this fixed vector the *Weyl vector* :

$$\bar{\rho} = (\Sigma_{\mathcal{P}} + \Sigma_{\mathcal{L}}\xi)/14 \tag{3.8}$$

We note some of the inner products between the special vectors that we need later : Let $(\rho_1, \cdots, \rho_{14}) = (x_1, \cdots, x_7, l_1\xi, \cdots, l_7\xi)$, so that σ interchanges ρ_s with $\rho_{7+s}\xi$ and $\bar{\rho}$ is the average of $\rho_1, \cdots, \rho_{14}$. Then we have $\langle \rho_s, \rho_t \rangle$ is equal to 0 or $\sqrt{2}$ according to whether the two nodes are joined or not joined in the diagram D. We have, for $s = 1, \cdots, 14$,

$$\langle \bar{\rho}, \rho_s \rangle = |\bar{\rho}|^2 = 1/(2 + 3\sqrt{2}) \tag{3.9}$$

From (3.3) and (3.4) we get

$$|w_{\mathcal{P}}|^2 = |w_{\mathcal{L}}|^2 = 2 \text{ and } \langle \bar{\rho}, w_{\mathcal{P}} \rangle = \langle \bar{\rho}, w_{\mathcal{L}}\xi \rangle = 1/\sqrt{2} \tag{3.10}$$

3.5.2 **The height of a root**

We use the Weyl vector $\bar{\rho}$ to define the *height of a root* r as

$$\text{ht}(r) = |\langle \bar{\rho}, r \rangle| / |\bar{\rho}|^2$$

The 14 roots of the diagram D have height equal to 1. Take the roots of the reflections generating $R(L)$, found in 3.4.4, given in the co-ordinate system $\Lambda \oplus$

H. Using the explicit isomorphism found in 3.2.5 we write them in co-ordinate system $3E_8 \oplus H$. We use the above definition of height and run a "height reduction algorithm" (see theorem 2.5.7) on the 81 generators for $R(L)$ found before to see that one can always get to a unit multiple of an element of D. (Sometimes one needs to perturb if the algorithm gets stuck - at-most one perturbation was enough in all cases). This proves

3.5.3 Theorem. *The 14 order 4 reflection of D generate the reflection group of L.*

Now we can prove the analog of Proposition 2.6.1. The proof is also exactly similar.

3.5.4 Proposition. *The 14 roots of the diagram D are the only roots (up to units) having the minimum height 1. All other roots have strictly bigger height. (In other words, the mirrors of the roots in D are the 14 mirrors closest to the vector $\bar{\rho}$).*

Proof. We need the following distance formulae for the metric on the quaternionic hyperbolic space $\mathbb{H}H^n$ (See [KP]). A positive norm vector x in the vector space determines a point in the hyperbolic space, also denoted by x. A negative norm vector r determine a totally geodesic hyperplane given by $r^{\perp}$. Let $c(u,v)^2 = \frac{|\langle u,v\rangle|^2}{|u|^2|v|^2}$. Then we have

$$\cosh^2(d(x,x')/2) = c(x,x')^2, \qquad \sinh^2(d(x,r^{\perp})/2) = -c(x,r)^2 \tag{3.11}$$

Two hyperplanes $r^{\perp}$ and $r'^{\perp}$ meet in the hyperbolic space if $c(r,r') < 1$, are asymptotic if $c(r,r') = 1$ and do not meet if $c(r,r') > 1$ in which case the distance between the hyperplanes is given by

$$\cosh^2(d(r^{\perp}, r'^{\perp})/2) = c(r,r')^2 \tag{3.12}$$

Let r be a root of the lattice L with $\text{ht}(r) = |\langle \bar{\rho}, r\rangle|/|\bar{\rho}|^2 \leq 1$. We want to prove that r is a unit multiple of one of the 14 roots of D.

Let x be a point in D. Either $|\langle x, r\rangle| \leq 2$, or using the triangle inequality $d(r^{\perp}, x^{\perp}) \leq d(r^{\perp}, \bar{\rho}) + d(x^{\perp}, \bar{\rho})$ along with the distance formulae (3.11) and (3.12) above we get

$$|\langle x, r\rangle| \leq 2\cosh(2\sinh^{-1}(|\bar{\rho}|/\sqrt{2})) \approx 2.32$$

So we must have $|\langle x, r\rangle|^2$ equal to 0, 2, or 4.

Similarly from $d(r^{\perp}, w_{\mathcal{P}}) \leq d(r^{\perp}, \bar{\rho}) + d(w_{\mathcal{P}}, \bar{\rho})$, (3.11) and (3.12) we get

$$|\langle w_{\mathcal{P}}, r\rangle| \leq 2\sinh(\sinh^{-1}(|\bar{\rho}|/\sqrt{2}) + \cosh^{-1}(1/2\bar{\rho})) \approx 2.26$$

It follows that $|\langle w_{\mathcal{P}}, r\rangle|^2$ is equal to 0, 2, or 4.

We can write

$$r = \sum_{x \in \mathcal{P}} -x\langle x, r\rangle/2 + w_{\mathcal{P}}\langle w_{\mathcal{P}}, r\rangle/2 \tag{3.13}$$

taking norm in (3.13) we get

$$-2 = \sum_{x \in \mathcal{P}} -|\langle x, r\rangle|^2/2 + |\langle w_{\mathcal{P}}, r\rangle|^2/2 \tag{3.14}$$

There are only few cases to consider. Changing r up-to units we may assume that $\langle w_{\mathcal{P}}, r\rangle$ is either 0 or p or 2. In the following let u_1, u_2 etc denote units in $\mathcal{H}^*$ and x_1, x_2 etc. denote points of D.

