GENUS TWO VEECH SURFACES ARISING FROM GENERAL QUADRATIC DIFFERENTIALS

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Abstract. We study Veech surfaces of genus 2 arising from quadratic differentials that are not squares of abelian differentials. We prove that all such surfaces of type $(2,2)$ and $(2,1,1)$ are arithmetic. In $(1,1,1,1)$ case, we reduce the question to abelian differentials of type $(2,2)$ on hyperelliptic genus 3 surfaces with singularities at Weierstrass points, and we give an example of a non-arithmetic Veech surface.

1. Introduction

A Veech surface is a Riemann surface with a quadratic differential, such that the derivatives of its affine deformations with respect to the flat structure induced by the quadratic differential, form a lattice in $PSL_2(\mathbb{R})$. Classification of Veech surfaces in genus 2 has been the subject of several recent works by McMullen ([8], [9], [10]) and Calta ([1]). These works classify all genus 2 Veech surfaces given by quadratic differentials that are squares of abelian differentials. In this paper, we attempt to classify genus 2 Veech surfaces given by a general quadratic differential.

Given a quadratic differential on a Riemann surface one can construct a double cover on which this quadratic differential pulls back to a square of an abelian differential. This provides a way to reduce the study of quadratic differentials to the study of abelian differentials on Riemann surfaces of higher genus. We extend this construction, and use it to describe genus 2 Veech surfaces given by general quadratic differentials. We note that a similar construction was recently obtained by McMullen in [11].

We prove that all Veech surfaces of genus 2 given by quadratic differentials with two double zeroes that are not squares of abelian differentials arise from tori:

Theorem 1.1. *All Veech surfaces in* $\mathcal{Q}M_2^-(2,2)$ *are arithmetic.*

Similarly all Veech surfaces of genus 2 with one double zero and two simple zeroes arise from tori:

Theorem 1.2. *All Veech surfaces in* $\mathcal{Q}M_2(2,1,1)$ *are arithmetic.*

In the case of Veech surfaces of genus 2 with four simple zeroes, we reduce the question to abelian differentials on genus 3 Veech surfaces:

Theorem 1.3. *There is a one-to-one correspondence between Veech surfaces of genus 2 in* $\mathcal{Q}M_2(1,1,1,1)$ *and hyperelliptic Veech surfaces of genus 3 in* $\Omega M_3(2,2)$ *with singularities at Weierstrass points.*

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2. Background

2.1. **Translation structures.**

A *translation structure* on a Riemann surface X is an atlas of coordinate charts $\{(U_i, \varphi_i \colon U_i \to \mathbb{C})\}$ covering X except maybe for some finite set of points $\{S_1, S_2, \dots S_k\}$, such that all transition maps $\varphi_j \circ \varphi_i^{-1} \colon \mathbb{C} \to \mathbb{C}$ are translations $z \mapsto z + a$. Weakening this condition to allow all maps of the form $z \mapsto \pm z + a$, we obtain a *half-translation structure* on X. Since these transition maps preserve families of parallel lines in $\mathbb{C}$, we obtain foliations F_φ of X for every direction φ. In the case of a translation structure, these foliations are orientable. These foliations will possibly have singularities at points $S_1, S_2, \dots, S_n$. By counting the number of prongs and multiplying it by π we can assign cone angles to singularities.

An important class of translation surfaces arises from polygonal billiard tables in which each angle is a rational multiple of π. Such a billiard table defines a translation surface via an unfolding construction ([7]).

2.2. **Quadratic differentials.**

There is another way to describe (half-)translation structures. In the case of a translation structure, differentials dz in each chart paste together to give a holomorphic differential ω on $X \backslash \{S_1, S_2, \dots, S_n\}$. This differential can be extended to X: a cone singularity of angle $2k\pi$ will give rise to a zero of ω of order $k-1$. In the case of a half-translation structure, quadratic differentials dz^2 in each chart paste together to give a (possibly meromorphic) quadratic differential q on X with zeroes of order $c-2$ at each cone singularity of angle $c\pi$. If $c = 1$ then q will have a pole of order 1.

Conversely, an abelian differential ω on X defines a translation structure by considering charts in which ω is given by dz. Similarly, a meromorphic quadratic differential with poles of order not larger than 1 defines a half-translation structure on X. We would like to point out that in this paper, a quadratic differential will mean a holomorphic quadratic differential, unless it is specified otherwise.

Denote by ΩM_g (resp. $\mathcal{Q}M_g$) the moduli space of genus g Riemann surfaces with a choice of an abelian (resp. quadratic) differential. These moduli spaces are further stratified by the orders of zeroes of the corresponding differentials. These strata will be denoted by $\Omega M_g(\varepsilon_1, \dots, \varepsilon_n)$ and $\mathcal{Q}M_g(\varepsilon_1, \dots, \varepsilon_n)$, where $\varepsilon_1, \dots, \varepsilon_n$ are the orders of zeroes. We will think of ΩM_g as a sub-space of $\mathcal{Q}M_g$ via $(X, \omega) \mapsto (X, \omega^2)$. We will also use the notation $\mathcal{Q}M_g^- = \mathcal{Q}M_g \backslash \Omega M_g$ for the moduli space of quadratic differentials that are not squares of abelian differentials.

Every element of $(X, q) \in \mathcal{Q}M_g$ can be thought of as a Riemann surface X with a half-translation structure (U_i, φ_i) with no cone singularities of angle π. An element $A \in PSL_2(\mathbb{R})$ acts on (X, q), by changing each coordinate map φ_i to $A \circ \varphi_i$. This action preserves ΩM_g and the stratifications by the orders of zeroes.

Quadratic differentials on a Riemann surface can be naturally thought of as elements of the co-tangent space to the surface in the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g. Using the Teichmüller metric on $\mathcal{M}_g$ co-tangent space is identified with the tangent space. This way a quadratic differential gives rise to a tangent vector to the surface in the moduli space. The projection of $PSL_2(\mathbb{R})$ orbit of $(X, q) \in \mathcal{Q}M_g$ to $\mathcal{M}_g$ is precisely the complex geodesic through X with respect to the Teichmüller metric in the direction given by q.

2.3. **Veech surfaces.** An affine group $Aff^+(X, q)$ of $(X, q) \in \mathcal{Q}M_g$ is the group of all orientation-preserving diffeomorphisms $X \to X$ that are given by affine maps in each chart of the half-translation structure. The linear parts of these affine maps are the same up to multiplication by $\pm Id$. Moreover, their determinant is 1 because the total surface area (in the metric defined by q) is preserved. Hence we get a well defined map $D\colon Aff^+(X, q) \to PSL_2(\mathbb{R})$. The image of this map is denoted by $SL(X, q)$ and is called the *Veech group* of (X, q) (strictly speaking it should be denoted by $PSL(X, q)$, but we will allow ourselves an abuse of notation here). We have an exact sequence

$$0 \longrightarrow Aut(X, q) \longrightarrow Aff^+(X, q) \xrightarrow{D} SL(X, q) \to 0,$$

where $Aut(X, q)$ is the group of all holomorphic automorphisms of X preserving quadratic differential q.

In the case in which q is a square of an abelian differential ω, the map D is a well-defined map to $SL_2(\mathbb{R})$ and the group $SL(X, \omega) = SL(X, q)$ is a subgroup of $SL_2(\mathbb{R})$.

Surface (X, q) is called a *Veech surface* if $SL(X, q)$ is a lattice in $PSL_2(\mathbb{R})$, that is it is a discrete subgroup such that the quotient $PSL_2(\mathbb{R})/SL(X, q)$ has finite (hyperbolic) volume. (X, q) is a Veech surface if and only if its $PSL_2(\mathbb{R})$ orbit in $\mathcal{Q}M_g$ is closed (this was proved by Smillie, see [14] for a sketch of the proof). Veech surfaces satisfy the so called Veech dichotomy: geodesic flow in every direction is either periodic or uniquely ergodic ([13]).

A Veech surface (X, q) is called *primitive* if it cannot be realized as a branched cover over (X', q'), where the genus of X' is lower than the genus of X and the quadratic differential q' pulls back to q under the covering map. If (X, q) is a Veech surface that is not primitive, then the corresponding surface (X', q') is also Veech (see Theorem 3.3), and hence (X, q) can be constructed as a branched cover of a Veech surface of lower genus.

