

On Coxeter diagrams of complex reflection groups

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Abstract: We study complex Coxeter diagrams of some unitary reflection groups. Using solely the combinatorics of diagrams, we give a new proof of the classification of root lattices defined over $\mathcal{E} = \mathbb{Z}[e^{2\pi i/3}]$: there are only four such lattices, namely, the \mathcal{E} -lattices whose real forms are A_2 , D_4 , E_6 and E_8 . Next, we address the issue of characterizing the diagrams for unitary reflection groups, a question that was raised by Broue, Malle and Rouquier. To this end, we describe an algorithm which, given a unitary reflection group G , picks out a set of complex reflections. If G is the reflection group of a root lattice defined over \mathcal{E} , the reflections selected by the algorithm form the known diagram for G . For the reflection group of the complex Coxeter-Todd lattice $K_{12}^{\mathcal{E}}$, we find a new diagram that extends to an “affine diagram” with $\mathbb{Z}/7\mathbb{Z}$ symmetry. If G is a Weyl group, the algorithm immediately yields a set of simple roots. Otherwise, experimental evidences indicate that the algorithm selects a minimal generating set of reflections if G is primitive and G has a set of roots whose \mathbb{Z} -span is a discrete subset of the ambient vector space.

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1. INTRODUCTION

1.1. Background on unitary reflection groups: A unitary reflection group is a finite subgroup of the unitary group of a positive definite hermitian vector space V , generated by “complex reflections”. They were classified by Shepherd and Todd in [12]. For a self-contained proof of the classification (which is also similar in spirit to part of our work) see [6]. A convenient table of all these groups and their properties may be found in [5]. There is an infinite series, denoted by $G(de, e, r)$, and 34 others, denoted by G_4, G_5, \dots, G_{37} . Unitary reflection groups have many invariant theoretic properties that are similar to those of the orthogonal reflection groups. Most of these properties were initially established for the unitary groups, via case by case verification through Shepherd and Todd’s list. Recently, there has been a lot of progress in trying to find unified and more conceptual proofs. (For example, see [4] and the references therein.) But a coherent theory, like that of the classical Coxeter groups and Weyl groups, is still not in place and lot of mysteries still remain. One of these mysteries involve the diagrams for unitary reflection groups.

Coxeter-Dynkin diagrams for orthogonal reflection groups encode presentations of these groups. For each unitary reflection group G , there is a diagram D_G , that encodes a presentation of G (see [5]). Most of these diagrams were first introduced by Coxeter in [9]. The vertices of D_G correspond to complex reflections that form a minimal set of generators for G . Other than that, the definition of D_G is ad-hoc and case by case. It is curious that even though these diagrams do not have any uniform definition, in most cases they contain a lot of non-trivial information about the group G . We quote two sample results.

- (1) Let \mathcal{M} be the union of the fixed point sets of the reflections in G and let $X = V \setminus \mathcal{M}$. Then the universal cover of X/G is contractible and the fundamental group of X/G has a presentation encoded by the diagram D_G (see [4]).
- (2) Let G be a *well generated* unitary reflection group of rank n . (A unitary reflection group of rank n is well generated if it can be generated by n complex reflections). The *Coxeter number* of G is defined to be the largest positive integer h such that $e^{2\pi i/h}$ is an eigenvalue of an element of G . Then the product of the generators of G corresponding to the vertices of D_G , in certain order, has eigenvalues $e^{2\pi i(d_j-1)/h}$ where d_j are the invariant degrees of G . (see theorem 5.5 of [11]).

These results were first verified for most of the groups in the Shepherd and Todd's list by arguments split into many separate cases. Essentially classification free proofs are now known, by recent work of David Bessis (see [4]). But we still do not know of a way to characterize these diagrams.

In this article, we study the complex diagrams of a few unitary reflection groups. The main results are discussed below. They are motivated by analogies with Weyl groups.

1.2. Summary of results: Our approach is to view unitary reflection groups as sets of automorphisms of “complex lattices”. Let $\mathcal{E} = \mathbb{Z}[e^{2\pi i/3}]$. The main examples of unitary reflection groups that we want to study, act as automorphisms of a sequence of \mathcal{E} -lattices, namely, $A_2^\mathcal{E} \subseteq D_4^\mathcal{E} \subseteq E_6^\mathcal{E} \subseteq E_8^\mathcal{E}$. Our interest in these lattices stem from their importance in studying the complex hyperbolic reflection group with Y_{555} diagram and its conjectured connection with the bimonster (see [1], [3]). In section 2, we present a new proof of theorem 2.2 of [1], which states that $A_2^\mathcal{E}, D_4^\mathcal{E}, E_6^\mathcal{E}, E_8^\mathcal{E}$ are the only “ \mathcal{E} -root lattices”. Our proof is like the A - D - E classification of Euclidean root lattices and is similar in spirit to the arguments in [6] and [10]. It is purely a linear algebra argument that only uses the diagrams for the complex reflection groups. This proof should be viewed as an illustration of the usefulness of the “complex diagrams”.

Section 3 contains the main bulk of our work. Here we address the following question, that was raised in [5]: *How to characterize the diagrams for the unitary reflection groups?* To this end, we shall describe an algorithm (in 3.8 and 3.9) which, given a unitary reflection group, picks out a set of complex reflections. We find that these reflections form Coxeter's diagram in the examples of our main interest, namely the reflection groups related to the \mathcal{E} -root lattices. We also apply our algorithm to the reflection groups of the Coxeter-Todd lattice $K_{12}^\mathcal{E}$ and its sub-lattice $K_{10}^\mathcal{E}$, defined over \mathcal{E} . In these two cases, the diagrams produced by our algorithm are different from those in the literature. However, these reflection groups, known respectively as G_{33} and G_{34} , are part of a few exceptional cases, in which, the diagrams known in the literature do not have some of the desirable properties. (For example, see question 2.28 in [5] and the remark following it). So there seems to be some doubt whether the diagrams known in the literature for these examples, are the “right ones”.

When G is a Weyl group or a primitive unitary reflection group defined over $\mathbb{Q}(e^{2\pi i/3})$ or $\mathbb{Q}(i)$, the complex reflections selected by the algorithm form a minimal set of generators of G . Otherwise, the algorithm does not work, in the sense that the set of reflections selected by the algorithm usually do not form a minimal set of generators of G . Our algorithm is based on a generalization of a “Weyl vector”. We show that “Weyl vectors” exist for all unitary reflection groups (see theorem 3.5).

In section 4 we describe “affine diagrams” for unitary reflection groups defined over \mathcal{E} . The affine diagrams are obtained from the unitary diagrams by adding an extra node. They encode presentations for the corresponding affine complex reflection groups. For each affine diagram, we describe a “balanced numbering” on its vertices, like the ${}_{2\ 4}^3{}_{6\ 5\ 4\ 3\ 2\ 1}$ numbering on the affine E_8 diagram. The existence of a balanced numbering on a diagram implies that the corresponding reflection group is not finite. So an affine diagram cannot occur as a full sub-graph of a diagram for a unitary reflection group. These facts, coupled with a combinatorial argument, complete the classification of \mathcal{E} -root lattices. The affine diagrams are often more symmetric compared to the unitary diagrams, like in the real case. For example, in the case of $K_{12}^{\mathcal{E}}$, we get an affine diagram with rotational $\mathbb{Z}/7\mathbb{Z}$ symmetry.

The complex root systems that we use in computer experiments are described in the appendix (see A.1 to A.5). If the diagram found by our algorithm is new, then a presentation of the corresponding reflection group, encoded by this diagram, is given in appendix A.6.

We finish this section by introducing some basic definitions and notations to be used.

1.3. Reflection groups and root systems: Let V be a complex vector space with an hermitian form (always assumed to be linear in the second variable). If $x \in V$, then $|x|^2 = \langle x, x \rangle$ is called the *norm* of x . Given a vector x of non-zero norm and a root of unity $u \neq 1$, let

$$\phi_x^u(y) = y - (1 - u)\langle x, y \rangle |x|^{-2}x.$$

Then ϕ_x^u is an automorphism of the hermitian space V , called an *u -reflection* in x , or simply, a *complex reflection*. The hyperplane x^\perp (or its image in the projective space $\mathbb{P}(V)$), fixed by ϕ_x^u , is called the *mirror* of reflection. A *complex reflection group* G is a discrete subgroup of $\text{Aut}(V, \langle \cdot, \cdot \rangle)$, generated by complex reflections. A mirror of G is a hyperplane fixed by a reflection in G . A complex reflection (resp. complex reflection group) is called a *unitary reflection* (resp. *unitary reflection group*) if the hermitian form on V is positive definite. We shall omit the words “complex” or “unitary”, if they are clear from context. A unitary reflection group G acting on V is *reducible* (resp. *imprimitive*) if V is a direct sum $V = V_1 \oplus \cdots \oplus V_l$ such that $0 \neq V_1 \neq V$ and each V_j is fixed by G (resp. $\{V_j : 1 \leq j \leq l\}$ is fixed by G) as a set. Otherwise it is *irreducible* (resp. *primitive*). Unless otherwise stated, we always assume that G is irreducible.