If $\langle r, w_{\mathcal{P}}\rangle = 0$, from (3.14) we get $\sum|\langle x_i, r\rangle|^2 = 4$. Then the unordered tuple $(\langle x_1, r\rangle, \cdots, \langle x_7, r\rangle)$ is equal to $(2u_1, 0^6)$ or $((pu_1, pu_2, 0^5)$. So either r is an unit multiple of x_i (in which case it has height equal to one) or $r = (x_1pu_1 + x_2pu_2)/(-2)$. Using diagram automorphisms (which is 2-transitive on points of D) we can assume that $x_1 = a$ and $x_2 = c_1$ and then check that there is no such root r.

If $\langle w_{\mathcal{P}}, r\rangle = p$, then $\sum|\langle x_i, r\rangle|^2 = 6$; the unordered tuple $(\langle x_1, r\rangle, \cdots, \langle x_7, r\rangle)$ is equal to $(2u_1, u_2p, 0^5)$ or $(u_1p, u_2p, u_3p, 0^4)$. In the first case we get

$$r = 2x_1u_1/(-2) + x_2u_2p/(-2) + w_{\mathcal{P}}p/2$$

Taking inner product with $\bar{\rho}$ and using $\langle \bar{\rho}, w_{\mathcal{P}}\rangle/|\bar{\rho}|^2 = 3 + \sqrt{2}$ we get

$$\langle \bar{\rho}, r\rangle/|\bar{\rho}|^2 = -u_1 - u_2/\bar{p} + (3 + \sqrt{2})/\bar{p}$$

which clearly has norm greater than one.

In the second case we get $r = \sum_{i=1}^{3} x_i u_i p/(-2) + w_{\mathcal{P}} p/2$ which implies

$$\langle \bar{\rho}, r\rangle/|\bar{\rho}|^2 = (-u_1 - u_2 - u_3 + 3 + \sqrt{2})/\bar{p}$$

Again this quantity has norm at least one. We now show that the only way it can be equal to one is if r is a unit multiple of $l_1, \cdots, l_7$.

The only way one can have $\operatorname{ht}(r) = 1$ in the above paragraph is if r has inner product p with three of the points x_1, x_2, x_3 and orthogonal to others. If x_1, x_2, x_3 do not all lie on a line then there is a line l that avoids all these three points. Taking inner products with r in the equation $w_{\mathcal{P}} = l\bar{p} + \sum_{x \in l} x$ gives $p = p\langle l, r\rangle$ contradicting $L'p = L$. So x_1, x_2, x_3 are points on a line l_1. It follows that r and an unit multiple of l_1 has the same inner product with each element of $\mathcal{P}$ and with $w_{\mathcal{P}}$. So r is an unit multiple of l_1.

If $\langle w_{\mathcal{P}}, r\rangle = 2$, and $\sum |\langle x_i, r\rangle|^2 = 8$, the unordered tuple $(\langle x_1, r\rangle, \cdots, \langle x_7, r\rangle)$ is equal to $(2u_1, 2u_2, 0^5)$ or $(2u_1, u_2 p, u_3 p, 0^4)$ or $(u_1 p, \cdots, u_4 p, 0^3)$. Using similar calculation as above, we get $\langle r, \bar{\rho}\rangle/|\bar{\rho}|^2$ is equal to $(-u_1 - u_2 + 3 + \sqrt{2})$ or $(-u_1 - u_2/\bar{p} - u_3/\bar{p} + 3 + \sqrt{2})$ or $(-(u_1 + \cdots + u_4)/\bar{p} + 3 + \sqrt{2})$ respectively. Again each of these quantities are clearly seen to have norm strictly bigger than one. □

3.5.5 **Remarks**

1. The group generated by the Coxeter diagram $Inc(P^2(\mathbb{F}_2)$ when the vertices are made into reflections of order two is the group $O_8^-(2) : 2$ as was found by Simons in [Sim]. It would be interesting to understand the relation between these two groups in a more conceptual way.

2. The order of the "spider element" $sp = ab_1c_1ab_2c_2ab_3c_3$ in the reflection group $R(L)$ is 40.

3. Since we can find a M_{444} diagram in $L = \Lambda \oplus H$ with the 6 vectors in the three E_8 hands of the form $(\lambda; 1, *)$ with λ in the first shell of Leech lattice, it is likely that the reflection group of L in-fact equals the whole automorphism group.

It would probably follow after a little more work in the line of Theorem 6.2 of [All1] and using the fact that $\mathbb{T} \subseteq R(L)$. But I have not checked this.

4. Exactly as in Remark 2.3.4 the $\mathcal{H}$-lattice $L = 3E_8 \oplus H$ can be defined by the starting with a singular lattice corresponding to the vertices of the diagram D and quotienting out by the relations $l(ip) + \sum_{x \in l} x = l'(ip) + \sum_{x \in l'} x$.