A Veech surface is called *arithmetic* if its Veech group $SL(X, q)$ is commensurable to $SL_2(\mathbb{Z})$. As was proved by Gutkin and Judge, $(X, \omega) \in \Omega M_g$ is arithmetic if and only if it is a branched cover of a torus ([4, Theorem 5.5]). Similarly, $(X, q) \in \mathcal{Q}M_g$ is arithmetic if and only if the double cover given by q (see section 3.1) is a branched cover of a torus.

2.4. **Genus 2 Veech surfaces.** Recent works by McMullen ([9]) and Calta ([1]) classify all primitive Veech surfaces of genus 2 arising from abelian differentials with a double zero. Up to the action of $SL_2(\mathbb{R})$ all such surfaces are obtained from an L-shaped billiard table of specific dimensions as illustrated in Figure 1 ([9]).

Due to McMullen ([10]), in $\Omega M_2(1, 1)$ there is a unique primitive Veech surface up to the action of $SL_2(\mathbb{R})$. It is obtained by gluing the opposite sides of the regular

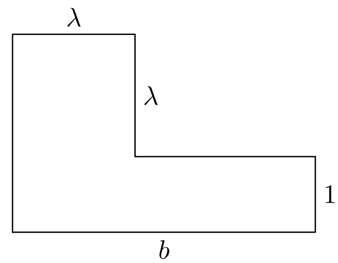

Figure 1. An L-shaped billiard table determines a Veech surface, provided that $b \in \mathbb{Z}$, $\lambda = (e+\sqrt{e^2+4b})/2$, $e = -1, 0$ or 1, $e+1 < b$, and if $e = 1$ then b is even.

decagon. McMullen's argument relies on a result about torsion divisors proved by Moller ([12, Cor. 3.4]).

2.5. **Hyperelliptic surfaces.** One of the main tools in our study of genus 2 Veech surfaces is the hyperelliptic involution. We would like to remind the reader of the main facts about hyperelliptic surfaces that we will need later in the paper (for proofs see [2, Section III.7]).

A Riemann genus g surface X is hyperelliptic if there exists a two sheeted covering $h\colon X \to \mathbb{CP}^1$. The corresponding sheet interchanging involution $i_h\colon X \to X$ is called the hyperelliptic involution. If genus of X is at least 2, then the hyperelliptic involution is uniquely defined and does not depend on the choice of h. Every surface of genus ≤ 2 is hyperelliptic. For hyperelliptic surfaces, the Weierstrass points are the points fixed under the hyperelliptic involution.

One can obtain every hyperelliptic surface X by starting with an affine curve

$$w^2 = (z-z_1)(z-z_2)\dots(z-z_{2g+2})$$

where all z_j's are distinct, and adding two points lying over $z = \infty$ by considering a second chart

$$z' = \frac{1}{z} \quad \text{and} \quad w' = \frac{w}{z^{g+1}}.$$

The hyperelliptic involution is given by $(z, w) \mapsto (z, -w)$. The Weierstrass points are $(z_1, 0), \dots, (z_{2g+2}, 0)$.

In coordinates (z, w) every holomorphic abelian differential $\omega \in \Omega(X)$ is given by

$$\frac{P(z)\,dz}{w}, \quad P(z) \in \mathbb{C}[z],\ deg(P) \leq g-1.$$

From this it is evident that $i_h(\omega) = -\omega$ for all $\omega \in \Omega(X)$. It can also be shown that the zero divisor of $\omega \in \Omega(X)$ is fixed by the hyperelliptic involution and the order of zero at each Weierstrass point is even. Conversely, any such divisor of degree g is a zero divisor of some holomorphic abelian differential.

The products of the holomorphic abelian differentials (taken two at a time) form a $(2g-1)$-dimensional subspace of the $(3g-3)$-dimensional space of quadratic differentials ([2, p. 104, corollary 2]). Since $2g-1 = 3g-3$ for $g = 2$, every quadratic differential on a genus 2 surface can be written as a product of two abelian differentials. In particular, every quadratic differential on a genus 2 surface is fixed by the hyperelliptic involution.

2.6. **Strata of quadratic differentials in genus 2.** Suppose (X, q) is a genus 2 surface with a quadratic differential q that is not a square of an abelian differential. The sum of orders of zeroes of q on X is 4. Therefore there are several possible cases for zero configuration: (4), $(3, 1)$, $(2, 2)$, $(2, 1, 1)$ and $(1, 1, 1, 1)$.

A single zero of order 4 is only possible if q is a square of an abelian differential. Indeed, q is fixed by the hyperelliptic involution, and hence the only zero of q will have to be a Weierstrass point. Therefore there is an abelian differential with a double zero at this point. The square of this abelian differential will give a quadratic differential with the same zero divisor as q, which means that it is proportional to q. Therefore $\mathcal{Q}M_2(4) = \Omega M_2(2)$.

The zero configuration $(3, 1)$ is impossible. If q was such a quadratic differential, then the hyperelliptic involution would have to fix the zero of order 1, and would permute the three horizontal directions coming out of this zero. A permutation of order 2 on three elements has to fix one of the elements, so the hyperelliptic involution would have to fix one of the horizontal directions and would have to have infinitely many fixed points, which is impossible. Therefore $\mathcal{Q}M_2(3, 1) = \varnothing$.

The goal of this paper is to study Veech surfaces in the strata $\mathcal{Q}M_2^-(2, 2)$, $\mathcal{Q}M_2(2, 1, 1)$ and $\mathcal{Q}M_2(1, 1, 1, 1)$.

3. Reduction to abelian differentials

3.1. Main Construction.

Notation 3.1. If $p\colon R \to S$ is a (ramified) double covering of Riemann surfaces, the corresponding sheet-interchanging involution on R will be denoted by i_p.

Let X be a hyperelliptic surface and q be a quadratic meromorphic differential on X with poles of order at most 1, which is not a square of an abelian differential. Assume furthermore that q is fixed by the hyperelliptic involution (this is automatically satisfied if X is a genus 2 surface and q is a holomorphic quadratic differential; see section 2.5).

Let $f\colon Y \to X$ be the double covering given by q, i.e. $f^*(q)$ is a square of an abelian differential $\alpha \in \Omega(Y)$ (cf. [6, p. 519]). The covering map f is branched over the odd-order zeroes and simple poles of q. It follows from Lemma 3.6 below that α is a holomorphic abelian differential even if q has some simple poles. The corresponding sheet-interchanging involution $i_f\colon Y \to Y$ sends α to $-\alpha$.

Since X is hyperelliptic, we have a ramified double covering $h_X\colon X \to \mathbb{CP}^1$. The involution $i_{h_X}\colon X \to X$ is the hyperelliptic involution.

One can think of Y as a set of pairs $(x, \tilde{\alpha})$, where x is a point of X and $\tilde{\alpha}$ is a holomorphic form defined locally around x, s.t. $\tilde{\alpha}^2 = q$. The hyperelliptic involution i_{h_X} acts naturally on such pairs by sending $(x, \tilde{\alpha})$ to $(i_{h_X}(x), i_{h_X *}(\tilde{\alpha}))$. Indeed, by our assumption the hyperelliptic involution i_{h_X} preserves q, therefore it maps $\tilde{\alpha}$ to another local square root of q. Hence the hyperelliptic involution on X can be naturally lifted via f to Y to give an involution i_g on Y. Factoring Y by this involution, we obtain a (ramified) double covering $g\colon Y \to Z$.