Let G be a unitary reflection group acting on \mathbb{C}^k with the standard hermitian form. Let F be the field of definition of G and \mathcal{O} be the ring of integers of F . Let \mathcal{O}^* be the group of units of \mathcal{O} . A vector r in an \mathcal{O} -module K is *primitive* if $m^{-1}r \in K$ and $m \in \mathcal{O}$ implies $m \in \mathcal{O}^*$. Let Φ be a set of primitive vectors in \mathcal{O}^k such that:

- Φ is stable under the action of G ,
- $\{r^\perp : r \in \Phi\}$ is equal to the set of mirrors of G , and
- given $r \in \Phi$, $ur \in \Phi$ if and only if u is an unit of \mathcal{O} .

Such a set of vectors will be called a (*unitary*) *root system* for G , defined over \mathcal{O} . The group of units \mathcal{O}^* , acts on Φ by multiplication. An orbit is called a *projective root*. A set of projective roots for G is denoted by Φ_* or $\Phi_*(G)$.

1.4. Lattices and their reflection groups: Let F be a number field. Let \mathcal{O} be the ring of integers of F . For simplicity, we assume that \mathcal{O} is a unique factorization domain. We shall fix an embedding of F in \mathbb{C} and identify F and \mathcal{O} as subsets of \mathbb{C} via this embedding. The three cases when \mathcal{O} forms a discrete set in \mathbb{C} are the integers, the Gaussian integers $\mathcal{G} = \mathbb{Z}[i]$

and the Eisenstein integers $\mathcal{E} = \mathbb{Z}[e^{2\pi i/3}]$. These are the rings that will be important for us. To fix ideas, one may take $\mathcal{O} = \mathcal{E}$, since we will mostly be dealing with this case.

A *lattice* K , defined over \mathcal{O} , is a free \mathcal{O} -module of finite rank with an \mathcal{O} -valued hermitian form. Let $V = \mathbb{C} \otimes_{\mathcal{O}} K$ be the complex vector space underlying K . The *dual lattice* of K , denoted by K' , is the set of vectors $y \in V$ such that $\langle y, x \rangle \in \mathcal{O}$ for all $x \in K$.

A *root* of K is a primitive vector $r \in K$ of non-zero norm such that $\phi_r^u \in \text{Aut}(K)$ for some root of unity $u \neq 1$. The *reflection group* of K , denoted by $R(K)$, is the subgroup of $\text{Aut}(K)$ generated by reflections in the roots of K . The projective roots of K are in bijection with the mirrors of $R(K)$. If K is positive definite, then the roots of K form a unitary root system, denoted by Φ_K , for the unitary reflection group $R(K)$.

1.5. Diagrams: Consider the permutation matrices acting on $k \times k$ hermitian matrices by conjugation. An orbit D of this action is called a *diagram*. If $M = ((m_{ij}))$ is a representative of an orbit D , then we say that M is a *gram matrix* of D . Let $\Delta = \{r_1, \dots, r_k\}$ be a subset of a hermitian space V . The matrix $((\langle r_i, r_j \rangle))$ is called a *gram matrix* of Δ and the corresponding diagram is denoted by $D(\Delta)$. Let Φ be a root system for a unitary reflection group G . If Δ is a subset of Φ such that $\{\phi_r^{u_r} : r \in \Delta\}$ is a minimal generating set for G (for some units u_r), then we say that $D(\Delta)$ is a *diagram* for G .

Pictorially, a diagram D is conveniently represented by drawing a directed graph D with labeling of vertices and edges, as follows: Let $v(D) = \{x_1, \dots, x_k\}$ be the set of vertices of D . We remember the entry g_{ii} by labeling the vertex x_i with g_{ii} . We remember the entry g_{ij} by drawing a directed edge from x_j to x_i labeled with g_{ij} or equivalently, by drawing a directed edge from x_i to x_j labeled with \bar{g}_{ij} (but not both).

Let D be a diagram with gram matrix $((m_{ij}))$. Assume that $m_{ij} \in \mathcal{O}$ for all i and j . Define $L(D)$ to be the \mathcal{O} -lattice generated by linearly independent vectors $\{x_1, \dots, x_k\}$ with $\langle x_i, x_j \rangle = m_{ij}$. Let L be an \mathcal{O} -lattice having a set of roots Δ which form a minimal spanning set for L as an \mathcal{O} -module. Then the diagram $D(\Delta)$ is called a *root diagram* or simply a diagram for L . A diagram for L will be denoted by D_L . We shall usually denote the vertices of D_L and the corresponding vectors of L by the same symbol. If D is a root diagram of L , then $L(D)$ surjects onto L preserving the hermitian form. If, further, the gram matrix of D is positive definite, then $L(D) \simeq L$.

Two diagrams D and D' are *equivalent* if $L(D) \simeq L(D')$. In this case, we write $D \simeq D'$. Let $L = L(D)$, $v(D) = \{x_1, \dots, x_k\}$ and u_1, \dots, u_k be units. Then there is a diagram for L whose vertices correspond to the generators $\{u_1 x_1, \dots, u_k x_k\}$. These two diagrams are equivalent. The only difference between them is in the edge labeling, which may differ by units.

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2. THE EISENSTEIN ROOT LATTICES

Let $\omega = e^{2\pi i/3}$, $\theta = \omega - \bar{\omega}$ and $p = 1 - \bar{\omega}$. Let $\mathcal{E} = \mathbb{Z}[\omega]$. In this section we shall classify the \mathcal{E} -root lattices, which we define following Daniel Allcock (see [1]).

2.1. Definition. An *Eisenstein root lattice* or \mathcal{E} -root lattice is a positive definite \mathcal{E} -lattice K , generated by vectors of norm 3, such that $K \subseteq \theta K'$. A root lattice is *indecomposable* if it is not a direct sum of two proper non-zero root lattices.

Let D be a diagram with gram matrix $((m_{ij}))$. The following assumptions about D will remain in force for the rest of this section. We assume that $m_{ij} \in \mathcal{E}$ for all i and j . We assume that $m_{ii} = 3$ for all i . So we omit the labels on the vertices. If $m_{ij} = -p$, we omit the label on the edge going from j to i . If $m_{ij} = 0$, we omit the edge $\{i, j\}$. The connectedness of a diagram is considered with these conventions. Each \mathcal{E} -root lattice has at-least one diagram. Any diagram for an indecomposable \mathcal{E} -root lattice is connected.

2.2. Remark. Let K be an \mathcal{E} -lattice satisfying $K \subseteq \theta K'$. The following observations are immediate: If $r \in K$ has norm 3, then r is a root of K . The order 3 reflections in r , denoted by $\phi_r = \phi_r^\omega$ and $\phi_r^{-1} = \phi_r^{\bar{\omega}}$, belong to the reflection group of K . Let x and y be two linearly independent vectors of K of norm 3. If K is positive definite, then one must have $|\langle x, y \rangle| \leq \sqrt{3}$. So either $\langle x, y \rangle = 0$, which imply $\phi_x \phi_y = \phi_y \phi_x$, or $\langle x, y \rangle \in \theta \mathcal{E}^*$, which imply $\phi_x \phi_y \phi_x = \phi_y \phi_x \phi_y$.

2.3. Definition. Let $E_{2k}^\mathcal{E}$ be the \mathcal{E} -lattice having a basis x_1, \dots, x_k such that $|x_i|^2 = 3$ for $i = 1, \dots, k$, $\langle x_i, x_{i+1} \rangle = -p$ for $i = 1, \dots, k-1$ and $\langle x_i, x_j \rangle = 0$ if $j > i+1$. The lattice $E_{2k}^\mathcal{E}$ has a diagram $D_{E_{2k}^\mathcal{E}} = (x_1 \leftarrow x_2 \leftarrow \dots \leftarrow x_k)$. The \mathbb{Z} -modules underlying $E_2^\mathcal{E}$, $E_4^\mathcal{E}$, $E_6^\mathcal{E}$ and $E_8^\mathcal{E}$, with the bilinear form $\frac{2}{3} \text{Re}\langle x, y \rangle$, are the lattices A_2 , D_4 , E_6 and E_8 respectively.

We use the complex diagrams to give a new proof of the following theorem (theorem 2.2 of [1]):

2.4. Theorem. *The only indecomposable Eisenstein root lattice are $E_{2i}^\mathcal{E}$ with $i = 1, 2, 3, 4$.*

The lattices $E_2^\mathcal{E}$, $E_4^\mathcal{E}$, $E_6^\mathcal{E}$ and $E_8^\mathcal{E}$ are positive definite. So if some Eisenstein lattice L has the root diagram of $E_{2j}^\mathcal{E}$, for some $1 \leq j \leq 4$, then $L \simeq E_{2j}^\mathcal{E}$. Thus it suffices to classify the equivalence classes of root diagrams of indecomposable \mathcal{E} -root lattices and show that there are only four classes. The proof of this classification, given below, is like the well-known classification of A - D - E root systems.