5. The calculations needed for this paper are done using the gp calculator and the codes (for finding the explicit isomorphism from $3E_8 \oplus H$ to $\Lambda \oplus H$, or height reduction algorithm to show that the 14 nodes of D generate $R(L)$) are contained in the file

`quat.gp`

The programs are available on the my web-site
http://www.math.berkeley.edu/~tathagat

Chapter 4

A few other examples

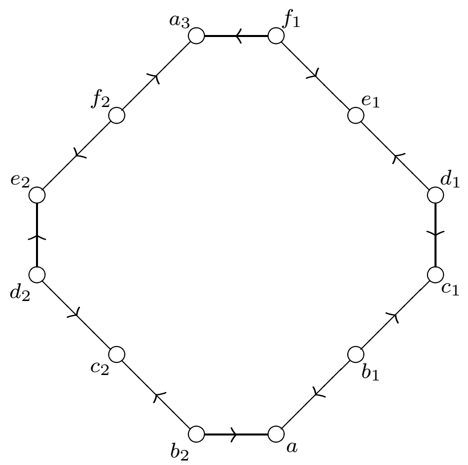
In this chapter we study the automorphism groups of a few more complex Lorentzian lattices L over $\mathcal{E}$. In the next three sections we shall take L to be $E_8 \oplus E_8 \oplus H$, $E_8 \oplus H$, and a finite index subgroup of $E_8 \oplus H$. In each of these examples we shall describe generators for the reflection group of these lattices and we shall see that they have features similar to those studied in the previous chapters about the Leech lattice. These generators either braid or commute and we form the Coxeter diagram D for the reflection group in the usual way. We find that there is a Weyl vector $\bar{\rho}$ in the sense of Chapter 2. The roots corresponding to the diagram D are precisely the ones whose mirrors are the closest to the Weyl vector.

We do not know if these examples are rather special or if there are lot of other examples of this kind. However, a "true" conceptual definition of the vector $\bar{\rho}$ still eludes us. That is why we have to stop with a few examples rather than having a theory of the diagrams of (an appropriate class of) complex Lorentzian reflection groups.

We looked at the lattices studied in the following examples, because they were easy to work with based on the previous computation. For example, when L is $E_8 \oplus H$ or $E_8 \oplus E_8 \oplus H$ we do not need to prove that the simple reflections generate the reflection group because this follows from [All1] where a minimal set of generators for these reflection groups are given and it is shown that the reflection groups equal the automorphism groups. In the remaining third example our proof is similar to the proof of 2.5.7. For computational purposes, in each of these examples we identify our lattices inside $3E_8 \oplus H$. We also use the terminology of points and lines et cetera from the example of $3E_8 \oplus H$.

4.1 An example in dimension 10

Let $L = E_8 \oplus E_8 \oplus H$. L is an 10 dimensional Lorentzian $\mathcal{E}$-lattice. For convenience we identify L inside $3E_8 \oplus H$. Twelve vertices of the 26 node diagram for $3E_8 \oplus H$ belong to $E_8 \oplus E_8 \oplus H$. They are $f_2, e_2, d_2, c_2, b_2, a, b_1, c_1, d_1, e_1, f_1, a_3$. (see figure 2.3). The Coxeter diagram D formed by these 12 roots is the affine A_{11} Dynkin diagram or the 12-gon with arrows going from lines to points.



Allcock proved in [All1] that the automorphism group of L is generated by triflections in the roots $f_2, e_2, d_2, c_2, b_2, a, b_1, c_1, d_1, e_1, f_1$. An extension of the dihedral group of a 12-gon acts as diagram automorphism. Let τ be the automorphism that fixes a, a_3 and interchanges the two copies of E_8. Let σ be the automorphism that interchanges points and lines, namely

$$\sigma(f_1) = a_3 \quad \sigma(f_2) = e_1 \quad \sigma(d_1) = e_2 \quad \sigma(d_2) = c_1 \quad \sigma(b_1) = c_2 \quad \sigma(b_2) = a$$

and $\sigma^2 = -\omega$. Let M be the group of diagram automorphisms generated by τ and σ. Modulo scalars M is the dihedral group acting faithfully on the complex hyperbolic space $\mathbb{C}H^9$. Let $\mathcal{P} = \{a, c_i, e_i, a_3\} = \{x_1, \cdots, x_6\}$ the set of points and $\mathcal{L} = \{b_i, d_i, f_i\} = \{l_1, \cdots, l_6\}$ be the lines. Let $(\rho_1, \cdots, \rho_{12}) = (x_1, \cdots, x_6, \xi l_1, \cdots, \xi l_6)$.

4.1.1 Lemma. *The Weyl vector* $\bar{\rho} = (1/12)\sum_{i=1}^{12} \rho_i$ *is the unique point in* $\mathbb{C}H^9$ *fixed by the group* M *of diagram automorphisms.*

Proof. Clearly $\bar{\rho}$ is fixed by the set of diagram automorphisms. As we know the inner products between ρ_i's we can calculate $\langle \bar{\rho}, \rho_i \rangle = (2\sqrt{3} - 3)/12$ and hence

$$\langle \bar{\rho}, \rho_i \rangle = |\bar{\rho}|^2 = (2\sqrt{3} - 3)/12$$

So $\bar{\rho}$ is the Weyl vector.

Let y be a point in the vector space $L \otimes \mathbb{C}$ whose image in $\mathbb{C}H^9$ is fixed by M. We have $\tau y = \pm y$. Suppose, if possible, $\tau y = -y$. The -1 eigenspace of τ is $\text{span}\{c_1 - c_2, d_1 - d_2, e_1 - e_2, f_1 - f_2\}$. Writing $y = \lambda_c(c_1 - c_2) + \lambda_d(d_1 - d_2) + \lambda_e(e_1 - e_2) + \lambda_f(f_1 - f_2)$ we get

$$\begin{aligned} &\lambda_c\omega(b_1 - d_2) + \lambda_d(e_2 - c_1) + \lambda_e\omega(d_1 - f_2) + \lambda_f(a_3 - e_1) = \\ &\pm \xi((\lambda_c(c_1 - c_2) + \lambda_d(d_1 - d_2) + \lambda_e(e_1 - e_2) + \lambda_f(f_1 - f_2)) \end{aligned}$$

Looking at the co-ordinates of the vector, the above equation successively implies, $\lambda_f = 0$, $\lambda_c = 0$, $\lambda_e = 0$ and finally $y = 0$. So if y is a fixed point of diagram automorphisms in $\mathbb{C}H^9$ then $\tau y = y$.