Lemma 3.1. *Z is hyperelliptic and i_f descends to i_{h_Z}, the hyperelliptic involution on Z (i.e. $g \circ i_f = i_{h_Z} \circ g$).*

Proof. Pick a point $z \in Z$. Using notations above, there are two preimages of z under g: $(x, \tilde{\alpha})$ and $(i_{h_X}(x), i_{h_X *}(\tilde{\alpha}))$ for some $x \in X$. Therefore $f \circ g^{-1}(z)$

consists of two (possibly coinciding) points x and $i_{h_X}(x)$, which are sent to the same point under h_X. Therefore there is a well-defined map $h_Z\colon Z \to \mathbb{CP}^1$ making the following diagram commute:

$$
\begin{array}{ccccc}
 & & Y & & \\
 & \swarrow^{f} & & \searrow^{g} & \\
X & & & & Z \\
 & \searrow_{h_X} & & \swarrow_{h_Z} & \\
 & & \mathbb{CP}^1 & &
\end{array}
\tag{3.1}
$$

It remains to check that h_Z is a (ramified) double covering. Take any point $p \in \mathbb{CP}^1$, s.t. h_X is not ramified at p and p is not an image of a zero of q under h_X. Then $(h_X \circ f)^{-1}(p)$ consists of four points $(x, \tilde{\alpha})$,$(x, -\tilde{\alpha})$,$(i_{h_X}(x), i_{h_X*}(\tilde{\alpha}))$ and $(i_{h_X}(x), -i_{h_X*}(\tilde{\alpha}))$:

$$
\begin{array}{ccc}
(x, -\tilde{\alpha}) & \overset{i_g}{\longleftrightarrow} & (i_{h_X}(x), -i_{h_X*}(\tilde{\alpha})) \\
{\scriptstyle i_f}\updownarrow & & \updownarrow{\scriptstyle i_f} \\
(x, \tilde{\alpha}) & \overset{i_g}{\longleftrightarrow} & (i_{h_X}(x), i_{h_X*}(\tilde{\alpha}))
\end{array}
\tag{3.2}
$$

As indicated on the diagram (3.2), involution i_g interchanges the columns while i_f interchanges the rows. Therefore ${h_Z}^{-1}(p) = (g \circ f^{-1} \circ {h_X}^{-1})(p)$ consists of two points. Moreover, i_f descends to i_{h_Z} on Z. Thus h_Z defines a ramified double covering of $\mathbb{CP}^1$, which proves that Z is hyperelliptic.

□

Consider involutions i_f^* and i_g^* on the space of holomorphic forms $\Omega(Y)$. As it is evident from the diagram (3.2), involutions i_f and i_g commute on Z, and therefore i_f^* and i_g^* are commuting linear involutions on the linear space $\Omega(Y)$. Hence $\Omega(Y)$ decomposes into the sum of 1-dimensional subspaces, on which i_f^* and i_g^* act by multiplication by 1 or -1. If i_f^* fixes a form $\omega \in \Omega(Y)$, then ω descends to a holomorphic form on X. But i_g descends to the hyperelliptic involution on X. Therefore $i_g^*(\omega) = -\omega$. Similarly, if $i_g^*(\omega) = \omega$, then $i_f^*(\omega) = -\omega$. This discussion leads to the following conclusion:

Lemma 3.2. **(a)** $\Omega(Y) = f^*(\Omega(X)) \oplus g^*(\Omega(Z))$.
(b) i_f^* *restricts to* Id *on* $f^*(\Omega(X))$ *and to* $-$Id *on* $g^*(\Omega(Z))$.
(c) i_g^* *restricts to* $-$Id *on* $f^*(\Omega(X))$ *and to* Id *on* $g^*(\Omega(Z))$.

Corollary 3.1. *Genus of* Y *is the sum of genera of* X *and* Z.

Proof. Follows from Lemma 3.2. □

Corollary 3.2. Y *is a hyperelliptic surface with the hyperelliptic involution* i_{h_Y} *given by* $i_f \circ i_g$. *In particular, the hyperelliptic involution of* Y *descends to the hyperelliptic involutions on* X *and* Z.

Proof. By the lemma $(i_f \circ i_g)^*(\omega) = -\omega$ for any holomorphic form ω on Y. The first statement will follow from the following lemma.

Lemma 3.3. *Assume S is a Riemann surface and $i\colon S \to S$ is an involution, such that $i^*(\omega) = -\omega$ for any holomorphic form ω on S. Then S is hyperelliptic, and i is a hyperelliptic involution.*

Proof. Consider $R = S/i$. If R is not $\mathbb{CP}^1$ then there exists a non-zero holomorphic form ω on R. But then the lift of ω to S will be fixed by i, contradicting the hypothesis. □

Since i_f is the sheet-interchanging involution of the covering $f\colon Y \to X$ and i_g descends to i_{h_X} on X, the hyperelliptic involution $i_{h_Y} = i_f \circ i_g$ descends to the hyperelliptic involution i_{h_X}. The same argument shows that i_{h_Y} descends to i_{h_Z} via $g\colon Y \to Z$. □

Since $i_f(\alpha) = -\alpha$, Lemma 3.2 implies that $\alpha \in g^*(\Omega(Z))$. Therefore α descends to an abelian differential ω on Z. This fact is fundamental to the rest of the paper. It has been known before that one can reduce the study of a quadratic differential to a study of an abelian differential on the corresponding double cover. Now, in the case of a hyperelliptic surface, we will be able to reduce this even further to a study of an abelian differential on a surface of genus lower than the genus of the corresponding double cover. We summarize the results of this section in the following theorem:

Theorem 3.1 (Main Construction)**.** *Let X be a hyperelliptic surface with a quadratic meromorphic differential q with poles of order at most 1. Assume q is not a square of an abelian differential. Assume furthermore that q is fixed by the hyperelliptic involution of X (this is automatically satisfied if genus of X is 2 and q is holomorphic). Consider the double covering $f\colon Y \to X$ given by q. The quadratic differential q lifts to a square of an abelian differential α on Y. The hyperelliptic involution of X can be naturally lifted to an involution i_g of Y. Let Z be the factor of Y by the involution i_g. Then Z is a hyperelliptic Riemann surface, and α descends to an abelian differential ω on Z.*

One can also obtain the same translation surface (Z, ω) from (X, q) by following the path $X \xrightarrow{h_X} \mathbb{CP}^1 \xleftarrow{h_Z} Z$ in the diagram (3.1) on page 6:

Theorem 3.2. *The quadratic differential q descends via the map h_X to a (possibly meromorphic) quadratic differential $\check{q}$ on $\mathbb{CP}^1$. Then (Z, ω) can be obtained from the quadratic differential $\check{q}$ on $\mathbb{CP}^1$ via the double cover construction, i.e. $h_Z^*(\check{q}) = \omega^2$.*

Proof. The quadratic differential q is fixed by the hyperelliptic involution h_X, hence it descends to a quadratic differential $\check{q}$ on $\mathbb{CP}^1$. Using the commutative diagram (3.1) on page 6, we see that $g^*(h_Z^*(\check{q})) = f^*(h_X^*(\check{q})) = f^*(q) = \alpha^2 = g^*(\omega^2)$. Since $g\colon Y \to Z$ is a covering map, the map g^* is injective on the spaces of quadratic differentials. Therefore $h_Z^*(\check{q}) = \omega^2$.

□

Remark 3.1. Recently, similar constructions were independently discovered by C. McMullen ([11]) and E. Lanneau.

3.2. **Relation between (X, q) and (Z, ω).** In this section we will assume that q is a holomorphic differential, though the results can be extended to the case in which q is allowed to have simple poles (see Remark 3.2).

To establish a relation between the Veech groups of (X, q) and (Z, ω), we need to relate the Veech groups of the flat surfaces one of which covers the other. This has been done independently by Gutkin and Judge ([4]), and by Vorobets ([15]).

Definition 3.1. Let (R, τ) be a flat surface given by a quadratic differential τ on a Riemann surface R. Let B be a finite subset of R. Denote by (R, τ, B) a flat surface obtained from (R, τ) by considering points in B to be additional singularities. Consequently, $Aff^+(R, \tau, B)$ is the group of all orientation-preserving affine diffeomorphisms of (R, τ) mapping the set $B \cup Z(\tau)$ into itself, where $Z(\tau)$ is the set of zeroes of τ, and $SL(R, \tau, B)$ is the group of derivatives of the diffeomorphisms in $Aff^+(R, \tau, B)$.