2.5. Definition. Let D be a connected root diagram for an \mathcal{E} -lattice L . A *balanced numbering* on D is a function from $v(D)$ to \mathcal{E} , denoted by $x \mapsto n_x$, such that $n_x = 1$ for some $x \in v(D)$ and

$$\sum_{x \in v(D)} n_x \langle a, x \rangle = 0 \quad (1)$$

for each vertex a of D . If $\{n_x : x \in v(D)\}$ is a balanced numbering on D , then the vector $y = \sum_{x \in v(D)} n_x \cdot x \in L$ is orthogonal to each $x \in v(D)$. So $|y|^2 = 0$. A diagram that admits a balanced numbering and is minimal with this property, is called an *affine diagram*.

Figure 1 shows a few affine diagrams, each with a balanced numbering. The number shown next to a vertex x is n_x . Given $a \in v(D)$, if b_1, \dots, b_m are the vertices connected to a and c_j is the label on the directed edge going from b_j to a , that is, $\langle a, b_j \rangle = c_j$, then to verify equation (1), one needs to check that $-3n_a = \sum_{j=1}^m c_j n_{b_j}$. This is easily verified.

We say that a connected diagram D is *indefinite* if D cannot appear as a full sub-graph of a diagram of an Eisenstein root lattice. Otherwise, we say that D is *definite*.

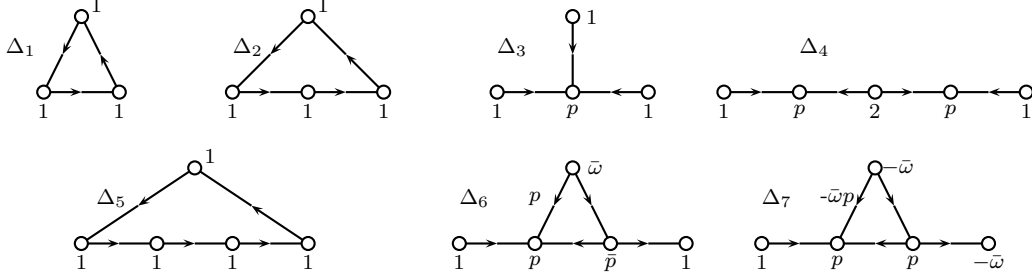


FIGURE 1. A few affine diagrams with balanced numbering. Choose any vertex v from Δ_i such that $\Delta_i \setminus \{v\}$ is connected. Then $\Delta_i \setminus \{v\}$ is a root diagram for $E_4^\mathcal{E}$ if $i = 1$, $E_6^\mathcal{E}$ if $i = 2, 3$ and $E_8^\mathcal{E}$ if $i = 4, 5, 6, 7$.

2.6. Lemma. *Let D be a diagram that admits a numbering $\{n_x : x \in v(D)\}$ such that $y = \sum_{x \in v(D)} n_x \cdot x$ is a norm zero vector of $L(D)$ and $n_{x_j} = 1$ for some $x_j \in v(D)$. Then D is indefinite. In particular, if D admits a balanced numbering, then D is indefinite.*

Proof. Suppose K is an \mathcal{E} -root lattice with a root diagram D_K . If $D \subseteq D_K$, then $y = \sum_{x \in v(D)} n_x \cdot x \in K$ has norm 0. Since K is positive definite, $y = 0$. If $n_{x_j} = 1$, then $x_j = -\sum_{x \neq x_j} n_x \cdot x$, contradicting the fact that $v(D_K)$ is a minimal generating set for K . \square

2.7. Definition. Let D be a diagram with vertices x_1, \dots, x_k and edges $\{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_{k-1}, x_k\}$ and $\{x_k, x_1\}$, that is, a circuit of length k . Changing the vectors x_1, \dots, x_k by units if necessary, we may assume that $\langle x_i, x_{i+1} \rangle = -p$ for $1 \leq i \leq k-1$ and $\langle x_1, x_k \rangle = -up$ for some $u \in \mathcal{E}^*$. We denote this diagram by $\text{Circ}_{k,u}$.

2.8. Lemma. (a) *Let $D = \text{Circ}_{3,-1}$ or $D = \text{Circ}_{3,\bar{\omega}}$. Then $D \simeq D_{E_6^\mathcal{E}}$.*

(b) *Let $D = \text{Circ}_{4,\bar{\omega}}$. Then $D \simeq D_{E_8^\mathcal{E}}$.*

(c) *Suppose $D = \text{Circ}_{k,u}$ with $k = 3, 4$ or 5 but D is not one of the three circuits considered in (a) and (b). Then D is indefinite.*

(d) *Suppose D is one of the diagrams given in figure 1. Then D is indefinite.*

Proof. (a) Let $v(D) = \{x_1, x_2, x_3\}$. Note that $\langle x_1, \bar{u}x_3 \rangle = \langle x_1, x_2 \rangle = -p$. If we take $x'_3 = x_2 - \bar{u}x_3$, then $\langle x_1, x'_3 \rangle = 0$. One checks that $|x'_3|^2 = 6 + 2 \text{Re}(\bar{u}p) = 3$ and $\langle x_2, x'_3 \rangle = 3 + \bar{u}p \in p\mathcal{E}^*$, for $u = -1$ and $\bar{\omega}$. So D is equivalent to the diagram $D_{E_6^\mathcal{E}}$ formed by the roots x_1, x_2, x'_3 .

(b) Let $x'_4 = \bar{\omega}x_2 - px_3 - x_4$. One checks that x_1, x_2, x_3 and x'_4 form the diagram $D_{E_8^\mathcal{E}}$.

(c) For $\nu = 1, \dots, 6$, let $(n_{1,\nu}, \dots, n_{5,\nu})$ be equal to $(1, -\omega, \bar{\omega}, \bar{\omega}, \bar{\omega})$, $(1, -\omega, \bar{\omega}, -1, -1)$, $(1, -\omega, \bar{\omega}, -1, \omega)$, $(1, -\omega, \bar{\omega}, -1, \omega)$, $(1, 1, 1, 1, 1)$ and $(1, -\omega, -\omega, -\omega, -\omega)$ respectively. One checks that the vector $y = \sum_{i=1}^k n_{i,\nu} x_i$ has norm zero in $L(\text{Circ}_{k,e^{2\pi i\nu/6}})$, for $k = 3, 4, 5$ and $\nu = 1, \dots, 6$, except for the three cases considered in part (a) and (b). Part (c) now follows from lemma 2.6. Part (d) also follows from lemma 2.6. \square

Proof of the theorem 2.4. Let D be a root diagram for an indecomposable \mathcal{E} -root lattice K . We shall repeatedly use lemma 2.8 in two ways. First, it implies that D cannot contain the diagrams mentioned in part (c) and (d) of the lemma. Secondly, from the proof of lemma 2.8, we observe the following:

If $\text{Circ}_{3,u}$ or $\text{Circ}_{4,u}$ is a sub-graph of D , then we are in one of the cases considered in part (a) or (b) of the lemma and we can change one of the vertices to get an equivalent diagram,

where one of the edges has been removed. However this may introduce new edges elsewhere in the graph.

Since K is indecomposable, D must be connected. Let $v(D) = \{x_1, \dots, x_k\}$. If $k = 1$ (resp. $k = 2$), then clearly $K \simeq E_2^\mathcal{E}$ (resp. $K \simeq E_4^\mathcal{E}$). If $k = 3$, then either $D \simeq D_{E_6^\mathcal{E}}$ or $D \simeq \text{Circ}_{3,u}$ with $u = -1$ or $\bar{\omega}$, which are again equivalent to $D_{E_6^\mathcal{E}}$.

Let $k = 4$. We may assume that the diagram D' formed by $\{x_1, x_2, x_3\}$ is $D_{E_6^\mathcal{E}}$. If possible, suppose D is not equivalent to $D_{E_8^\mathcal{E}}$. Also suppose that $\{x_2, x_4\}$ is an edge of D . Since D cannot be the affine diagram Δ_3 , either $\{x_1, x_2, x_4\}$ or $\{x_2, x_3, x_4\}$ is a circuit in D . Without loss, suppose $\{x_1, x_2, x_4\}$ is a circuit. Then we can change x_4 by adding a multiple of x_1 to get an equivalent diagram where $\{x_2, x_4\}$ is not an edge. So $D \simeq E_8^\mathcal{E}$ or D is a circuit. In the latter case, $D \simeq \text{Circ}_{4,\bar{\omega}}$, since all other circuits of length 4 are indefinite, by lemma 2.8(c). Lemma 2.8(b) implies that $\text{Circ}_{4,\bar{\omega}} \simeq E_8^\mathcal{E}$.