Since $\sigma^2 = -\omega$ we must have $\sigma y = \pm\xi y$. Suppose $\sigma y = \xi y$. Using τ and σ alternatively we get $\langle y, a \rangle = \langle \sigma y, \sigma a \rangle = \langle \xi y, -\omega b_2 \rangle = \langle y, \xi b_2 \rangle = \langle \tau y, \tau \xi b_2 \rangle = \langle y, \xi b_1 \rangle = \langle \sigma y, \sigma \xi b_1 \rangle = \langle \xi y, \xi c_2 \rangle = \langle y, c_2 \rangle = \langle \tau y, \tau c_2 \rangle = \langle y, c_1 \rangle = \cdots$. In short $\langle y, \rho_i \rangle$ must be a constant, hence y and $\bar{\rho}$ determine the same element in $\mathbb{C}H^9$.

Now suppose $\sigma y = -\xi y$. Let $(\rho_{1_-}, \cdots, \rho_{12_-}) = (x_1, \cdots, x_6, -\xi l_1, \cdots, -\xi l_6)$ and $\bar{\rho}_- = (1/12)\sum_1^{12} \rho_{i_-}$. $\langle \rho_{i_-}, \rho_{j_-} \rangle$ is equal to 0 or $-\sqrt{3}$ depending on whether there is an edge or not in the diagram. This gives $\langle \bar{\rho}_-, \rho_{i_-} \rangle = |\bar{\rho}_-|^2 = (-3 - 2\sqrt{3})/12$.

Meanwhile a chain of equations similar to the above show that $\langle y, \rho_{i_-}\rangle$ is constant, hence y must be a scalar multiple of $\bar{\rho}_-$ which does not give a point in $\mathbb{C}H^9$. $\square$

4.1.2 Proposition. *The mirrors closest to the Weyl vector $\bar{\rho}$ are precisely the mirrors orthogonal to the roots of the diagram D.*

Proof. Suppose r is a root with $\mathrm{ht}(r) = \langle \bar{\rho}, r\rangle / |\bar{\rho}|^2$ having absolute value less than or equal to 1. Let $w = (0^8; \omega, \omega^2)$. $|w|^2 = 3$ and $w^\perp \cap H = a\mathcal{E}$. The lemma 2.6.2 holds for our lattice $L = 2E_8 \oplus H$. So from equations (2.17) and (2.18) We have

$$|\langle r, \rho_i\rangle| \leq 3.077 \qquad\qquad |\langle r, w\rangle| \leq 4.238$$

So $\langle r, \rho_i\rangle$ and $\langle r, w\rangle$ are equal to a unit multiple of 0, θ or 3. Changing r by a unit if necessary we assume $\langle w, r\rangle$ is equal to 0, $-\theta\omega$ or 3. This only leaves a small number of cases to check.

Suppose $\langle r, w\rangle = 0$. We can write $r = (l; \alpha, \alpha\omega^2)$ with $\alpha \in \mathcal{E}$. $|r|^2 = -3$ implies $|l|^2 - 3|\alpha|^2 = -3$. So either $l = 0$ and $\alpha \in \mathcal{E}^*$, thus r is a unit multiple of a, or $\alpha = 0$, in which case l is a root of $2E_8$. We can check that, up-to units the only ones with height 1 among these are the ones in the diagram, namely c_i, d_i, e_i, f_i.

Suppose $\langle w, r\rangle = -\theta\omega$. Let $z = (0^8; -1, 0)$. We can write $r = (l; 0, 0) + z + \alpha a$, with $l \in 2E_8$ and $\alpha \in \mathcal{E}$. $\langle a, r\rangle = -3\alpha - \omega\theta$. If $\alpha = 0$ then r has the form $(l; -1, 0)$. If $\alpha \neq 0$ then $|\langle a, r\rangle| = |-3\alpha - \omega\theta| \leq 3$ implies α is equal to 1 or $-\omega^2$ and r has the form $(l; 0, \omega^2)$ or $(l; \omega, -\omega)$ respectively. So we need to check the roots $(l; -1, 0)$, $(l; 0, \omega^2)$ and $(l; \omega, -\omega)$. We find that the only roots among these with height less than on equal to 1 are b_1 and b_2.

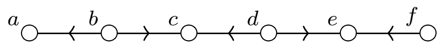
Finally suppose $\langle r, w\rangle = 3$. We can write $r = (l; 0, 0) + w + \alpha a$, with $l \in 2E_8$ and $\alpha \in \mathcal{E}$. Taking inner product with a we have $3|\alpha| \leq |\langle r, a\rangle| \leq 3$, that is $|\alpha| \leq 1$. $|r|^2 = -3$ gives $|l|^2 + 3 - 3|\alpha|^2 = -3$. So we have two possible cases. The first case is when $\alpha = 0$ and $|l|^2 = -6$. So we have to check all the roots of the form $(l; 0, 0) + w$. Either l has the form $(l_1, 0^4)$ where l_1 is a vector in E_8 of length -6 (there are 2160 vectors in the second shell of E_8) or $l = (l_1, l_2)$ where both l_1 and

l_2 are roots in E_8. (there are 240^2 such vectors to check). We find that only a_3 has height less than or equal to one among all these vectors.

The second case is when $|\alpha| = 1$ and $|l|^2 = -3$. So we have to check all the roots of the form $(l; 0, 0) + w + \alpha a$ with $\alpha \in \mathcal{E}^*$. None of these have height ≤ 1. $\square$

4.2 An example in dimension 6 ?

In this section we consider the lattice $L = E_8 \oplus H$ over $\mathcal{E}$. It is shown in [All1] that the reflection group of $E_8 \oplus H$ can be minimally generated by six reflections of order 3 forming the diagram D that looks like the A_6 Dynkin diagram.