Theorem 3.3. *Let $p\colon R \to S$ be a ramified finite covering of Riemann surfaces branched over a finite set B. If τ is a holomorphic quadratic differential on S, then $SL(S, \tau, B)$ and $SL(R, p^*(\tau))$ are commensurate.*

Proof. It was proved independently by Gutkin and Judge ([4, Theorem 4.9]), and by Vorobets ([15, Theorem 5.4]) that in the case of coverings ramified at singularities, the Veech groups of the base and the cover are commensurate. Therefore, $SL(S, \tau, B)$ and $SL(R, p^*(\tau), p^{-1}(B))$ are commensurate (cf. [5, Lemma 3]). It remains to be noticed that $p^*(\tau)$ vanishes at the preimage of each branching point, and therefore $SL(R, p^*(\tau), p^{-1}(B)) = SL(R, p^*(\tau))$.

□

Theorem 3.3 implies that $SL(X, q)$ and $SL(Y, \alpha)$ are commensurate, and hence (X, q) is a Veech surface if and only if (Y, α) is. Indeed, the set B_f of branching points of $f\colon Y \to X$ is the subset of zeroes of q, and therefore $SL(X, q, B_f) = SL(X, q)$. However, the set B_g of branching points of $g\colon Y \to Z$ is not necessarily a subset of zeroes of ω. Applying Theorem 3.3 to $g\colon Y \to Z$, we obtain the main tool for finding hyperelliptic Veech surfaces given by quadratic differentials:

Theorem 3.4. *Assume q is a holomorphic quadratic differential satisfying the hypothesis of Theorem 3.1. Let B_g be the set of branching points of covering map $g\colon Y \to Z$. Then $SL(X, q)$ is commensurate to $SL(Z, \omega, B_g)$. Hence (X, q) is a Veech surface if and only if (Z, ω) is a Veech surface and $SL(Z, \omega, B_g)$ is a finite index subgroup of $SL(Z, \omega)$.*

We will establish in the subsequent sections when $SL(Z, \omega, B_g)$ is a finite index subgroup of $SL(Z, \omega)$ (see Lemma 4.1 and Lemma 5.1.)

Remark 3.2. Theorem 3.4 can be extended to the case in which q is allowed to have simple poles. If P is the set of (simple) poles of q, then the group $SL(X, q)$ is commensurate to $SL(Z, \omega, B_g \cup g(f^{-1}(P)))$.

3.3. **Reconstructing (X, q) from (Z, ω).** We will need to reverse the main construction. For this take any holomorphic differential $\bar{\omega}$ on X. We can lift it to a holomorphic differential $\bar{\alpha}$ on Y. By Lemma 3.2, $i_g(\bar{\alpha}) = -\bar{\alpha}$. Therefore $\bar{\alpha}^2$ descends to a possibly meromorphic quadratic differential $\bar{q}$ on Z with poles of order at most 1 (see Lemma 3.6). Since $\bar{\alpha}$ is fixed by i_f and i_f descends to the hyperelliptic involution of Z (Lemma 3.1), we get that $\bar{q}$ is fixed by the hyperelliptic involution of Z. Hence we can apply the main construction to $(Z, \bar{q})$. Since i_f descends to the hyperelliptic involution on Z, the lift of the hyperelliptic involution of Z to Y as described in section 3.1 is either i_f or $i_f \circ i_g$. We know that this lift has to preserve $\bar{\alpha}$, hence it has to be i_f. Therefore the main construction applied to $(Y, \bar{q})$ will produce $(X, \bar{\omega})$.

In other words, we have shown that there is a one-to-one correspondence between triples $(X, q, \bar{\omega})$ and $(Z, \omega, \bar{q})$, where q and $\bar{q}$ are fixed by the hyperelliptic involutions of X and Z and have poles of order at most 1.

3.4. **Some technical results.**

Lemma 3.4. *Let τ be a quadratic differential on a hyperelliptic surface R. Assume τ is fixed by the hyperelliptic involution i. Then τ cannot have odd-degree zeroes or odd-degree poles at Weierstrass points of R.*

Proof. Assume τ has an odd-degree singularity at a Weierstrass point $P \in R$. We can choose a local complex coordinate z on R, s.t. $z(P) = 0$ and $i(z) = -z$. Locally around P we can write $\tau = Q(z)(dz)^2$. Since $i_*(q) = q$, we see that $Q(z)$ is an even function, and hence cannot have an odd-degree singularity at 0.

□

To apply Theorem 3.4 we need a description of the branching points of $g\colon Y \to Z$ or, equivalently, the fixed points of i_g.

Lemma 3.5. *The set of fixed points of i_g consists of pre-images under f of all zeroes of q of order 2 mod 4 that are also Weierstrass points.*

Proof. Recall that we can think of Y as a set of pairs $(x, \tilde{\alpha})$, where $\tilde{\alpha}$ is one of the two locally defined around $x \in X$ square roots of q. Then i_g sends $(x, \tilde{\alpha})$ to $(i_{h_X}(x), i_{h_X *}(\tilde{\alpha}))$. For this point to be fixed under i_g, we first of all need that $i_{h_X}(x) = x$, which means that x is a Weierstrass point. Then the map i_g permutes the fiber $f^{-1}(x)$. Lemma 3.4 shows that x cannot be an odd-degree zero or a simple pole of q. Hence f is not branched at x, and $f^{-1}(x)$ consists of two points. Choose a local complex coordinate z on X, s.t. $z(x) = 0$ and $i_{h_X}(z) = -z$. Locally around x we can write $q = z^{2k}Q(z)(dz)^2$, where $k \geq 0$ and $Q(0) \neq 0$. Since the hyperelliptic involution i_{h_X} fixes q, and $i^*_{h_X}(q) = z^{2k}Q(-z)(dz)^2$, we should have that $Q(z) = Q(-z)$. Locally around x, q has two square roots $\pm z^k\sqrt{Q(z)}dz$. These square roots are fixed under $i_{h_X}(z) = -z$ if and only if k is odd, i.e. x is a zero of q of order 2 mod 4.

□

The following calculation shows what happens to the singularities of abelian and quadratic differentials when they are pulled back via a double covering.

Lemma 3.6. *Let $p\colon R \to S$ be a double covering, θ is an abelian differential on S and τ is a quadratic differential on S.*
(a) *Outside of the set of branching points each singularity of θ and τ on S gives rise to two singularities of the same order on R.*

If $s \in S$ is a branching point of p, then
(b) $ord_{p^{-1}(s)}\, p^*(\theta) = 2\, ord_s\, \theta + 1$
(c) $ord_{p^{-1}(s)}\, p^*(\tau) = 2\, ord_s\, \tau + 2$

Proof. Part (a) is obvious. To prove (b) we can choose local coordinates around s and $p^{-1}(s)$ in which the map p is given by $w(z) = z^2$. If $\theta = w^k S(w)dw$, $S(0) \neq 0$ and $k = ord_s\, \theta$, then $p^*(\theta) = z^{2k}S(z^2)d(z^2) = z^{2k+1}S(z^2)dz$. This proves part (b). Part (c) is checked similarly. □

4. Veech surfaces in $\mathcal{Q}M_2^-(2,2)$

Let $(X,q) \in \mathcal{Q}M_2^-(2,2)$. Since f is ramified only at odd-order zeroes of q, the covering $f\colon Y \to X$ is unramified. Using Riemann-Hurwitz formula, we obtain that Y is a genus 3 surface. Corollary 3.1 implies that Z is a torus.

Each of the two zeroes of q are Weierstrass points. Otherwise, they would have to be interchanged under the hyperelliptic involution, and therefore there would exist an abelian differential with simple zeroes at both zeroes of q (see section 2.5). The square of this abelian differential would then have the same zero divisor as q, which would imply that it is proportional to q, contradicting our assumption that q is not a square of an abelian differential.

Applying Lemma 3.5, we see that each point in the fibers over the two double zeroes of q is fixed under i_g. The double covering $g\colon Y \to Z$ is ramified at these four points. Lemma 3.6 implies that the abelian differential ω on Z has no zeroes (which is not surprising, since Z is a torus).