Let $k = 5$. We may assume that the diagram D' formed by $\{x_1, x_2, x_3, x_4\}$ is $D_{E_8^\mathcal{E}}$. Since D cannot be the affine diagram Δ_4 , there must be at-least two edges joining x_5 with D' . Since the diagrams of the form $\text{Circ}_{5,u}$ are not definite, D must contain a circuit of length 3 or 4. If $\{x_1, x_5\}$ is an edge, it is part of a circuit of length 3 or 4 and as before, we can remove it by shifting to an equivalent diagram. The remaining possibilities are shown in figure 2.

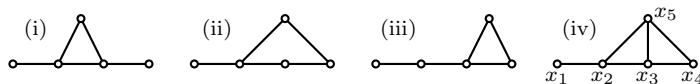


FIGURE 2. An undirected edge between x_i and x_j means $|\langle x_i, x_j \rangle| = |p|$.

(The arrows on edges are not important here, so they have been omitted). In cases (iii) and (iv), we may add a multiple of x_3 to x_5 and disconnect x_4 from x_5 (note that this does not introduce an edge between x_1 and x_5). So we are reduced to the first two cases. But (i) is affine (either Δ_6 or Δ_7) and (ii) contains the affine diagram Δ_3 (See figure 1). \square

3. AN ATTEMPT TO CHARACTERIZE THE DIAGRAMS FOR UNITARY REFLECTION GROUPS

In this section we want to address a question that was raised in [5]: *How to characterize the diagrams for unitary reflection groups?* We maintain the definitions and notations introduced in 1.3, 1.4 and 1.5. In 3.8-3.9, we shall describe an algorithm which, given a unitary reflection group, picks out a set of reflections. For the Weyl groups and the primitive unitary reflection groups defined over $\mathbb{Q}(\omega)$ and $\mathbb{Q}(i)$, the reflections selected by the algorithm form a minimal set of generators. There are ten primitive unitary reflection groups defined over $\mathbb{Q}(\omega)$ and $\mathbb{Q}(i)$. Most of these can be viewed as sets of automorphisms of some special complex lattices defined over \mathcal{E} or \mathcal{G} , namely, $D_4^\mathcal{E}, E_6^\mathcal{E}, E_8^\mathcal{E}, K_{10}^\mathcal{E}, K_{12}^\mathcal{E}, D_4^\mathcal{G}$ and $E_8^\mathcal{G}$. For a description of $E_8^\mathcal{E}, E_8^\mathcal{G}$ and $K_{12}^\mathcal{E}$, see example 11b, 13b and 10a respectively, in chapter 7, section 8 of [8]). The lattice $K_{10}^\mathcal{E}$ is the orthogonal complement of any root in $K_{12}^\mathcal{E}$.

We take cue from the fact that for a Weyl group, vertices of the Dynkin diagram correspond to the simple roots, which are the positive roots having minimal inner product with a Weyl vector. Our algorithm is based on a generalization of the notion of Weyl vector.

3.1. Definition. Let G be an irreducible unitary reflection group with a root system Φ , defined over \mathcal{O} . Let $\Phi_* = \Phi/\mathcal{O}^*$ be the set of projective roots. Given a projective root r ,

let $o(r)$ denote the order of the subgroup of G generated by reflections in r . Let K be the \mathcal{O} -lattice spanned by Φ . Let V be the vector space underlying K and $\mathcal{M} = \cup_{r \in \Phi_*} r^\perp$ be the union of the mirrors. Define a function $\alpha : V \setminus \mathcal{M} \rightarrow V$, by

$$\alpha(w) = \sum_{r \in \Phi_*} o(r)^{-2} \frac{\langle r, w \rangle}{|\langle r, w \rangle|} \frac{r}{|r|} \quad (2)$$

- 3.2. *Remark.* (1) Note that the quantity $o(r)^{-2} \frac{\langle r, w \rangle}{|\langle r, w \rangle|} \frac{r}{|r|}$ does not change if we change r by a scalar. So the function α is well defined and only depends on the reflection group G and not on the choice of the roots. The function α descends to a function $\bar{\alpha} : \mathbb{P}(V \setminus \mathcal{M}) \rightarrow \mathbb{P}(V)$. Also note that α is G -equivariant, that is, $\alpha(gw) = g\alpha(w)$ for all $g \in G$. So α induces a function from $\mathbb{P}(V \setminus \mathcal{M})/G$ to $\mathbb{P}(V)/G$.
- (2) If all the roots have the same norm, then the factor $|r|$ in the denominator is unnecessary and can be omitted from the definition of α .
- (3) The exponent -2 on $o(r)$ was found by experimenting with the example G_{26} , which has reflections of order two and three. We have included it to indicate one of the ways in which our method can be modified to possibly include other examples. If the stabilizer of each mirror have the same order, then the factor $o(r)^{-2}$ can be omitted from the definition of α . For example, if we consider reflection group of an Euclidean root lattice (resp. \mathcal{E} -root lattice), then $o(r)$ is always equal to 2 (resp. 3).

3.3. **The case of Weyl groups:** Let W be a Weyl group acting on the real vector space $V_{\mathbb{R}}$ via its standard representation. Let $\mathcal{M}_{\mathbb{R}}$ be the union of the mirrors of W . Then W can be viewed as a unitary reflection group acting on $V = V_{\mathbb{R}} \otimes \mathbb{C}$.

Claim: Let $w, w' \in V_{\mathbb{R}} \setminus \mathcal{M}_{\mathbb{R}}$. If w and w' are in the same Weyl chamber, then $\alpha(w) = \alpha(w')$. Further, w and $\alpha(w)$ belong to the same Weyl chamber. So $\alpha(w)$ is a fixed point of α .

Proof. For this argument, assume that the factor $o(r)^{-2} = \frac{1}{4}$ is absent in the definition of α . Let $\Phi_+(w)$ be the set of roots having strictly positive inner product with w . Then $\Phi_+(w)$ can be chosen as a set of representatives for the projective roots, so $\alpha(w) = \sum_{r \in \Phi_+(w)} r/|r|$. Now, w and w' are in the same Weyl chamber if and only if $\Phi_+(w) = \Phi_+(w')$, so $\alpha(w) = \alpha(w')$.

To check that w and $\alpha(w)$ belong to the same Weyl chamber, first, let Φ be a root system of type A_n, D_n, E_6, E_7 or E_8 . Let $w \in V_{\mathbb{R}} \setminus \mathcal{M}_{\mathbb{R}}$. Then $\rho = \frac{1}{2} \sum_{r \in \Phi_+(w)} r$ is a Weyl vector which belong to the same Weyl chamber as w . Note that $\alpha(w) = \sqrt{2}\rho$ and $\alpha(\sqrt{2}\rho) = \sqrt{2}\rho$.

Now consider the non-simply laced case. Since α is G -equivariant, it is enough to check that w and $\alpha(w)$ are in the same Weyl chamber, for a single chamber. We show the calculation for B_n . The Weyl group of type C_n is isomorphic to the group of type B_n . Calculations for type G_2 and F_4 are only little more complicated and will be omitted.

Let e_j be the j -th unit vector in \mathbb{Z}^n . The roots of B_n are $\{\pm e_j, \pm e_i \pm e_j : j < i\}$. Choose $w = (w_1, \dots, w_n)$ such that $0 < w_1 < \dots < w_n$. Then $\Phi_+(w) = \{e_j, e_i \pm e_j : j < i\}$. So

$$\rho^{(n)} = \alpha(w) = \sum_{r: \langle r, w \rangle > 0} r/|r| = (1, 1 + \sqrt{2}, 1 + 2\sqrt{2}, \dots, 1 + (n-1)\sqrt{2}). \quad (3)$$

Observe that $\rho^{(n)}$ and w are in the same Weyl chamber. So $\alpha(\rho^{(n)}) = \alpha(w) = \rho^{(n)}$. \square

3.4. **Definition.** Let Φ be a unitary root system defined over \mathcal{O} . In view of 3.3, a fixed point of α will be called a *Weyl vector*, when the root system is not defined over \mathbb{Z} . One

may try to find a fixed point of α by iterating the function. Our method for selecting a set of generating reflections for G , is based on this notion of Weyl vector. Before describing it we show that Weyl vectors exist.

3.5. Theorem. *Let Φ be a root system for a unitary reflection group G . Then the function α , defined in (2), has a fixed point.*

For notational simplicity, let $\mu_r = |r|^{-1} \cdot o(r)^{-2}$, so that,

$$\alpha(w) = \sum_{r \in \Phi_*} \mu_r |\langle r, w \rangle|^{-1} \langle r, w \rangle r. \quad (4)$$

The argument given below actually shows, that for any sequence of non-zero positive numbers μ_r , a function of the form (4) has a fixed point. We need a lemma, which converts the problem of finding a fixed point of α to a maximization problem.

