Math 199 Introduction to Analysis and Linear Algebra, Section 43

Homework # 4 , due January 31

Problem 1
Read Chapter 3.5 about the Construction of the real numbers in *"Foundations of Mathematical Analysis"* by Paul J. Sally.

Problem 2
The Well-Ordering Principle on $\mathbb{Z}$ has an important consequence, which is called the

Division Algorithm: Let $a, b \in \mathbb{Z}$, $b > 0$. Then there are integers $q \in \mathbb{Z}$ (the *quotient*) and $r \in \mathbb{Z}$ (the *remainder*) such that

$$a = qb + r \text{ and } 0 \leq r < b.$$

Moreover q and r are unique.
Prove the Division Algorithm:

(1) First prove that q, r are unique. Show that if there exist integers q_1, q_2, r_1, r_2 such that $a = q_1 b + r_1 = q_2 b + r_2$ and $0 \leq r_1 < b$ and $0 \leq r_2 < b$, then $q_1 = q_2$ and $r_1 = r_2$.
(2) Consider the set

$$S = \{m \in \mathbb{Z} \,|\, m = ak + b \text{ for some } k \in \mathbb{Z}, \text{ and } m > 0\}.$$

By the well ordering principle the set S contains a least element r. Show that r is of the form $r = a - qb$ and $0 \leq r < b$, then conclude the Division Algorithm.

Problem 3
Let $a, n \in \mathbb{Z}$, $n > 0$. By the Division Algorithm we know that there exists unique $q, r \in \mathbb{Z}$ such that $a = qn + r$ and $0 \leq r < n$. Remember the relation "congruence modulo n".

(1) Show that $a \equiv r \mod n$.
(2) Using the Division Algorithm, we see that given $0 \leq c < n$ the congruence class of c modulo n is precisely the set of all integers $a \in \mathbb{Z}$, which by applying the Division Algorithm can be written as $a = qn + r$ with $0 \leq r < n$, with the remainder r being equal to c. List the congruence classes modulo 3 and there elements.
(3) Prove that if $a \equiv b \mod n$ and $c \equiv d \mod n$, then $(a + c) \equiv (b + d) \mod n$ and $(ac) \equiv (bd) \mod n$.

Problem 4
Let $|\cdot| : \mathbb{Q} \to \mathbb{Q}$ be the absolute value. Suppose $x, y \in \mathbb{Q}$.

(1) Prove the triangle inequality: $|x + y| \leq |x| + |y|$.
(2) Prove the following inequality: $|x + y| \geq |x| - |y|$.

Problem 5

Let (a_n), (b_n) be Cauchy sequences in $\mathbb{Q}$. Let $p, q \in \mathbb{Q}$ be two fixed rational numbers. Show that the sequence $(c_n) = (pa_n + qb_n)$ is again a Cauchy sequence in $\mathbb{Q}$. Write the proof carefully.

Problem 6

Prove that the sequence (a_n) defined by $a_n = \frac{1}{n+1}$ is a Cauchy sequence in $\mathbb{Q}$ which converges to $0 \in \mathbb{Q}$.

Problem 7

We defined the real numbers $\mathbb{R}$ in class as follows: Let

$$\mathcal{C} = \{(a_n) \,|\, (a_n) \text{ is a Cauchy sequence in } \mathbb{Q}\}$$

where we define the equivalence relation $\sim$ on $\mathcal{C}$ by:

$$(a_n) \sim (b_n) \text{ if and only if } (a_n - b_n) \text{is a null sequence .}$$

Then as a set we define $\mathbb{R} = \{$ equivalence classes of $\sim\}$. We defined the addition and the multiplication by:

$$[(a_n)] + [(b_n)] = [(a_n + b_n)] \text{ and } [(a_n)] \cdot [(b_n)] = [(a_n \cdot b_n)]$$

(1) Show that addition is well defined, closed and associative.
(2) **Bonus - not required** Show that multiplication is well-defined, closed and associative.