Let $B_g \subset Z$ be the 4 branching points of $g\colon Y \to Z$. By Theorem 3.4, $SL(X,q)$ and $SL(Z,\omega,B_g)$ are commensurate. Since $SL(Z,\omega,B_g) \subset SL(Z,\omega)$ and $SL(Z,\omega) \cong SL_2(\mathbb{Z})$, we get the following theorem:

Theorem 4.1. *All Veech surfaces in $\mathcal{Q}M_2^-(2,2)$ are arithmetic.*

To find all Veech surfaces in $\mathcal{Q}M_2^-(2,2)$, according to Theorem 3.4, we need to establish when $SL(Z,\omega,B_g)$ is a finite index subgroup in $SL(Z,\omega) \cong SL_2(\mathbb{Z})$

Definition 4.1. A finite subset S in a torus T is called *rational* if one can identify T with $\mathbb{C}/\Lambda$ for some lattice $\Lambda \subset \mathbb{C}$, so that all points of S have rational coordinates with respect to some (and therefore any) basis of Λ.

Lemma 4.1. *$SL(Z,\omega,B_g)$ is a finite index subgroup of $SL(Z,\omega) \cong SL_2(\mathbb{Z})$ if and only if B_g is a rational subset of Z.*

Proof. For a finite subset $F \subset Z$, denote by $Aff_F^+(Z,\omega)$ the affine diffeomorphisms of (Z,ω) that fix F pointwise, and by $SL_F(Z,\omega)$ the linear parts of these diffeomorphisms. It is clear that $SL_F(Z,\omega)$ is a finite index subgroup of $SL(Z,\omega,F)$. Indeed, a suitable power of a diffeomorphism permuting F will fix F pointwise.

Therefore $SL_{B_g}(Z,\omega)$ is a finite index subgroup of $SL(Z,\omega,B_g)$. Hence it suffices to prove that $SL_{B_g}(Z,\omega)$ is a finite index subgroup of $SL(Z,\omega)$ if and only if B_g is a rational subset of Z.

Pick a point $b \in B_g$ and a lattice $\Lambda \subset \mathbb{C}$, so that $Z = \mathbb{C}/\Lambda$ and $b \mapsto 0+\Lambda$. Under the map $\varphi \mapsto D\varphi$, the group $Aff_{\{b\}}^+(Z,\omega)$ is identified with $SL(Z,\omega) \cong SL_2(\mathbb{Z})$: every orientation-preserving affine map of (Z,ω) that fixes point b can be lifted to a linear map of $\mathbb{C}$ preserving lattice Λ. The group $SL_{B_g}(Z,\omega) = Aff_{B_g}^+(Z,\omega)$ is the pointwise stabilizer of B_g under the action of $SL_2(\mathbb{Z})$ on $\mathbb{C}/\Lambda$. Therefore it is a finite index subgroup of $SL(Z,\omega)$ if and only if every point of B_g has a finite orbit under the action of $SL_2(\mathbb{Z})$, which is equivalent to every point of B_g having rational coordinates with respect to Λ.

□

We have associated to every Veech surface in $\mathcal{Q}M_2^-(2,2)$ a rational four-point subset $B_g \subset Z$. We need to reconstruct the original surface (X,q) from (Z,ω,B_g). As explained in section 3.3, this can be achieved by choosing an abelian differential $\bar{\omega}$ on X. We can do this so that $\bar{\omega}$ has a double zero at one of the zeroes of q

(such $\bar{\omega}$ exists since both zeroes of q are Weierstrass points). Lemma 3.6 implies that the corresponding quadratic differential $\bar{q}$ has simple zeroes at two points of B_g and simple poles at the other two points of B_g. Let P_1^0 and P_2^0 be the zeroes of $\bar{q}$ and P_1^∞ and P_2^∞ be the poles of $\bar{q}$. Note that the involution i_f descends to a hyperelliptic involution i_{h_Z} of Z (Lemma 3.1) such that $i_{h_Z}(P_1^0) = P_2^0$ and $i_{h_Z}(P_1^\infty) = P_2^\infty$. Pick a lattice $\Lambda \subset \mathbb{C}$ so that $Z = \mathbb{C}/\Lambda$ and points of B_g have rational coordinates with respect to Λ. Hyperelliptic involution i_{h_Z} of $\mathbb{C}/\Lambda$ lifts to $z \mapsto -z+a$ on $\mathbb{C}$ for some $a \in \mathbb{C}$. Since i_{h_Z} permutes B_g, a has rational coordinates with respect to Λ. By composing projection $\pi\colon \mathbb{C} \to \mathbb{C}/\Lambda$ with a translation on the torus, we can assume that $a = 0$ and B_g still has rational coordinates with respect to Λ. Since $i_{h_Z}(P_1^0) = P_2^0$, three points P_1^0, P_2^0 and $\pi(0)$ are colinear. Moreover, since coordinates of these three points are rational with respect to Λ, any geodesic through these three points is closed. Since we are interested in describing Veech surfaces up to the action of $SL_2(\mathbb{R})$, we can act by a suitable element from $SL_2(\mathbb{R})$ to map Λ into the standard lattice $\mathbb{Z}^2$, so that the line through the three points P_1^0, P_2^0 and $\pi(0)$ is the image of the imaginary axis. Depending on whether all four points $P_1^0, P_2^0, P_1^\infty, P_2^\infty$ are colinear, we will get two possible configurations illustrated in Figure 2.

Zeroes of $\bar{q}$, as well as poles of $\bar{q}$, are symmetric about all 4 points of order 2 on the torus $\mathbb{C}/\mathbb{Z}^2$.

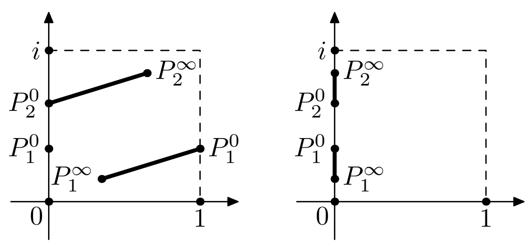

FIGURE 2. Zeroes and poles of quadratic differential $\bar{q}$ on the standard torus $\mathbb{C}/\mathbb{Z}^2$.

Now assume we start with a standard torus $Z = \mathbb{C}/\mathbb{Z}^2$ and four rational points P_1^0, P_2^0, P_1^∞ and P_2^∞ as above. There exists unique (up to multiplication by a scalar) meromorphic differential $\bar{q}$ having simple zeroes at P_1^0, P_2^0 and simple poles at P_1^∞, P_2^∞. Indeed, involution $i_{h_Z}\colon z \mapsto -z$ defines a 2-to-1 map $h_Z\colon Z \to \mathbb{CP}^1$. By composing it with a Moebius transformation on $\mathbb{CP}^1$ we can assume that $h_Z(P_1^0) = h_Z(P_2^0) = 0$ and $h_Z(P_1^\infty) = h_Z(P_2^\infty) = \infty$. Then $\bar{q} = h_Z(z)\,dz^2$. It is clear that $\bar{q}$ is fixed under the hyperelliptic involution $z \mapsto -z$. Hence we can apply the main construction to $(Z, \bar{q})$ in order to describe (X, q).

To describe the covering $g\colon Y \to Z$ we need to see when the monodromy along a closed curve $\gamma\colon [0,1] \to Z$ avoiding branching points $P_1^0, P_2^0, P_1^\infty, P_2^\infty$ is trivial. Since $g\colon Y \to Z$ is the double covering given by $\bar{q} = h_Z(z)\,dz^2$, the monodromy along γ will be trivial if the curve $h_Z \circ \gamma\colon [0,1] \to \mathbb{C}$ has even winding number around zero. If we cut $\mathbb{CP}^1$ from zero to infinity along some simple path σ avoiding branching points of h_Z, then the monodromy of γ is trivial if and only if $h_Z \circ \gamma$ intersects σ even number of times or, equivalently, if γ intersects $h_Z^{-1}(\gamma)$ even number of times (see Figure 3).