We call a, c, e points and b, d, f lines. Note that this diagram does not have enough automorphisms to pick out a single fixed point - the only diagram automorphism σ_1 interchanges points and lines and forms the cyclic group $\mathbb{Z}/12\mathbb{Z}$. The eigenvalues of σ_1 are $\pm\xi$ with three dimensional eigenspace span$\{a \pm \xi f, c \pm \xi d, e \pm \xi b\}$. So there is no unique fixed point. We tried to find a symmetric diagram for L but did not succeed. This attempt actually led to the example studied in the next section.

However, since there are six simple roots in the diagram D and L is six dimensional we can find a vector that has inner product 1 with each simple root and then scale it so that we can find a unique vector $\bar{\rho}$ that is equidistant from the six simple mirrors. This, we shall call the Weyl vector. We leave it to the discretion of the reader if he would consider this example in the same league with the other four. We shall move quickly to the next example after noting that the statement of 4.1.2 holds in this example too. That is,

4.2.1 Proposition. *The mirrors closest to the Weyl vector $\bar{\rho}$ are precisely the six mirrors orthogonal to the roots of the diagram D.*

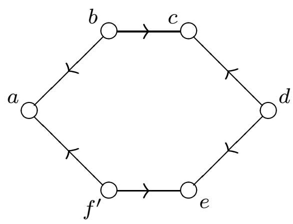
Proof. The same calculation done in the proof of 4.1.2 works. Let $(\rho_1, \cdots, \rho_6) = (a, c, e, \xi f, \xi d, \xi b)$. By direct computation check that $|\bar{\rho}|^2 = \langle \bar{\rho}, \rho_i \rangle \approx .022 > 0$. Let $w = (0^4; 1, \omega)$. If r is a root of $E_8 \oplus H$ with $|\langle \bar{\rho}, r \rangle| \leq |\bar{\rho}|^2$, the inequalities in the lemma 2.6.2 give the bound $|\langle r, \rho_i \rangle| \leq 3.05$ and $|\langle r, w \rangle| \leq 2.293$. So changing r by a unit if necessary we may assume $\langle r, w \rangle$ is equal to 0 or θ. Also, up-to unit multiples $|\langle r, \rho_i \rangle|$ equals $0, \theta$, or 3. There are only a few cases to check. □

4.3 Another example in dimension 6

In this section we consider a sub-lattice L of $E_8 \oplus H$ over $\mathcal{E}$. The Weyl vector in this example can again be characterized as the unique fixed point of the diagram automorphisms. As we noted when working with $E_8 \oplus H$ in the previous section, the diagram for $E_8 \oplus H$ does not have enough automorphisms to pick out a single fixed point.

4.3.1 **The lattice L with hexagon diagram**

We can change f by adding $(0^4; -1, 0)$ to get a symmetric hexagonal diagram.



In this section we let L be the Lorentzian lattice generated by the 6 root vectors of the diagram D given above. Note that the hexagon D can be thought of as the incidence graph of "$P^2(\mathbb{F}_1)$". The lattice L is in the kernel of the map $E_8 \oplus H \to \mathcal{E}/2\mathcal{E}$ given by $(w_1, \cdots, w_6) \mapsto w_4 + w_5 + \omega w_6 \bmod 2$ because each of its 6 basis vectors are. Let $u = (0^4; 1, 0)$. Notice that L together with u generate $E_8 \oplus H$ and $2u \in L$. Hence $\{0, u, \omega u, \omega^2 u\}$ form a complete set of coset representative of L in $E_8 \oplus H$. So L is the kernel of the map given above. Thus L is a sub-lattice of index 4 in $E_8 \oplus H$.

$z_1 = (0, -\omega, -\omega, -\omega; 1, \omega)$ is a primitive null vector in L. We find a hyperbolic cell $H_1 \simeq H$ containing z_1. $H_1 = \text{span}\{z_1, z_2\}$, $|z_1|^2 = |z_2^2| = 0$, $\langle z_1, z_2 \rangle = \bar{\theta}$. (for example, we take $z_2 = \omega^2(z_1 + e)$). Since E_8 is the only even unimodular positive definite $\mathbb{Z}$-lattice in dimension 8, the orthogonal complement Λ_1 of H_1 in $E_8 \oplus H$ is isomorphic to E_8. One can check that the orthogonal complement of H_1 in L is isomorphic to $\Lambda = D_4^{\mathcal{E}} \oplus D_4^{\mathcal{E}}$. This provides a second co-ordinate system for L in which we would be able to write down a set of generators. To summarize we have

4.3.2 Lemma. $L \simeq D_4^{\mathcal{E}} \oplus D_4^{\mathcal{E}} \oplus H = \Lambda \oplus H$. *The underlying* $\mathbb{Z}$*-lattice of* $D_4^{\mathcal{E}}$ *with the bilinear form* $(-2/3)\,\text{Re}\langle x, y\rangle$ *is the lattice* D_4 *with minimal norm 2. The covering norm and minimal norm of* Λ *is -3.*

Proof. The proof is straight forward calculation, though things can be a bit confusing because the real and complex forms use different scaling. For someone wishing to do the calculations we provide some details.

