Sheet 11: Cauchy Sequences

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We will continue to discuss sequences, introducing yet another way to characterize convergence.

Definition 1 (Cauchy sequence)**.** We say that a sequence (a_n) is a *Cauchy* sequence if for each $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that if $n, m \geq N$, then $|a_n - a_m| < \epsilon$.

As an interesting note, an alternative way to construct the reals from $\mathbb{Q}$ is to let $\mathbb{R}$ be the set of all Cauchy sequences of rational numbers, up to a certain equivalence relation. Addition and multiplication of such sequences is well-defined, and you can verify the ordered field axioms and the least upper bound axiom.

Lemma 2. Every convergent sequence is Cauchy.

Lemma 3. Every Cauchy sequence is bounded.

Exercise 4. Let (a_n) be a sequence and let (b_k) be a subsequence that converges to a. Then (a_n) does not necessarily converge to a.

Lemma 5. Let (a_n) be a Cauchy sequence and let (b_k) be a subsequence. Prove that if (b_k) converges to a, then so does (a_n).

Theorem 6 (Cauchy convergence theorem)**.** A sequence of real numbers is convergent if and only if it is Cauchy.

Lemma 7. A set $A \subset \mathbb{R}$ is closed if and only if any Cauchy sequence of real numbers contained in A converges to some element of A.

Theorem 8. A set $A \subset \mathbb{R}$ is compact if and only if any sequence in A has a convergent subsequence that converges to some element of A.

Here we introduce another definition that provides yet one more way to determine whether or not a sequence converges.

Definition 9 (Lim sup)**.** Let (a_n) be a bounded sequence, and let A be the set of its accumulation points. Define the *limit superior*, denoted either $\limsup_{n\to\infty} a_n$ or $\overline{\lim_{n\to\infty}}\, a_n$, to be the last point of A.

Exercise 10. Define the *limit inferior* of a bounded sequence (a_n), denoted $\liminf_{n\to\infty} a_n$ or $\underline{\lim_{n\to\infty}}\, a_n$, and prove that

$$\underline{\lim_{n\to\infty}}\, a_n \leq \overline{\lim_{n\to\infty}}\, a_n.$$

Theorem 11. Let (a_n) be a bounded sequence. Prove that $\underline{\lim_{n\to\infty}}\, a_n = \overline{\lim_{n\to\infty}}\, a_n$ if and only if (a_n) is convergent, in which case

$$\lim_{n\to\infty} a_n = \underline{\lim_{n\to\infty}}\, a_n = \overline{\lim_{n\to\infty}}\, a_n.$$