3.6. Lemma. *Let Φ be a unitary root system and let \mathcal{M} be the union of mirrors. Consider the function $S : \mathbb{P}(V) \rightarrow \mathbb{R}$, defined by*

$$S(w) = |w|^{-1} \langle \alpha(w), w \rangle = \sum_{r \in \Phi_*} \mu_r |w|^{-1} |\langle r, w \rangle|. \quad (5)$$

For $w \notin \mathcal{M}$, one has $\partial_w(S) = 0$ if and only if $\alpha(w) = |w|^{-1} S(w)w$. (here ∂_w denotes the holomorphic derivative with respect to w).

Proof. Let $n(w) = \sum_{\Phi_*} \mu_r |\langle r, w \rangle|$, so that $S(w) = n(w)/|w|$. We fix a basis for the vector space V and write $\langle r, w \rangle = r^* M w$, where r^* is the conjugate transpose of r and M is the matrix of the hermitian form. Differentiating, we get $\partial_w(|\langle r, w \rangle|^2) = \langle w, r \rangle r^* M$. So

$$\partial_w(\mu_r |\langle r, w \rangle|) = \mu_r (2|\langle r, w \rangle|)^{-1} \langle w, r \rangle r^* M.$$

Summing over Φ_* , one gets, $\partial_w n(w) = \frac{1}{2} \alpha(w)^* M$. Similarly we get $\partial_w(|w|) = \frac{1}{2|w|} w^* M$. It follows that

$$\partial_w(S(w)) = \partial_w \left(\frac{n(w)}{|w|} \right) = \frac{\frac{1}{2} |w| \alpha(w)^* M - \frac{1}{2|w|} n(w) w^* M}{|w|^2} = \frac{\alpha(w)^* - \frac{S(w)}{|w|} w^*}{2|w|} M.$$

Since M is an invertible matrix, the lemma follows. \square

Proof of theorem 3.5. We first prove the following claim:

Claim: If $w \in V$ lies on a mirror, then the function S can not have a local maxima at w .

Fix a root $r_0 \in \Phi$ and $w \in r_0^\perp$. Assume $|w| = 1$. Take $w' = w + \epsilon \xi r_0$, where ξ is a complex root of unity, $\epsilon > 0$ and $\epsilon^2 = 0$. We shall show that $S(w') > S(w)$, for suitable choice of ξ . Note that $|w'|^2 = |w|^2 + 2 \operatorname{Re}(\epsilon \langle w, r_0 \rangle) = |w|^2 = 1$. Let $\Psi_0 = \Phi_* \cap w^\perp$ and $\Psi_1 = \Phi_* \setminus \Psi_0$. We can break up the sum in S as follows:

$$S(w') = \sum_{r \in \Psi_0} \mu_r |\langle r, w' \rangle| + \sum_{r \in \Psi_1} \mu_r |\langle r, w' \rangle| = \epsilon \sum_{\Psi_0} \mu_r |\langle r, r_0 \rangle| + \sum_{\Psi_1} \mu_r |\langle r, w \rangle + \epsilon \xi \langle r, r_0 \rangle|.$$

Let $a = \langle r, w \rangle \neq 0$ and $b = \xi \langle r, r_0 \rangle$. Using the first order expansion of $(1+x)^{1/2}$, we have,

$$|a + \epsilon b| = \sqrt{|a|^2 + 2 \operatorname{Re}(\epsilon b \bar{a})} = |a| + \operatorname{Re}(\epsilon b \bar{a} |a|^{-1}).$$

It follows that

$$S(w') = S(w) + \epsilon \sum_{\Psi_0} \mu_r |\langle r, r_0 \rangle| + \epsilon \operatorname{Re} \left(\xi \sum_{\Psi_1} \mu_r \langle r, r_0 \rangle \overline{\langle r, w \rangle} |\langle r, w \rangle|^{-1} \right).$$

Note that the third term can be made non-negative by choosing ξ suitably, and the second term is positive, since $r_0 \in \Psi_0$. This proves the claim.

The function S is continuous on $\mathbb{P}(V)$, so it attains its global maximum, say at w_0 . The claim we just proved implies that $w_0 \notin \mathcal{M}$, so $\partial_{w_0}(S) = 0$. Let $w_1 = |w_0|^{-1} S(w_0) w_0$. Lemma 3.6 implies that $\alpha(w_1) = \alpha(w_0) = w_1$. \square

3.7. Definition. Let $(\mathcal{O}, G, \Phi, K, V)$ be as in 3.1. Given $w \in V$, let (r_1, \dots, r_N) be the projective roots of G arranged so that $d(r_1^\perp, w) \leq \dots \leq d(r_N^\perp, w)$, where d is the Fubini-Study metric on $\mathbb{P}(V)$. So

$$d(r^\perp, w) = \sin^{-1}(|\langle r, w \rangle| / |r||w|).$$

Let k be the minimum number of reflections needed to generate G . Define $\Delta(w) = \{r_1, \dots, r_k\}$. In other words, $\Delta(w)$ consists of k projective roots, whose mirrors are closest to w .

3.8. Method to obtain a set of “simple reflections”: We now describe a computational procedure in which, the input is a unitary reflection group G , (or equivalently, a projective root system Φ_* for G and the numbers $\{o(r) : r \in \Phi_*\}$) and the output is either the empty set or a non-empty set of projective roots, to be called the *simple roots*. The reflections in the simple roots are called *simple reflections*. A set of simple reflections form a *simple system*.

- (1) From the set $\{w : \alpha(w) = w\}$, choose w such that $S(w)$, defined in (5), is maximum.
- (2) Let $\Delta(w) = \{r_1, r_2, \dots, r_k\}$.
- (3) If r_1, \dots, r_k are linearly dependent, then return the empty set.
- (4) If r_1, \dots, r_k are linearly independent, then return $\Delta(w)$ (the simple roots).

In practice, for each group G to be studied, we execute the following algorithm many times.

3.9. Algorithm: Start with a random vector $w_0 \in V$. Generate a sequence w_n by $w_n = \alpha(w_{n-1})$. If the sequence stabilizes, then say that the algorithm converges and let $w = \lim w_n$. Note down $\Delta(w)$ and $S(w)$.

For computer calculation, we assume that w_n stabilizes if $|w_{n+1} - w_n|^2 / |w_n|^2$ becomes small, say less than 10^{-8} , and remains small and decreasing for many successive values of n . Note that if $\alpha(w) = w$, then $S(w) = |w|$. From the values of $S(w)$, the maximum, denoted by S_{\max}^G , can be found. (In all the examples that we have studied, $S(w)$ takes at most two values on the set of fixed points of α found experimentally, so finding the maximum is not difficult). Each instance of the algorithm, that produced a vector w with $S(w) = S_{\max}^G$, now yields a simple system $\Delta(w)$ (provided that $\Delta(w)$ is a linearly independent set).

3.10. Observations for Weyl groups: First, suppose that Φ is a root system for a Weyl group W . We maintain the notations of 3.3 and ignore the factor $o(r)^{-2} = \frac{1}{4}$ in the definition of α . Given $w \in V_{\mathbb{R}}$, let $\delta_w(r) = |\langle r, w \rangle| / |r|$. Arranging the roots according to increasing distance from w (according to the spherical metric or the Fubini-Study metric) is equivalent

to arranging them according to increasing order of $\delta_w(r)$.

Claim: Let W be a Weyl group, $w_0 \in V_{\mathbb{R}} \setminus \mathcal{M}_{\mathbb{R}}$ and $w = \alpha(w_0)$. Then the algorithm 3.9 converges in one iteration and yields a simple system $\Delta(w)$. For Weyl groups, the definition of a set of simple roots given in 3.8 agrees with the classical notion. Further, for each simple root $r \in \Delta(w)$, we have $\delta_w(r) = 1$. So the simple roots are equidistant from w .

Proof. We saw in 3.3 that $w = \alpha(w_0)$ is a fixed point of α , so $w = \alpha(w_0) = \alpha^2(w_0) = \dots$, that is, the algorithm converges in one iteration.

Suppose Φ is a root system of type A , D or E . Then $w = \alpha(w_0) = \sqrt{2}\rho$, where ρ is a Weyl vector. So $\Delta(w) = \Delta(\rho)$ consists of a set of simple roots and for each $r \in \Delta(w)$, one has $\langle r, \rho \rangle = 1$, so $\delta_w(r) = 1$.