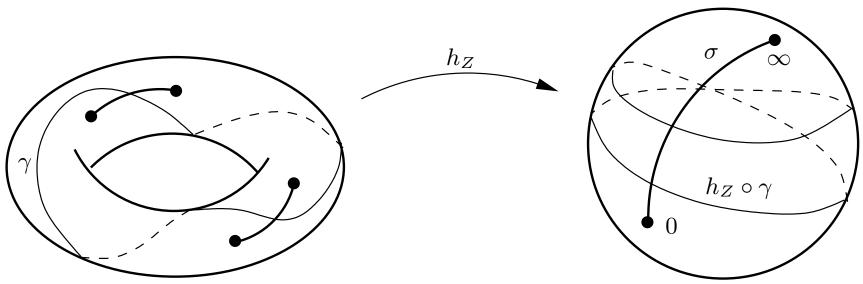

FIGURE 3. Curve γ with trivial monodromy.

We can make parallel cuts on Z as shown in Figure 2. Under h_Z these cuts get identified into one curve connecting zero to infinity.

From the discussion above we see that we can think of Y as two copies of the torus Z glued along the cuts, so that when we cross one of the cuts we go to the other copy of Z: $Y = (Z_1 \sqcup Z_2)/(\partial Z_1 \sim \partial Z_2)$.

Now that we understand the structure of Y, we can explain how to get X from Z. X is obtained from Y by factoring by the involution $i_f \colon Y \to Y$. By Lemma 3.1, i_f descends to the hyperelliptic involution i_{h_Z} on Z. We have also seen that i_f has no fixed points. This implies that i_f has to interchange $Z_1 \backslash \partial Z_1$ and $Z_2 \backslash \partial Z_2$. Indeed, say a point A from $Z_1 \backslash \partial Z_1$ was mapped to another point $B \in Z_1 \backslash \partial Z_1$ symmetrical about the center of the unit square. Then we can connect A to B avoiding the cuts by a path symmetrical about the center of the unit square (the path does not have to be straight). By continuity, along this path i_f will have to equal $i_{h_{Z_1}}$, which will mean that the center of the square is fixed under i_f, contradicting that i_f has no fixed points. Since i_f glues interiors of two copies of Z, we just need to see how it acts on the sides of the cuts. As easily verified, we get the following statement:

Theorem 4.2. *Up to the action of $SL_2(\mathbb{R})$ any Veech surface in $\mathcal{Q}M_2^-(2,2)$ can be obtained by making two non-intersecting (but possibly colinear) parallel cuts of the same length with rationally related vertices on the standard torus $(\mathbb{C}/\mathbb{Z}^2, dz)$ and gluing as indicated in Figure 4.*

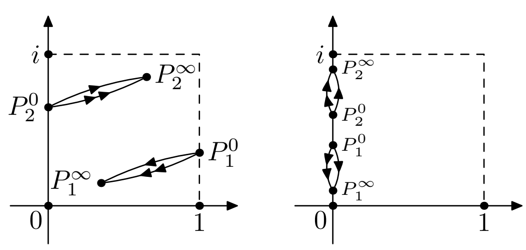

FIGURE 4. Veech surface in $\mathcal{Q}M_2^-(2,2)$. Non-colinear (left) and colinear (right) cuts.

It is easy to check that there will be infinitely many configurations of four points $\{P_1^0, P_2^0, P_1^\infty, P_2^\infty\}$ up to the action of $SL_2(\mathbb{R})$ on the standard torus $\mathbb{C}/\mathbb{Z}^2$. Therefore there are infinitely many Veech surfaces in $\mathcal{Q}M_2^-(2,2)$.

5. Veech surfaces in $\mathcal{Q}M_2(2,1,1)$

Assume quadratic differential q has one double zero and two simple zeroes. Then the covering $f\colon Y \to X$ is branched over two simple zeroes of q. Riemann-Hurwitz formula implies that Y is a genus 4 surface, and hence by Corollary 3.1 Z is a genus 2 surface.

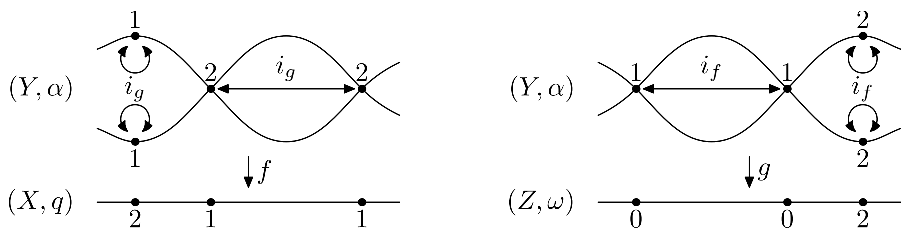

Figure 5. Structure of the covering maps $f\colon Y \to X$ and $g\colon Y \to Z$, and corresponding involutions i_f and i_g, along with the orders of zeroes of the forms q, α and ω.

Hyperelliptic involution h_X preserves q, hence the double zero of q is a Weierstrass point. The simple zeroes of q are conjugate under the hyperelliptic involution. Indeed, if they were fixed under the hyperelliptic involution, then they would have to be Weierstrass points. Since q is a product of two abelian differentials, the order of zero of q at a Weierstrass point has to be even (see section 2.5), therefore q can not have a simple zero at a Weierstrass point.

By Lemma 3.5, i_g has two fixed points on Y, namely the points in the fiber over the double zero of q. Applying Lemma 3.6, we get that abelian differential α on Y has two double zeroes over the simple zeroes of q, and two simple zeroes over the double zero of q. Consequently, abelian differential ω on Z has one double zero.

Let B_g be the set of branching points of $g\colon Y \to Z$. As indicated on the figure above B_g consists of two points. According to Theorem 3.4, (X,q) is Veech if and only if (Z,ω) is Veech and $SL(Z,\omega,B_g)$ is a finite index subgroup of $SL(Z,\omega)$. Assume (X,q) and (Z,ω) are Veech. The following lemma shows that the two points of B_g have to be periodic under the action of $Aff^+(Z,\omega)$.

Lemma 5.1. *Let τ be a quadratic differential on a closed Riemann surface R of genus at least 2, and B be a finite subset of R. Then $SL(R,\tau,B)$ is a finite index subgroup of $SL(R,\tau)$ if and only if B has a finite orbit under the action of $Aff^+(R,\tau)$ or equivalently, that every point of B is periodic under the action of $Aff^+(R,\tau)$.*

Proof. Consider the exact sequence:

$$0 \longrightarrow Aut(R,\tau) \longrightarrow Aff^+(R,\tau) \xrightarrow{D} SL(R,\tau) \to 0$$

Since R is a genus 2 surface, there are finitely many holomorphic automorphisms of R, and therefore $Aut(R,\tau)$ is finite. The group $Aff^+(R,\tau,B)$ is mapped onto $SL(R,\tau,B)$ under D. Using the exact sequence above and that $Aut(R,\tau)$ is finite, we see that $SL(R,\tau,B)$ is a finite index subgroup of $SL(R,\tau)$ if and only if $Aff^+(R,\tau,B)$ is a finite index subgroup of $Aff^+(R,\tau)$. Since B is a finite set,

$Aff_B^+(R,\tau)$, the pointwise stabilizer of B under the action of $Aff^+(R,\tau)$, is a finite index subgroup of $Aff^+(R,\tau,B)$. The group $Aff_B^+(R,\tau)$ has finite index in $Aff^+(R,\tau)$ if and only if B has a finite orbit under the action of the affine group $Aff^+(R,\tau)$. □

Periodic points on translation surfaces have been studied in a paper by Gutkin, Hubert and Schmidt ([3]). They prove that under certain conditions all Weierstrass points on a hyperelliptic surface are periodic. In a more recent paper, Möller showed that the converse is always true in the case of a primitive Veech surface of genus 2 in $\Omega M_2(2)$.

Theorem 5.1 (Möller, [12, Theorem 5.1])**.** *The only periodic points on a primitive Veech surface in $\Omega M_2(2)$ are the Weierstrass points.*

Since i_f descends to the hyperelliptic involution h_Z on Z, two points of B_g are interchanged under h_Z. Hence they are not Weierstrass. This along with the Theorem 5.1 means that (Z,ω) cannot be primitive, thus it has to be a cover of a torus. Since $SL(X,q)$ and $SL(Z,\omega)$ are commensurate, we get the following theorem:

Theorem 5.2. *All Veech surfaces in $\mathcal{Q}M_2(2,1,1)$ are arithmetic.*

6. Veech surfaces in $\mathcal{Q}M_2(1,1,1,1)$

6.1. **General situation.** Assume quadratic differential q has four simple zeroes on X. Then the map $f\colon Y\to X$ is branched over four points. Therefore Y has genus 5. The map $g\colon Y\to Z$ has no branching points, and hence Z has genus 3. The holomorphic form α has four zeroes of order 2 on Y, which are mapped under $g\colon Y\to Z$ into two double zeroes of ω. Both zeroes of ω are Weierstrass points of Z.