In the notation of chapter 2, $D_4^{\mathcal{E}}$ is the 2 dimensional $\mathcal{E}$-lattice with the A_2 Dynkin diagram. In other words it has an $\mathcal{E}$-basis consisting of u and v with $|u|^2 = |v|^2 = -3$, $\langle u, v\rangle = -\omega\theta$. Check that $\{u, v, \omega^2(u+v), -(u-\omega v)\}$ is a $\mathbb{Z}$-basis of the lattice. The integer Gram matrix of this basis with respect to the bilinear form $(-2/3)\,\text{Re}\langle x, y\rangle$ is given by

$$\begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & & \\ -1 & & 2 & \\ -1 & & & 2 \end{pmatrix}$$

which is the Cartan matrix for D_4 Dynkin diagram. Alternatively one can find the roots in $H_1^{\perp} \cap L$ and identify it with the root system $D_4 \oplus D_4$ showing that $D_4 \oplus D_4 \subseteq H_1^{\perp} \cap L$. Next we check that the discriminant of $D_4 \oplus D_4 \oplus H_1$ and L are equal showing that they are equal. (The absolute discriminant is 108 in our scaling, where minimal norm is 3) ∎

We can use the co-ordinate system $2D_4^{\mathcal{E}} \oplus H$ for L to write down a set of generators for $R(L)$.

4.3.3 Lemma. *The statements of 2.4.2 and 2.4.5 hold when the* Λ *is the Eisenstein lattice* $2D_4^{\mathcal{E}}$ *instead of the complex Leech lattice.*

Proof. The proofs of above mentioned propositions go though word for word when we denote $2D_4^{\mathcal{E}}$ instead of the Leech lattice by Λ and d is equal to 4 instead of 12. The properties of the lattice Λ required are the following. The minimal norm and covering radius are both -3, $\Lambda \subseteq \theta\Lambda'$ and there are vectors u, v in Λ with $\langle u, v\rangle = -\omega\theta$. Note again, that the covering norm -3 is the critical value to make the proof of 2.4.2 work. It would fail if the covering norm was any larger. □

4.3.4 **The diagram automorphisms**

Let σ and τ be the two following diagram automorphisms :

$$\begin{array}{llllll} \sigma(a) = -\omega f' & \sigma(b) \quad = e & \sigma(c) = -\omega d & \sigma(d) \quad = c & \sigma(e) = -\omega b & \sigma(f') \quad = a \\ \tau(a) = a & \tau(b) \quad = f' & \tau(c) = e & \tau(d) \quad = d & \tau(e) = c & \tau(f') \quad = b \end{array}$$

$\tau^2 = 1$ and $\sigma^2 = -\omega$. These act on the hexagon as reflections if we think of the nodes as mirrors i.e. forget the arrows. They generate the group M of diagram automorphisms. Modulo the center, M is isomorphic to the dihedral group of an hexagon.

4.3.5 **Fixed point under diagram automorphisms**

The group of diagram automorphisms M has a unique fixed point $\bar{\rho}$ in the hyperbolic space $\mathbb{C}H^5$. To see this note that the eigenspaces of τ with eigenvalue ± 1 are, respectively span$\{a, d, c+e, b+f'\}$ and span$\{c-e, b-f'\}$. Suppose $v = \lambda_1 a + \lambda_2 d_2 + \lambda_3(c+e) + \lambda_4(b+f')$ is an eigenvector of σ. Since $\sigma^2 = -\omega$ the eigenvalues of σ are $\pm\xi$. Then writing out $\sigma v = \pm\xi v$ and equating coefficients we get that σ fixes only the lines containing $\bar{\rho}_{\pm} = ((a+c+e) \pm \xi(b+d+f'))/6$ in $+1$ eigenspace of τ. Similarly see that σ does not fix any point in the -1 eigenspace of τ. $\bar{\rho}_{-}$ has negative norm while $\bar{\rho}_{+} = \bar{\rho}$ has positive norm. So there is a unique fixed point under diagram automorphisms given by the image of

$$\bar{\rho} = ((a+c+e) + \xi(b+d+f'))/6$$

We note that

$$\langle \bar{\rho}, \rho_j \rangle = |\bar{\rho}|^2 = (2\sqrt{3} - 3)/6$$

where $(\rho_1, \cdots, \rho_6) = (a, c, e, \xi f', \xi d, \xi b)$. The calculations are easy using the fact that $\langle \rho_i, \rho_j \rangle = \sqrt{3}$ or 0 depending up-to whether there is a edge between the corresponding nodes or not.

4.3.6 Proposition. *The mirrors closest to the Weyl vector $\bar{\rho}$ are precisely the mirrors orthogonal to the roots of the diagram D.*

Proof. Copy the proof of 4.1.2. As in all the examples the simple roots satisfy $\langle \bar{\rho}, \rho_i \rangle = |\bar{\rho}|^2$. Let $w = (0^4, 1, \omega)$ be the norm 3 vector. If r is a root with $\text{ht}(r) \leq 1$, using the triangle inequality for the metric on $\mathbb{C}H^5$ we get the bounds $|\langle r, \rho_i \rangle| \leq 3.155$ and $|\langle r, w \rangle| \leq 2.585$. We only have to check a few cases on the computer. ☐

4.3.7 Proposition. *The six reflections forming the hexagonal diagram D generate the reflection group of $L = 2D_4^{\mathcal{E}} \oplus H$.*

Proof. Using 4.3.3 we can write down 18 reflections in $R(L)$ that generate the reflection group of L. The actual generators we use are obtained as follows. All the calculations are done in the co-ordinate system for $E_8 \oplus H$. We find z_1 and z_2 of norm zero with $\langle z_1, z_2 \rangle = \bar{\theta}$, such that the orthogonal complement of z_1 and z_2 in L is $2D_4^{\mathcal{E}}$. In the co-ordinates $2D_4^{\mathcal{E}} \oplus H$, z_1 and z_2 will be equal to $(0^4, 1, 0)$ and $(0^4, 0, 1)$. So the roots $r_1 = (0^4; 1, \omega^2)$ and $r_2 = (0^4; 1, -\omega)$ of theorem 3.1 in [All1] are equal to $z_1 + \omega^2 z_2$ and $z_1 - \omega z_2$. We find 4 roots $d_3, \cdots, d_6$ that form $\mathcal{E}$-basis of $\text{span}\{z_1, z_2\}^{\perp} \simeq 2D_4^{\mathcal{E}}$. Let $\lambda_1, \cdots, \lambda_8$ be the $\mathbb{Z}$-basis of $2D_4^{\mathcal{E}}$ given by $d_i, \omega d_i$.