In type B_n , one has $w = \rho^{(n)} = \alpha(\rho^{(n)}) = \dots$, where $\rho^{(n)}$ is given in (3). Observe that the function $\delta_w(r)$ is minimum for the simple roots $\Delta(w) = \{e_1, e_2 - e_1, \dots, e_n - e_{n-1}\}$ and $\delta(r) = 1$ for each $r \in \Delta(w)$. The claim was verified for G_2 and F_4 without difficulty. \square

3.11. Observations for complex root systems: The primitive unitary reflection groups defined over $\mathbb{Q}(\omega)$ (resp. $\mathbb{Q}(i)$) are $\{G_4, G_5, G_{25}, G_{26}, G_{32}, G_{33}, G_{34}\}$ (resp. $\{G_8, G_{29}, G_{31}\}$). In the following discussion, let G be one of these groups. In each case, a projective root system $\Phi_*(G)$ for G is chosen. These are described in the appendix (see A.1, A.2, A.3). For each of these groups G , we have run algorithm 3.9 at least one thousand times and obtained many simple systems by method 3.8. The calculations were performed using the GP/PARI calculator. The computer codes are contained in the files **finite.gp** and **ei-linear-alg.gp** available online at <http://www.math.uchicago.edu/~tathagat/codes/index.html>. The main observations made from the computer experiments are the following:

The sequence $\{w_n\}$ stabilizes in each trial for each G mentioned above. The simple reflections generate G and the simple roots form the same diagram every time. For $G_4, G_5, G_8, G_{25}, G_{26}, G_{32}$, these are Coxeter's diagrams, which can be found, for example, in [5]. In the exceptional cases G_{29}, G_{31}, G_{33} and G_{34} new diagrams are obtained. These diagrams and the presentations of the groups obtained from these diagrams are given in appendix A.6. The "affine" diagrams of type G_{33} and G_{34} are shown in figure 3. The diagrams for the unitary groups are obtained by omitting the extending vertex (the vertex joined by dotted lines).

Further observations from the computer experiments are summarized below.

- (1) If G is not G_{29}, G_{31} or G_{32} , then for all w such that $\alpha(w) = w$, the function $S(w)$ attains the same value. So each trial of the algorithm yields a maxima w for S . For G_{29}, G_{31} and G_{32} , the function $S(w)$ attains two values on the fixed point set of α . In these three cases, the \mathbb{Z} -span of the roots form the E_8 lattice.
- (2) If $G \neq G_{33}$, then in each trial of the algorithm, we find that the vectors $\{r_1, \dots, r_k\}$ are linearly independent. So each choice of a maxima w for the function S , yields a set of simple roots $\Delta(w)$. For $G_{33} = R(K_{10}^{\mathcal{E}})$, in most of the trials, we find that $\{r_1, \dots, r_5\}$ are linearly dependent. So most trials do not yield a set of simple roots $\Delta(w)$. In an experiment with 5000 trials, only 297 yielded simple systems.
- (3) Let G be $G_{33} = R(K_{10}^{\mathcal{E}})$ or $G_{34} = R(K_{12}^{\mathcal{E}})$ or G_{29} . Let g_1, \dots, g_k be a set of simple reflections of G obtained by method 3.8. For these three groups, we have verified the following result. (almost a re-statement of sample theorem (1), quoted in 1.1):

Fix a permutation π so that the order of the product $p = g_{\pi_1} \cdots g_{\pi_k}$ is maximum (over all permutations). Then, the order of p is equal to the Coxeter number of G (denoted by h) and the eigenvalues of either p or p^{-1} are $e^{2\pi i(d_1-1)/h}, \dots, e^{2\pi i(d_k-1)/h}$, where d_1, \dots, d_k are the invariant degrees of G .

For G_{29} , G_{33} and G_{34} the invariant degrees are $(4, 8, 12, 20)$, $(4, 6, 10, 12, 18)$ and $(6, 12, 18, 24, 30, 42)$ respectively. In all three cases, h is equal to the maximum degree. These three are the well generated groups for which the diagrams obtained by method 3.8 are different from the ones in the literature. The remaining group G_{31} is not well generated.

- (4) Start with $w_0 \in V$ and consider the sequence defined by $w_n = \alpha(w_{n-1})$. Roughly speaking, each iteration of the function α makes the vector w_n more symmetric with respect to the set of projective roots Φ_* . The function S measures this symmetry. So the fixed points w of α are often the vectors that are most symmetrically located with respect to Φ_* . Based on this discussion, an alternative definition of a Weyl vector may be suggested, namely, a vector $w \in V$, such that $d(w, \mathcal{M})$ is maximum. (This was suggested to me by Daniel Allcock).

It seems harder to compute these vectors, so we have not experimented much with this alternative definition. However, we would like to remark, that for some complex and quaternionic Lorentzian lattices, similar analogs of Weyl vectors and simple roots, are useful. (One such example is studied in [3]; other examples are studied in [2]. In these examples of complex and quaternionic Lorentzian lattices, the simple roots are again defined as those whose mirrors are closest to the ‘‘Weyl vector’’.)

- (5) The method 3.8 fails for the primitive unitary reflection groups that are not defined over \mathbb{Q} , $\mathbb{Q}(i)$ or $\mathbb{Q}(\omega)$ and for the imprimitive groups $G(de, e, n)$, except when they are defined over \mathbb{Q} , that is, for the cases $G(1, 1, n+1) \simeq A_n$, $G(2, 1, n) \simeq BC_n$ and $G(2, 2, n) \simeq D_n$. It fails in the sense that the set $\Delta(w)$ does not in general form a minimal set of generators for the group. This was found by experimenting with $G(de, e, n)$ for small values of (d, e, n) and also with G_6 and G_9 (see appendix A.4). Although method 3.8 fails, the following observation holds for $G(de, e, n)$:

Let k be the minimum number of reflections needed to generate $G(de, e, n)$ ($k = n$ or $n+1$). There exists a vector $w \in V$ such that, if $r_1^\perp, \dots, r_k^\perp$ are the mirrors closest to w , then reflections in $\{r_1, \dots, r_k\}$ generate $G(de, e, n)$.

For $G(de, e, n)$, one can take $w = \rho^{(n)}$ (the vector we obtained for the Weyl group B_n ; see equation (3)). It is easy to check that $\Delta(\rho^{(n)})$ forms the known diagrams for $G(de, e, n)$. We found $\rho^{(n)}$ by using an algorithm that tries to find a point in V whose distance from \mathcal{M} is at a local maximum. For small values of d, e and n , we find that this algorithm always converge to $\rho^{(n)}$.

4. THE AFFINE REFLECTION GROUPS

4.1. In this section we shall describe affine diagrams for the primitive unitary reflection groups defined over \mathcal{E} , except for G_5 . (Including G_5 would further complicate notations). An affine diagram is obtained by adding an extra node to the corresponding ‘‘unitary diagram’’ and each admit a balanced numbering. A unitary diagram can be extended to an affine diagram

in many ways. We have chosen one that makes the diagram more symmetric. The Weyl vector, that yielded the unitary diagram, is often fixed by the affine diagram automorphisms.

The discussion below and the lemma following it are direct analogs of the corresponding results for Euclidean root lattices. We have included a proof since we could not find a convenient reference.

Let Φ be a unitary root system defined over \mathcal{E} , for an unitary reflection group G . Let K be the \mathcal{E} -lattice spanned by Φ . Assume that the subgroup of $\text{Aut}(K)$ generated by reflections in Φ is equal to G . Let \mathcal{E}_0 be the one dimensional free module over \mathcal{E} with zero hermitian form. Let $\tilde{K} = K \oplus \mathcal{E}_0$. Let us write the elements of \tilde{K} in the form (y, m) with $y \in K$ and $m \in \mathcal{E}$. Let \tilde{G} be the subgroup of $\text{Aut}(\tilde{K})$ generated by the reflections in $\tilde{\Phi} = \{(x, m) : x \in \Phi, m \in \mathcal{E}\}$. Note that $\phi_{(x, m)}^u \in \tilde{G}$ if and only if $\phi_x^u \in G$. Consider the semi-direct product $K \rtimes G$, in which the product is defined by $(x, g) \cdot (y, h) = (x + gy, gh)$. The faithful action of $K \rtimes G$ on \tilde{K} via affine transformations is given by $(x, g)y = x + gy$.

4.2. Lemma. *Given the setup above, assume that $|r|^2 = 3$ for all $r \in \Phi$ and $K' \supseteq p^{-1}K$.*

- (a) *The affine reflection group \tilde{G} is isomorphic to the semi-direct product $K \rtimes G$.*
- (b) *Let r_0 be a root of Φ such that the orbit Gr_0 span K as a \mathbb{Z} -module. If $\phi_{r_1}, \dots, \phi_{r_k}$ generate G , then $\phi_{(r_1, 0)}, \dots, \phi_{(r_k, 0)}$, together with $\phi_{(r_0, 1)}$ generate \tilde{G} .*
- (c) *If ϕ_{r_0} commutes (resp. braids) with ϕ_{r_j} , then $\phi_{(r_0, 1)}$ commutes (resp. braids) with $\phi_{(r_j, 0)}$.*

Proof. (a) Identify K (resp. G) inside \tilde{K} (resp. \tilde{G}) via $y \mapsto (y, 0)$ (resp. $\phi_x \mapsto \phi_{(x, 0)}$). For $x \in K$, let us define

$$t_x(y, n) = (y, n - p^{-1}\langle x, y \rangle). \quad (6)$$

The automorphisms t_x of \tilde{K} are called translations. The subgroup of $\text{Aut}(\tilde{K})$ generated by translations is isomorphic to the additive group of K . For any root $(x, m) \in \tilde{\Phi}$, one has,

$$\phi_{(x, m)}^\omega(y, n) = (\phi_x^\omega(y), n - mp^{-1}\langle x, y \rangle) = \phi_{(x, 0)}^\omega \circ t_{\bar{m}x}(y, n).$$

Let us write $\phi_{(x, 0)} = \phi_x$. From the above equation, one has, in particular,

$$t_x = \phi_x^{-1} \circ \phi_{(x, 1)}. \quad (7)$$

From equation (7), it follows that $\phi_a t_x \phi_a^{-1} = t_{\phi_a(x)}$. So $(x, g) \mapsto t_x \circ g$ is an isomorphism from $K \rtimes G$ onto \tilde{G} .