Since the covering $g\colon Y\to Z$ is not ramified, Theorem 3.4 implies that (X,q) is Veech if and only if (Z,ω) is Veech. This way we get a map from $\mathcal{Q}M_2(1,1,1,1)$ to $\Omega M_3(2,2)$, which sends Veech surfaces to hyperelliptic Veech surfaces that have singularities at Weierstrass points.

Theorem 6.1. *There is a one-to-one correspondence between Veech surfaces of genus 2 in $\mathcal{Q}M_2(1,1,1,1)$ and hyperelliptic Veech surfaces of genus 3 in $\Omega M_3(2,2)$ with singularities at Weierstrass points.*

Proof. Every quadratic differential on a genus 2 surface is a product of abelian differentials (see section 2.5). Therefore the quadratic differential q is a product of two abelian differentials $\bar\omega_1$ and $\bar\omega_2$ of type $(1,1)$. Take one of them, say $\bar\omega_1$ and apply to it the construction from section 3.3. We will get a quadratic differential $\bar q$ on Z with a zero of order 6 at one of the zeroes of ω and a zero of order 2 at the other zero of ω. If we choose $\bar\omega_2$ instead of $\bar\omega_1$ at the beginning, then the zeroes of $\bar q$ will be switched.

Now suppose we start with any hyperelliptic genus 3 surface Z with an abelian differential ω of type $(2,2)$, where both zeroes are Weierstrass points. There exists an abelian differential that has a zero of order 4 at one of the zeroes of ω. Let $\bar q$ be the product of this differential and ω. Then $\bar q$ is a quadratic differential of type $(6,2)$, and it is the only quadratic differential with such zero divisor up to multiplication by a scalar. Moreover $\bar q$ is not a square of an abelian differential, because an abelian differential on a hyperelliptic surface cannot have a zero of odd order at a Weierstrass point. Furthermore, $\bar{q}$ is fixed by the hyperelliptic involution because it is a product of abelian differentials. Applying ideas from section 3.3 to $(Z, \omega, \bar{q})$ we get a triple $(X, q, \bar{\omega})$. It can be easily verified using results from section 3.1, that X is a genus 2 surface and q is a quadratic differential with four simple zeroes.

□

We will use the following description of a double covering given by a quadratic differential to better understand the correspondence between Veech surfaces in $\mathcal{Q}M_2(1,1,1,1)$ and $\Omega M_3(2,2)$.

Lemma 6.1. *Assume X is a hyperelliptic Riemann surface of genus at least 2 with the hyperelliptic involution $i\colon X \to X$. Let q be a quadratic differential on X with two zeroes of even orders at Weierstrass points P_1 and P_2. Assume additionally that q is not a square of an abelian differential. Let $Y \to X$ be the double covering given by q. Connect P_1 to P_2 by a simple path σ not passing through other Weierstrass points of X. Then Y can be obtained by taking two copies of X, cutting them along a closed loop $\sigma \cup i(\sigma)$ and re-gluing them in the standard way to obtain a double cover of X.*

Proof. Let $P_1, P_2, \ldots, P_{2g+2}$ be all Weierstrass points of X. Let $z\colon X \to \mathbb{CP}^1$ be the double cover defined by the hyperelliptic involution i, chosen so that $z(P_j) \neq \infty$, $j = 1, \ldots, 2g+2$. Consider the function $w = \sqrt{\prod_{j=1}^{2g+2}(z - z(P_j))}$. Then X is the Riemann surface defined in complex coordinates (z, w) by $w^2 = \prod_{j=1}^{2g+2}(z - z(P_j))$.

Assume $deg_{P_1}(q) = 2a$ and $deg_{P_2}(q) = 2b$, $a, b \in \mathbb{Z}$. The total number of zeroes of q is $2a + 2b = 4g - 4$. Therefore $a + b$ is even, and a and b have the same parity. If they were both even, then q would be a square of an abelian differential with the zero divisor $aP_1 + bP_2$. Hence both a and b are odd.

In coordinates (z, w) we can express

$$q = \frac{(z - z(P_1))^a (z - z(P_2))^b (dz)^2}{w^2} = \frac{(z - z(P_1))^{a-1} (z - z(P_2))^{b-1} (dz)^2}{(z - z(P_3)) \ldots (z - z(P_{2g+2}))} \tag{6.1}$$

Take any closed path γ on X. We would like to see when the monodromy of the covering $Y \to X$ along γ is non-trivial, or equivalently, when the extension of $\sqrt{q}$ along γ changes sign. Since a and b are odd, the numerator in (6.1) is a square of a well-defined abelian differential. Therefore the monodromy along γ is non-trivial if and only if the analytical extension of $\sqrt{(z - z(P_3)) \ldots (z - z(P_{2g+2}))}$ along γ changes sign. This will happen if and only if the total winding number of $z(\gamma)$ about points $z(P_3), \ldots, z(P_{2g+2})$ is odd. The winding number of $z(\gamma)$ about the images of all Weierstrass points $z(P_1), \ldots, z(P_{2g+2})$ is always even (this is the reason why the function $w\colon X \to \mathbb{CP}^1$ is well-defined). Therefore the winding number of $z(\gamma)$ about $z(P_3), \ldots, z(P_{2g+2})$ is odd if and only if the winding number of $z(\gamma)$ about $z(P_1)$ and $z(P_2)$ is odd.

We are given a path σ connecting P_1 to P_2. The winding number of $z(\gamma)$ about $z(P_1)$ and $z(P_2)$ is odd if and only if the intersection number of $z(\gamma)$ and $z(\sigma)$ is odd, or equivalently the intersection number of γ and $\sigma \cup i(\sigma)$ is odd. This finishes the proof of the lemma.

□

Consider a genus 3 surface (Z, ω) and a quadratic differential $\bar{q}$ as in the proof of Theorem 6.1. Denote zeroes of ω (and $\bar{q}$) by P_1 and P_2. Connect P_1 to P_2 by a saddle connection σ. Following the construction in Theorem 6.1, we get a genus 5 surface Y which is a double cover of Z given by $\bar{q}$. The pair $(Z, \bar{q})$ satisfies the hypothesis of Lemma 6.1. Therefore Y is obtained by taking two copies of Z, cutting them along $\sigma \cup i_{h_Z}(\sigma)$ and gluing together along the cuts: $Y = (Z_1 \sqcup Z_2)/(\partial Z_1 \sim \partial Z_2)$, where $\partial Z_i = \sigma_i \cup i_{h_{Z_i}}(\sigma_i),\ i = 1, 2$.