$\lambda_i^2 = -3$. The roots generating $R(L)$ may be taken to be the 18 vectors r_1, r_2, $T_{\lambda_i,\theta/2}(r_1)$ and $T_{\lambda_i,\theta/2}(r_2)$ for $i = 1, \cdots, 8$. To calculate these vectors quickly, note that $T_{\lambda,z}(z_1) = \lambda + z_1 + \bar{\theta}^{-1}(z - \lambda^2/2)z_2$ and $T_{\lambda,z}(z_2) = z_2$. Further substituting $\lambda = \lambda_i$ and $z = \theta/2$ the first equation simplifies to $T_{\lambda_i,\theta/2}(z_1) = \lambda_i + z_1 + \omega z_2$.

The result is verified by running a height reduction algorithm as in the proof of 2.5.7. The height reduction algorithm terminates for all the 18 generators. Unlike in 2.5.7 we did not need to perturb. However we do not have a proof that the height of a root can always be decreased by a simple reflection. □

4.4 Concluding remarks

We shall end with a few miscellaneous remarks and some speculations about possible lines of further development of this work.

4.4.1 The $3E_8$ holy construction of the complex Leech lattice

Let us recall the holy constructions of the Leech lattice ([SPLAG] Chapter 24) : Let N be a Niemeier lattice, one of the 23 even unimodular integral negative definite lattice of dimension 24 with a rank 24 root system. Let H_1 be the hyperbolic cell over $\mathbb{Z}$ with Gram matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $N \oplus H_1$ is isomorphic to $II_{1,25}$, the unique even unimodular Lorentzian integer lattice of signature $(1, 25)$. Let ρ_N be a Weyl vector of N and h be the Coxeter number of (any component of) the root system of N. Then $\rho = (\rho_N, h, h+1)$ is a norm zero vector of Leech type, that is $\rho^\perp/\rho$ is isomorphic to the Leech lattice.

Let us now come back to Hermitian lattices over Eisenstein integers, that is, the setup of chapter 2. Λ is the Leech lattice over $\mathcal{E}$. The computation given in 2.8.2 implies that the method of holy construction given above can be used, to find a norm zero vector of Leech type inside $3E_8 \oplus H$. In 2.8.2 we saw that there exists vectors $b_i, c_i, d_i, e_i, f_i, i = 1, 2, 3$ inside $\Lambda \oplus H$, forming the extended $3E_8$ complex diagram, and each of them have absolute inner product $\sqrt{3}$ with $(0^{12}; 0, 1)$.

Now let us work in the complex lattice $3E_8 \oplus H$. Let $\{\alpha_1, \cdots, \alpha_5\} = \{b', c, d, e, f\}$. Let ρ_{E_8} be the Weyl vector having inner product θ with each of the simple roots b, c, d, e, f of affine E_8, where $b = b' + (0^4; -1, 0)$. We have the balanced numbering $(c_i) = (1, 2+\omega, 2, 2+\omega, 1)$, such that $\sum c_i \alpha_i = 0$. So $h = \sum c_i = 8 + 2\omega$. It follows that the norm zero vector $\rho = (\rho_{E_8}, \rho_{E_8}, \rho_{E_8}; h, k)$ (with k chosen to make ρ have norm zero) is of type Λ, that is, $\rho^{\perp}/\rho \simeq \Lambda$.

4.4.2 **How to characterize the diagram or the Weyl vector?**

The vague idea that governs us is that the Weyl vector is a point of symmetry for the mirror configuration. The simple mirrors are the ones closest to the Weyl vector. In-fact the same idea works in characterizing the known Coxeter diagrams for a few finite complex reflection groups too, that we have checked so far. The trouble is that we have not been able to decide on a precise meaning for "a point of symmetry". Thus we are left in a rather unsatisfactory position of not having a real definition for either the Weyl vector or the diagrams.

If we had a definition for the Weyl vector we could possibly define the simple mirrors to be the ones closest to the Weyl vector, and thus define a diagram for the group. In the key example of chapter 2 we showed that the Weyl vector $\bar{\rho}$ is "a hole" for the mirror configuration, that is, the distance from the mirrors achieves a local maxima at $\bar{\rho}$. This is probably true for the other examples of hyperbolic reflection groups and the finite complex reflection group too, though we have not checked this.

On the other hand if we had a definition of what is really meant by the diagram of a complex reflection group we could maybe define the Weyl vector as a point which is equidistant from them. Here is a rather naive attempt.

Let D be a set of roots of a lattice L. Given a set of roots that either braid or commute we call it an *affine diagram* if the diagram is connected and there is a labeling of the vertices by nonzero complex numbers that cannot be changed by any simple reflection. The well known examples are the "balanced numbering" of the vertices of the affine Dynkin diagrams. The affine E_8 diagram over $\mathcal{E}$ has a balanced numbering given in figure 2.1. An affine diagram for the E_6 lattice over $\mathcal{E}$ looks like the usual D_4 Dynkin diagram with arrows pointing towards the central vertex. If we label central vertex with $(2+\omega)$ and the other three vertices with 1 we get a balanced numbering.

