Math 199 Introduction to Analysis and Linear Algebra, Section 43

Homework # 3 , due January 24

Problem 1
Read Chapter 1.5 of *"Foundations of Mathematical Analysis"* by Paul J. Sally.

Problem 2
Let R be a ring.

(1) In class we defined the rules of order as follows: The ring R is *ordered* if there is a relation $S \subset R \times R$, which we denote by $<$ (that means that $(a, b) \in S$ if and only if $a < b$) satisfying the following *rules of order*:
(O1) (Trichotomy) If $a, b \in R$ then one and only one of the following holds
(a) $a < b$, that is $(a, b) \in S$
(b) $b < a$, that is $(b, a) \in S$
(c) $a = b$.
(O2) (Transitivity) If $a, b, c \in R$ with $a < b$ and $b < c$, then $a < c$.
(O3) (Additivity) If $a, b, c \in R$ with $a < b$, then $a + c < b + c$.
(O4) (Multiplication by positive elements) If $a, b, c \in R$ with $a < b$ and $c > 0$, then $ac < bc$.

(2) Compare this to the *rules of order* presented as follows: The ring R is *ordered* if there exists a set $P \subset R$ (the set of positive elements) satisfying the following *rules of order*:
(O1') (Trichotomy) If $a \in R$ then one and only one of the following holds
(a) $a \in P$
(b) $-a \in P$
(c) $a = 0$.
(O2') (Closed under Addition) If $a, b \in P$, then $a + b \in P$.
(O3') (Closed under Multiplication) If $a, b \in P$, then $ab \in P$.

Show that the two definitions of order are equivalent. That is, you have to show that a ring R has a subset P satisfying (O1') - (O3') if and only if R has a relation $S \subset \times R$ satisfying (O1) - (O4).

Problem 3
Show that an ordered commutative ring R with identity satisfies the Cancellation Law:
(C) If $a, b, c \in R$ with $a \neq 0$ and $ab = ac$, then $b = c$.

Problem 4
Let $a, b \in \mathbb{Z}$ and assume that $a > 0$ and $b > 0$. Show that if $b|a$ (that is b divides a), then $b \leq a$.

Bonus Problem - not required
1) Let $a \in \mathbb{Z}$. Show that $2|a$ if and only if $2|a^2$.
2) Let $a \in \mathbb{Z}$. We call a *even* if $2|a$, that is if there exists $n \in \mathbb{Z}$ such that $a = 2n$ and *odd* if there exists $n \in \mathbb{Z}$ such that $a = 2n + 1$. Let a be odd. Show that then $a^2 \equiv 1 \mod 4$.
Problem 5
Let R be an ordered commutative ring with identity. Let $a, b \in R$. Show that $2ab \leq (a^2 + b^2)$.
Problem 6
Remember that we defined the rational numbers $\mathbb{Q}$ on the last homework assignment as follows. Consider the set

$$F := \{(a, b) \,|\, a, b \in \mathbb{Z} \text{ and } b \neq 0\}.$$

Then we let $\mathbb{Q}$ be the equivalence classes in F of the equivalence relation: $(a, b) \sim (c, d)$ if and only if $ad = bc$. We defined the addition and multiplication on $\mathbb{Q}$ by

$$\{(a, b)\} + \{(c, d)\} = \{(ad + bc, bd)\}$$

and

$$\{(a, b)\} \cdot \{(c, d)\} = \{(ac, bd)\}.$$

Now we define an order on $\mathbb{Q}$ by

$$\{(0, 1)\} < \{(a, b)\} \text{ if and only if } 0 < ab.$$

With these operations and this order, $\mathbb{Q}$ is an ordered field. Show the following:

(1) Identify the additive identity and show that the element you identified is indeed the additive identity.
(2) Identify the additive inverse of the element $\{(a, b)\}$ and show that the element you identified is indeed the additive inverse of $\{(a, b)\}$.
(3) Identify the multiplicative identity and show that the element you identified is indeed the multiplicative identity.
(4) Identify the multiplicative inverse of the non-zero element $\{(a, b)\}$ and show that the element you identified is indeed the mutliplicative inverse of $\{(a, b)\}$.
(5) Show that the relation $<$ is well-defined (does not depend on the representative in the equivalence class).
(6) Show that the order given above satisfies the rules of order $(01) - (04)$ (You can also use Problem 2 and show set of positive elements in $\mathbb{Q}$ satisfies the rules $(O1') - (O3')$.)
(7) Show that the Well-Ordering Principle (1.5.12 in the text) does not hold for $\mathbb{Q}$

Bonus Problem - not required
Show that there is no element $x \in \mathbb{Q}$ such that $x^2 = 2$.

Problem 7

Use Mathematical Induction to prove the following statement. Let n be a positive integer (we will write $n \in \mathbb{N}$ for this), then :

$$\sum_{i=1}^{n} (2i - 1) = n^2.$$

(So the sum of the first n odd numbers is always a square). Write the proof very carefully.

Bonus Problem - not required

Use Mathematical Induction to prove the following statements. Let $n \in \mathbb{N}$, then

(1) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.
(2) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.

Problem 8

Remember that we defined the congruence classes mod n, denoted by $\mathbb{Z}_n$, and showed that $\mathbb{Z}_n$ is a commutative ring with identity.

(1) Show that $\mathbb{Z}_n$ is not ordered.
(2) For what values of n is $\mathbb{Z}_n$ a field. Explain your answer.