SOLUTION TO 4.5, PROBLEM 60

60. The cost of erecting an office building is \$1,000,000 for the first story, \$1,100,000 for the second, \$1,200,000 for the third, and so on. Other expenses are \$5,000,000. Assume that the net annual income is \$200,000 per story. How many stories will provide the greatest return on investment?

Solution 1. Unfortunately, this questions is worded abysmally. Greatest return on investment, in light of our discussion on Monday, is most naturally interpreted as profit, which we know is total revenue minus total cost. Let's solve the problem using this interpretation:

First let's compute our cost function in terms of n, the number of stories we build. If we build n stories, our cost per floor is given by $1,000,000 + 1,100,000 + \cdots + (1,000,000+100,000n) = 1,000,000n+100,000(1+2+\cdots+n-1) = 1,000,000n+ \frac{(n-1)(n)}{2}100,000 =$. Thus, our total cost, once we factor in the additional \$5,000,000 fixed cost, is given by $C(n) = 1,000,000n+50,000(n^2-n)+5,000,000 = 50,000n^2+ 950,000n + 5,000,000$.

Our revenue is provided in the problem as \$200,000 per story per year. Thus, if we want to express our total revenue in dollars as a function of n, we have to have an expression for how many years we're maximizing over. Since this information isn't provided in the problem as concrete data, call it y for years, and imagine that it's a constant. Our total revenue is then given by $R(n) = 200,000yn$ and is a function of n.

Profit is then given by the function $P(n) = R(n) - C(n) = 200,000yn - (50,000n^2+950,000n+5,000,000) = -50,000n^2+(200,000y-950,000)n-5,000,000$. To find a candidate for the number of stories that maximizes $P(n)$, compute $P'(n)$ and solve for $P'(n) = 0$. The answer you get will involve the value y, and it will in fact be $n = \frac{4y-19}{5}$. This definitely provides a local maximum by the Second Derivative Test, since $P''(n) = -250,000 < 0$ for all n. Notice that we will only start making a profit after 4 years and 9 months have passed. This, to my mind, is a perfectly correct solution to the problem as stated. □

Solution 2. The thing that makes us nervous about Solution 1 is the fact that nowhere in the problem was time expressed as a function of n; *i.e.,* we were not given a concrete value for y. This suggests that the text authors were actually fishing for a different interpretation of the unclear expression "greatest return on investment." An alternative way of parsing this expression that yields a function that is independent of how many years we own the building is to maximize the ratio dollars made per every dollar spent, or the function $E(n) = R(n)/C(n)$.

If we do this, our computation of the initial cost remains the same; however, we now only care about the upfront initial receipts we expect to collect. Our revenue upfront consists actually solely of the price that we set on buying a condo in our building, which is $R(n) = 200,000n$. (You can interpret this as a price function, if you so choose.) Thus, $E(n) = \frac{200{,}000n}{50{,}000n^2+950{,}000n+5{,}000{,}000} = \frac{4n}{n^2+19n+100}$. Computing via the quotient rule, $E'(n) = \frac{-n^2+100}{(n^2+19n+100)^2}$ and setting it equal to 0 yields $n = 10$. Here, the First Derivative Test is easier to apply, since for $n = 9$, $E'(n) > 0$ and for $n = 11$, $E'(n) < 0$. Since the derivative changes from positive to negative at $n = 10$, this is indeed a local maximum. □

Remark. The text authors do get credit for preferring the interpretation given in Solution 2, since we usually think of "returns" as unitless quantities in arenas such as gambling. We speak of the returns on a particular roulette game being 3:1, for example. Note that our function $E(n)$ is indeed unitless, since $R(n)$ and $C(n)$ are both expressed in dollars. That said, I would accept either solution as correct (and had instructed the grader to do the same before tossing the problem out).

Remark. When *I* write problems, I will be absolutely certain to state a well-posed question using the precise language and vocabulary that we developed together in class.