Say that D is a *diagram* for $R(L)$ if

1. If $r, r' \in D$ then ϕ_r and ϕ'_r either commute or braid and $R(L) = \langle \phi_r : r \in D \rangle$.

2. For all $D_1 \subseteq D$ such that the reflection group of $\operatorname{span}(D_1)$ is finite or affine, we have $R(\operatorname{span}(D_1)) = \langle \phi_r : r \in D_1 \rangle$.

3. For all $D_1 \subseteq D$ such that $\operatorname{span}(D_1)$ is affine $R(\operatorname{span}(D_1^{\perp})) = \langle \phi_r : r \in D_1^{\perp} \rangle$

4. D is minimal satisfying the conditions above.

These conditions probably hold true for the 26 node diagram D in chapter 2. The only connected affine sub-diagrams of D are the E_6 and E_8 affine diagrams mentioned above and the only connected finite diagrams are sub-diagrams of these affine diagrams.

4.4.3 **Further examples**

Working through the two examples in chapter 2 and 3 one might guess that the phenomenon studied in these examples is specific to the Leech lattice. Indeed the Leech lattice is a "sporadic lattice" that has many special properties. The next example to look for would be over Cayley's octonions.

However if one is ready to include the examples of the reflection group of $E_8^{\mathcal{E}} \oplus E_8^{\mathcal{E}} \oplus H$ and of $D_4^{\mathcal{E}} \oplus D_4^{\mathcal{E}} \oplus H$ in the same league then it is natural to ask what is the right class of complex hyperbolic reflection groups which provide examples of similar phenomenon.

If one also includes the example of $E_8^{\mathcal{E}} \oplus H$ in the same league then one can no longer hope to find the Weyl vector as the fixed point of diagram automorphisms.

If there is a large class of examples then it is probably true that such symmetries of the diagram would not be present in all the cases.

On a slightly different track one might try to start with "nice graphs" and make a Lorentzian lattice out of it by taking generating vectors corresponding to the vertices, the inner products between the vectors being governed by the adjacency matrix of the graph, and then quotient by appropriate relations to get a non-degenerate bilinear form. In particular one can mimic the construction of the lattice given in 2.3.4. Exact parallel of this construction works for the quaternionic example in chapter 3 too. We tried to mimic these two constructions starting with the incidence graph of $P^2(\mathbb{F}_5)$ but did not succeed.

4.4.4 **Fundamental group of the quotient of the mirror complement**

Let $X = \mathbb{C}H^{13}/\operatorname{Aut}(L)$, and let $\mathfrak{D}$ be the image of mirrors in X. In view of the Allcock's conjecture stated in 2.7.2, one would like to understand the fundamental group $\pi_1(X \setminus \mathfrak{D})$. For a finite Weyl group W acing on a real vector space V, the space $(V \setminus \{\text{mirrors of W}\}) \otimes \mathbb{C}/W$ is $K(\pi, 1)$ and the fundamental group is the Artin group of the corresponding Dynkin diagram. Allcock's Conjecture predicts a similar description for the fundamental group $\pi_1(X \setminus \mathfrak{D})$.

For complex reflection groups similar results are conjectured and known in many cases. For complex reflection groups we do not need to complexify the vector space. Let W be a finite complex reflection group acting on $\mathbb{C}^n$. It is conjectured (and known in all but some exceptional cases, see [Bes] and [BMR]) that $Y = (\mathbb{C}^n \setminus \{\text{mirrors of W}\})/W$ is $K(\pi, 1)$ and that the fundamental group of Y has a presentation as the "Artin group" of the complex Coxeter diagram for W.

For complex hyperbolic reflection group W acting on $\mathbb{C}H^n$, Allcock showed that $(\mathbb{C}H^n \setminus \{\text{mirrors of W}\})/W$ is $K(\pi, 1)$ if any two mirrors, whenever they meet inside $\mathbb{C}H^n$ meet orthogonally. However this is not true for the examples that we have studied and it is not known whether $X \setminus \mathfrak{D}$ has contractible universal cover.

For a finite complex reflection group W one strategy used to calculate $\pi_1((\mathbb{C}^n \setminus \{\text{mirrors of W}\})/W)$ is the following (See [BM] and the references given there).

The ring of invariants of W is a polynomial ring if and only if W is a reflection group. Hence the quotient $X = \mathbb{C}^n/W$ is isomorphic to $\mathbb{C}^n$. The image of the mirrors determine a hypersurface $\mathfrak{D}$, called the discriminant hypersurface in X. The fundamental group of $X \setminus \mathfrak{D}$ can be calculated by Zariski, Van-Kampen method. One uses the fact that for a generic enough complex line F one has as surjection from $\pi_1((X \setminus \mathfrak{D}) \cap F)$ to $\pi_1(X \setminus \mathfrak{D})$ and for a generic enough 2-plane F this is an isomorphism of π_1.

Coming back to our example $X = \mathbb{C}H^{13}/\operatorname{Aut}(L)$ is a quasi-projective variety. The Baily-Borel compactification $\tilde{X}$ of X is obtained by adding points at cusps, corresponding to orbits of primitive norm zero vectors in L. One can write down meromorphic automorphic forms for $U(1,n)$ with singularities along the mirrors of reflection. Allcock has done this in [All1] using Borcherds' machinery. If one can find an explicit uniformization of $\tilde{X}$ using automorphic forms and find a description of $\mathfrak{D}$ in terms of this embedding, it might be useful to get ones hand on the fundamental group of $X \setminus \mathfrak{D}$.

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