(b) Let G_1 be the subgroup of \tilde{G} generated by $\phi_{r_1}, \dots, \phi_{r_k}$ and $\phi_{(r_0, 1)}$. Then $G \subseteq G_1$. Let $x = gr_0$ be a root in the G -orbit of r_0 . Then $t_x = \phi_x^{-1} \phi_{(x, 1)} = \phi_x^{-1} g \phi_{(r_0, 1)} g^{-1} \in G_1$. Since the roots in the G -orbit of r_0 span K as a \mathbb{Z} -module, it follows that $t_x \in G_1$ for all $x \in K$. The translations, together with G , generate \tilde{G} . So $\tilde{G} = G_1$.

Part (c) follows from 2.2 since $\{(r_0, 1), (r_j, 0)\}$ and $\{r_0, r_j\}$ have same gram matrix. \square

4.3. Method to get an affine diagram: Lemma 4.2 applies to the root systems $\Phi(G_4)$, $\Phi(G_{25})$ and $\Phi(G_{32})$ and the corresponding lattices $D_4^\mathcal{E}$, $E_6^\mathcal{E}$ and $E_8^\mathcal{E}$. Similar result holds for the root systems $\Phi(G_{33})$ and $\Phi(G_{34})$ and the corresponding lattices $K_{10}^\mathcal{E}$ and $K_{12}^\mathcal{E}$, if one replaces order three reflections by two reflections and p by $-\omega$. In each of these cases, any root can be chosen as r_0 in part (b) of 4.2.

Further modifications are necessary for G_{26} . In this case $p^{-1}K$ is not a subset of K' but there is a sub-lattice $E_6^\mathcal{E} \subseteq K$ such that $p^{-1}E_6^\mathcal{E} \subseteq K'$. Accordingly, the translations t_x ,

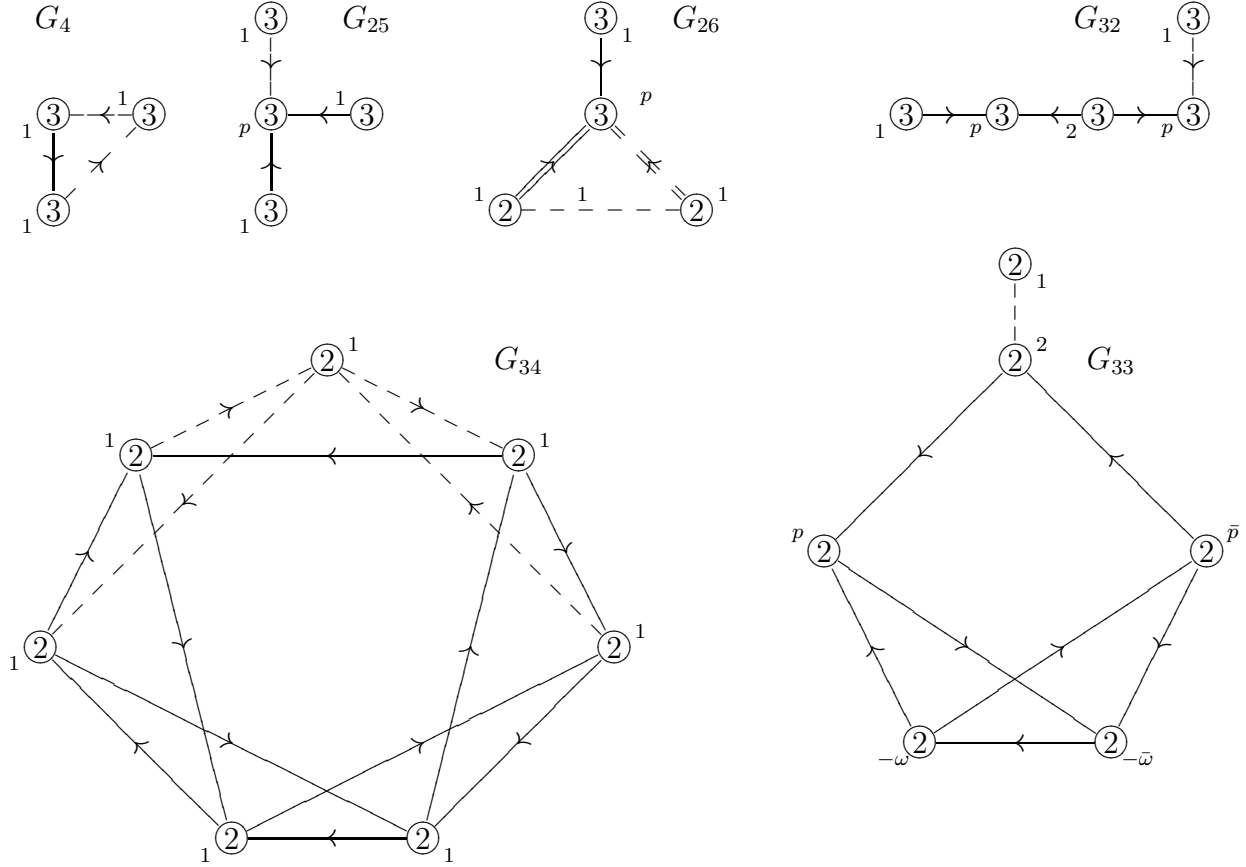


FIGURE 3. Affine diagrams with balanced numbering (shown next to the vertices). The number shown inside the vertex is the norm of the root as well as the order of a reflection in that root. An edge (resp. a double edge) between x and y implies the Coxeter relation $\phi_x\phi_y\phi_x = \phi_y\phi_x\phi_y$ (resp. $\phi_x\phi_y\phi_x\phi_y = \phi_y\phi_x\phi_y\phi_x$). An unmarked directed edge or double edge from y to x means that $\langle x, y \rangle = -p$ for $G_4, G_{25}, G_{26}, G_{32}$ and it means $\langle x, y \rangle = \omega$ for G_{33}, G_{34} .

given in (6), define automorphisms of \tilde{K} only for $x \in E_6^\mathcal{E}$. The conclusion in part (a) is that $\tilde{G} \simeq E_6^\mathcal{E} \rtimes G$. In part (b), any root of an order 3 reflection can be chosen as r_0 . The details are omitted.

Let $G \in \{G_4, G_{25}, G_{26}, G_{32}, G_{33}, G_{34}\}$. We take the diagram for G obtained by method 3.8 and extend it by adding an extra node corresponding to a suitable root, thus obtaining the corresponding affine diagram. These are shown in Figure 3. The extending node is joined with dotted lines. The vertices of an affine diagram of type G correspond to a minimal set of generators for the affine reflection group \tilde{G} . The edges indicate the Coxeter relations among the generators. Additional relations may be needed to obtain a presentation of \tilde{G} (like those given in appendix A.6).

G	$ \Phi_*(G) $	$\Phi_*(G)$	K
G_4	4	$(1, 1, \omega^j), (0, 0, p)$	$D_4^\mathcal{E}$
G_5	4 + 4	$\Phi_*(G_4)$ and $(p, p, 0), (1, 1, -2\omega^j)$	$D_4^\mathcal{E}$
G_{25}	12	$(\overline{p}, 0, 0), (1, \omega^j, \bar{\omega}^k)$	$E_6^\mathcal{E}$
G_{26}	9 + 12	$(\overline{1, -\omega^j, 0})$ and $\Phi_*(G_{25})$	
G_{32}	40	$(0, 1, \omega^j, \bar{\omega}^k), (1, \omega^j, \overrightarrow{-\bar{\omega}^k}, 0)$	$E_8^\mathcal{E}$

TABLE 1.

APPENDIX A. ROOT SYSTEMS FOR SOME UNITARY REFLECTION GROUPS

We describe a root system for each unitary reflection group considered in section 3. Notation: a set of co-ordinates marked with a line (resp. an arrow) above, means that these co-ordinates can be permuted (resp. cyclically permuted).