Surface (X, q) is obtained by factoring Y by the involution i_f. By Lemma 3.1, i_f descends to the hyperelliptic involution i_{h_Z} on Z. By Lemma 3.5, i_f interchanges the Weierstrass points of Z_1 and Z_2 that are not zeroes of $\bar{q}$. Surfaces Z_1 and Z_2 are connected, because Y is (equivalently, the curve $\sigma \cup i_{h_Z}(\sigma)$ is non-separating). Therefore by continuity, i_f interchanges interiors of Z_1 and Z_2. On the sides of the cuts ∂Z_i, i_f is given by the hyperelliptic involution $i_{h_{Z_i}}$. We have proved the following statement:

Theorem 6.2. *Every Veech surface in $\mathcal{Q}M_2(1,1,1,1)$ can be obtained from a hyperelliptic Veech surface in $\Omega M_3(2,2)$ with singularities at the Weierstrass points in the following way: take a saddle connection joining the two zeroes, act on it with the hyperelliptic involution to obtain a closed loop, then cut the surface along this loop and glue each opening shut via the hyperelliptic involution (see Figure 6).*

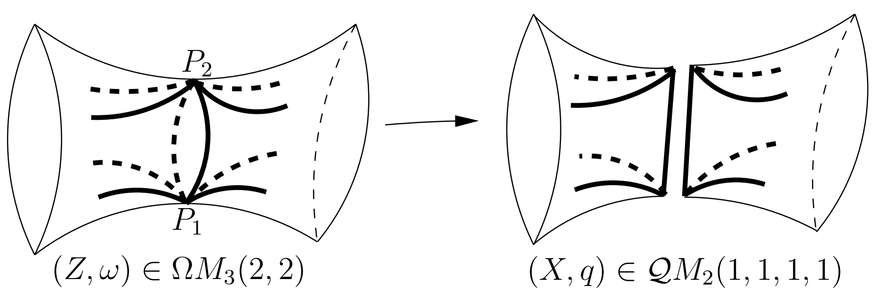

FIGURE 6. Correspondence between Veech surfaces in $\Omega M_3(2,2)$ and $\mathcal{Q}M_2(1,1,1,1)$. Bold lines indicate the foliation in the direction of a saddle connection from P_1 to P_2.

6.2. **Examples.** While the author is not aware of any primitive genus 3 Veech surfaces satisfying conditions of Theorem 6.1, we can construct some non-primitive, non-arithmetic examples by considering an unramified double covering over a primitive genus 2 Veech surface in $\Omega M_2(2)$.

According to [9], all such surfaces arise from L-shaped billiard tables of specific dimensions (see section 2.4). The surface itself is obtained by reflecting an L-shaped table 4 times across its sides until we get a 'Swiss cross', and then gluing the parallel sides. The hyperelliptic involution is given by rotating by π about the center of the cross. The Weierstrass points are shown on Figure 7.

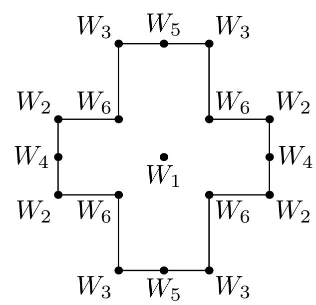

FIGURE 7. Weierstrass points on a Swiss cross surface $(S, \theta) \in \Omega M_2(2)$ arising from an L-shaped billiard table. W_6 is the double zero of θ.

Choose a Veech surface $(S, \theta) \in \Omega M_2(2)$. An unramified double covering $(Z, \omega) \to (S, \theta)$ corresponds to a choice of a quadratic differential τ on S with two double zeroes. We are interested in non-trivial double coverings, therefore τ should not be a square of an abelian differential. This implies that the two zeroes of τ have to occur at the Weierstrass points of S (see the beginning of section 4). To satisfy the hypothesis of Theorem 6.1, we need to have that the abelian differential ω has its zeroes at the Weierstrass points of Z. By Corollary 3.2 and Lemma 3.5, Weierstrass points of Z are the eight points lying over those Weierstrass points of S that are not zeroes of the quadratic differential τ. Therefore τ cannot have one of its zeroes at W_6, which is the zero of θ. Hence zeroes of τ can only occur at the five Weierstrass points $W_1, \ldots, W_5$. Having chosen two such Weierstrass points on S, we use Lemma 6.1 to obtain a concrete description of the surface $(Z, \omega) \in \Omega M_3(2, 2)$. Then we use Theorem 6.2 to obtain a Veech surface $(X, q) \in \mathcal{Q}M_2(1, 1, 1, 1)$.

There are ten different ways to choose two zeroes of τ, hence there are at most ten different Veech surfaces in $\mathcal{Q}M_2(1, 1, 1, 1)$ corresponding to a given Veech surface in $\Omega M_2(1, 1)$.

We will look in detail at the case in which W_1 and W_2 are the zeroes of τ.

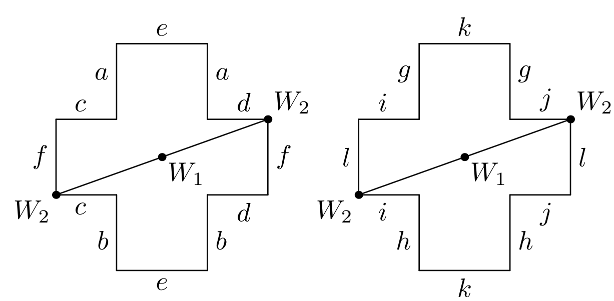

FIGURE 8. Two copies of a Veech surface $(S, \theta) \in \Omega M_2(2)$. Lowercase letters indicate which sides are identified.

Take two copies of a Veech surface $(S, \theta) \in \Omega M_2(2)$ (see Figure 8). Following Lemma 6.1, pick a path from W_1 to W_2 and act on it with the hyperelliptic involution to obtain a closed loop on each surface. Cut the surfaces along these loops and re-glue the surfaces along the loops switching the order to get a non-trivial double cover of S. This results in a genus 3 surface displayed in Figure 9.

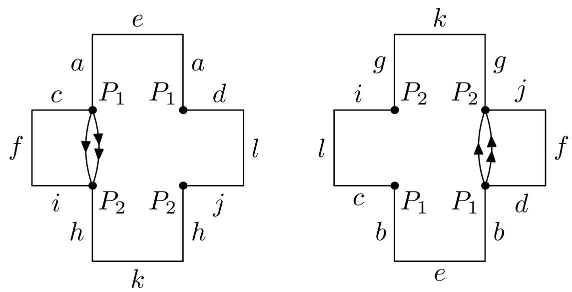

FIGURE 9. Genus 3 surface $(Z, \omega) \in \Omega M_3(2, 2)$.

Points P_1 and P_2 in Figure 9 are the two zeroes of ω. By Corollary 3.2, the hyperelliptic involution on (Z, ω) projects to the hyperelliptic involution on (S, θ). As mentioned above, Weierstrass points of Z are the eight points lying over the four Weierstrass points W_3, W_4, W_5, W_6. Hence the hyperelliptic involution of Z has no fixed points in the interiors of the two Swiss crosses. Therefore the hyperelliptic involution acts on Z by rotating each Swiss cross about its center by π and then switching the sheets. Following Theorem 6.2, pick a path from P_1 to P_2 and act on it with the hyperelliptic involution to obtain a closed loop. The two sides of this loop are indicated in Figure 9 using single and double arrows. We need to cut the surface along this loop and glue each opening shut via the hyperelliptic involution. This corresponds to identifying the segments marked by single arrows and the segments marked by double arrows. The resulting surface is displayed in Figure 10.

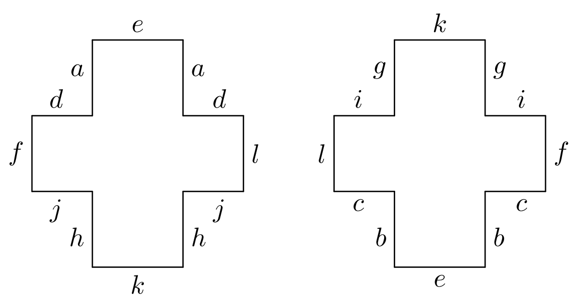

FIGURE 10. Genus 2 Veech surface $(X, q) \in \mathcal{Q}M_2(1, 1, 1, 1)$.

After performing some of the identifications in Figure 10, we will get an H-shaped surface displayed in Figure 11. Provided that we started with a non-arithmetic Veech surface in $\Omega M_2(2)$, Figure 11 is an example of a non-arithmetic Veech surface in $\mathcal{Q}M_2(1, 1, 1, 1)$.

Remark 6.1. The same example was recently constructed in [11].

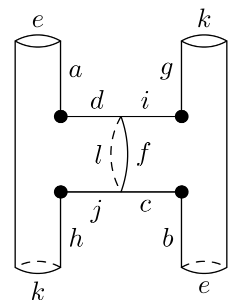

FIGURE 11. Genus 2 Veech surface $(X, q) \in \mathcal{Q}M_2(1,1,1,1)$. Sides marked with e and sides marked with k are identified with a twist by π so that the segments a and b, and g and h are aligned. The four marked points are the zeroes of q.

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