A.1. $\mathbf{G}_4, \mathbf{G}_5, \mathbf{G}_{25}, \mathbf{G}_{26}, \mathbf{G}_{32}$: For each of these groups, a set of projective roots and the lattices spanned by these roots are given in table 1. The groups G_5, G_{25} and G_{32} are reflection groups of the \mathcal{E} -root lattices $D_4^\mathcal{E}, E_6^\mathcal{E}$ and $E_8^\mathcal{E}$ respectively.

A.2. $\mathbf{G}_{33}, \mathbf{G}_{34}$: These are reflection groups of the \mathcal{E} -lattices $K_{10}^\mathcal{E}$ and $K_{12}^\mathcal{E}$ respectively, where $K_{12}^\mathcal{E}$ is a complex form of the Coxeter–Todd lattice (see example 10a in chapter 7, section 8 of [8]) and $K_{10}^\mathcal{E}$ is the orthogonal complement of any vector of minimal norm in $K_{12}^\mathcal{E}$. The minimal norm vectors of $K_{10}^\mathcal{E}$ and $K_{12}^\mathcal{E}$ form root systems of type G_{33} and G_{34} respectively.

A.3. $\mathbf{G}_8, \mathbf{G}_{29}, \mathbf{G}_{31}$: These are the primitive unitary reflection groups defined over $\mathcal{G} = \mathbb{Z}[i]$. Let $q = 1 + i$.

The reflection group of the two dimensional \mathcal{G} -lattice $D_4^\mathcal{G} = \{(x, y) \in \mathcal{G}^2 : x + y \equiv 0 \pmod{q}\}$ is G_8 . The six projective roots are $(q, 0), (0, q)$ and $(1, i^j)$. The reflection group contains order 4 and order 2 reflections in these roots.

A set of projective roots for G_{29} can be chosen to be

$$(\overline{2, 0, 0, 0}), (1, \overline{\pm 1, i, i}), (1, \overline{\pm 1, -i, -i}), (\overline{q, \pm q, 0, 0}), (1, \overline{\pm 1, i, -i}).$$

There are a total of $4 + 6 + 6 + 12 + 12 = 40$ projective roots. They span the \mathcal{G} -lattice $E_8^\mathcal{G} = \{(x_1, \dots, x_4) \in \mathcal{G}^4 : x_i \equiv x_j \pmod{q}, \sum_i x_i \equiv 0 \pmod{2}\}$ whose real form is E_8 . The minimal norm vectors of $E_8^\mathcal{G}$ form a root system for G_{31} . The projective roots of G_{31} can be chosen to be

$$(1, \pm 1, \pm 1, \pm 1), (1, \overline{\mp 1, \pm 1, \pm 1}), (\overline{q, \pm iq, 0, 0}) \text{ and } \Phi_*(G_{29}).$$

There are $2 + 6 + 12 + 40 = 60$ projective roots.

A.4. $\mathbf{G}_6, \mathbf{G}_9$: Let $\zeta_m = e^{2\pi i/m}$. The projective roots of G_6 (defined over $\mathbb{Z}[\zeta_{12}]$) can be chosen to be

$$\{(\overline{2, 0}), (\pm \zeta_{12} q, q), (\pm \omega q, q)\} \text{ and } \{(\pm 1, 1 + \zeta_{12}), (1 + \bar{\zeta}_{12}, \pm i)\}.$$

The group G_6 contains order 2 reflections in the six projective roots of norm 4 and order 3 reflections in the four projective roots of norm $(3 + \sqrt{3})$.

The projective roots of G_9 (defined over $\mathbb{Z}[\zeta_8]$) can be chosen to be

$$\{(\overline{2}, 0), (\pm 1 \pm i\sqrt{2}, 1), (1, \pm 1)\} \text{ and } \{(\pm 1, 1 + \sqrt{2}), (1 + \zeta_8, \pm i(1 + \zeta_8))\}.$$

There are 12 projective roots of norm 4 and six of norm $4 + 2\sqrt{2}$. The group G_9 contains order 2 reflections in all the roots and order 4 reflections in the roots of norm $(4 + 2\sqrt{2})$.

A.5. $\mathbf{G}(\mathbf{de}, \mathbf{e}, \mathbf{n})$: Let $\zeta_m = e^{2\pi i/m}$ and let e_j be the j -th unit vector in \mathbb{C}^n . The projective roots can be chosen to be $\{e_j, e_j - \zeta_{de}^t e_k : 1 \leq j < k \leq n, 1 \leq t \leq de\}$. For a detailed study of these groups, see [5].

A.6. The diagrams and presentations for G_{29} , G_{31} , G_{33} and G_{34} . The presentations given here were found by coset enumeration for which we used MAGMA. One reason for including these presentations is to observe the following fact: each extra relation needed beyond Coxeter's relations correspond to "deflating" a triangle or a square in the diagram in the sense of Conway [7]. If x, y, z are three vertices in a diagram forming a triangle, then let $\text{def}(x, y, z)$ be the "deflation" relation $xyzy = yzyx$. The Coxeter group of a triangle is affine. Adding the deflation relation $\text{def}(x, y, z)$ reduces it to a finite group.

Let $q = 1 + i$. Then a gram matrix for the diagram of G_{31} (obtained according to our method described in 3.8) is given by

$$2. \begin{pmatrix} 2 & \bar{q} & 1 & \bar{q} & 1 \\ & 2 & 1 & 1 & q \\ & & 2 & \bar{q} & q \\ & & & 2 & q \\ & & & & 2 \end{pmatrix}$$

The Coxeter relations implied by this gram matrix are $\phi_x \phi_y \phi_x = \phi_y \phi_x \phi_y$ if $\langle x, y \rangle = 2$ and $\phi_x \phi_y \phi_x \phi_y = \phi_y \phi_x \phi_y \phi_x$ if $|\langle x, y \rangle| = 2\sqrt{2}$. Let a_1, \dots, a_5 be the simple reflections forming this diagram for G_{31} . Adding the relations

$$\text{def}(a_2, a_4, a_1), \text{def}(a_1, a_5, a_2), \text{def}(a_1, a_3, a_4), \text{def}(a_2, a_4, a_5), a_3 a_4 a_5 = a_4 a_5 a_3 \\ \text{and } a_1 a_2 a_3 a_2 a_1 a_2 = a_2 a_3 a_2 a_1 a_2 a_3$$

to the Coxeter relations, yields a presentation of G_{31} . Omitting the generator a_5 one obtains the diagram for G_{29} (found by method 3.8). Omitting a_5 and the relations involving a_5 , one obtains the corresponding presentation for G_{29} .

The diagrams for G_{33} and G_{34} are described in figure 3. Starting from top left and going counter-clockwise, let us denote the simple reflections of the diagram for G_{34} by a_1, \dots, a_6 . A presentation of G_{34} is obtained by adding the relations $\text{def}(a_i, a_{i+1}, a_{i+2})$ for $i = 1, 2, 3, 4$ and

$$a_1 a_3 a_5 a_6 a_5 a_3 a_1 a_3 a_5 = a_3 a_5 a_6 a_5 a_3 a_1 a_3 a_5 a_6$$

to the Coxeter relations. A presentation for G_{33} is obtained by deleting the generator a_2 and the relations involving a_2 .

REFERENCES

- [1] D. Allcock, *on the Y_{555} complex reflection group*, Preprint, 2008, arXiv:0802.1082.
- [2] T. Basak, *Complex reflection groups and Dynkin diagrams*, U.C. Berkeley, PhD thesis, 2006.
- [3] T. Basak, *Complex Lorentzian Leech lattice and bimonster*, Journal of Algebra 309 (2007) 32-56.
- [4] D. Bessis, *Finite complex reflection arrangements are $K(\pi, 1)$* , Preprint, arXiv:math.GT/0610777.
- [5] M. Broue, G. Malle, R. Rouquier, *Complex reflection groups, braid groups, Hecke algebras*, J. reine angew. Math. 500 (1998), 127-190.
- [6] A. M. Cohen, *Finite complex reflection groups*, Ann. Scient. Ec. Norm. Sup. 9 (1976), 379-436.
- [7] J. H. Conway, C. S. Simons, *26 implies the bimonster*, J. Alg. 235 (2001), 805-814.
- [8] J. H. Conway, N. Sloane, *Sphere Packings, Lattices and Groups 3rd ed.*. Springer-Verlag (1998)
- [9] H. S. M. Coxeter, *Finite groups generated by unitary reflections*, Abh. math. Sem. Univ. Hamburg 31 (1967), 125-135.
- [10] M. C. Hughes, *Complex reflection groups*, Communications in algebra, 18(12), 3999-4029 (1990)
- [11] P. Orlik, L. Solomon, *Unitary reflection groups and cohomology* Invent. Math. 59 (1980) 77-94
- [12] G. C. Shephard, J. A. Todd, *Finite unitary reflection groups* Canad. J. Math. 6 (1954), 274-304.