Math 199 Introduction to Analysis and Linear Algebra, Section 43

Homework # 5 , due February 7

Problem 1
Read Chapter 3.1. in *"Foundations of Mathematical Analysis"* by Paul J. Sally

Problem 2
Let R be a commutative ring with identity. For $a \in R$ we denote by a^k as usual the element of R which is obtained by multiplying a with itself k-times, i.e. $a^k = a \cdot \dots \cdot a$ (k-times). For $k \in \mathbb{N}$ and $a \in R$ we denote by $k \cdot a$ the element of R which is obtained by adding a to itself k-times, i.e. $k \cdot a = a + \cdots + a$ (k-times).

(1) (Binomial Theorem) Prove the following statement by Mathematical Induction. Let $a, b \in R$. Then for every $n \in \mathbb{N}$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

$\big($Here for $n, k \in \mathbb{N}$, $\binom{n}{k}$ is the binomial coefficient which is defined as $\binom{n}{k} = \frac{n \cdot (n-1) \cdot \dots \cdot (n-k+1)}{1 \cdot \dots \cdot k}\big)$.

(2) Let $R = \mathbb{Z}$. Use the Binomial Theorem to show

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

(3) Let $R = \mathbb{Q}$. Let $a \in \mathbb{Q}$ and assume $(1-a) \neq 0$. Use Mathematical Induction to show that

$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$$

Problem 3
We defined the real numbers $\mathbb{R}$ in class as follows: Let

$$\mathcal{C} = \{(a_n) \mid (a_n) \text{ is a Cauchy sequence in } \mathbb{Q}\}$$

where we define the equivalence relation $\simeq$ on $\mathcal{C}$ by:

$$(a_n) \simeq (b_n) \text{ if and only if } (a_n - b_n) \text{ converges to } 0.$$

Then as a set we define $\mathbb{R} = \{$ equivalence classes of $\simeq$ in $\mathcal{C}\}$. We defined the addition $[(a_n)] + [(b_n)] = [(a_n + b_n)]$, multiplication $[(a_n)] \cdot [(b_n)] = [(a_n \cdot b_n)]$ and the order relation < 0 by:

$$0 < [(a_n)]$$

if and only if

- there exists some $N \in \mathbb{N}$ such that if $n > N$, then $a_n > 0$, and
- (a_n) does not converge to 0.

For $a = [(a_n)], b = [(b_n)] \in \mathbb{R}$ we define $a < b$ if and only if $b - a = [(b_n - a_n)] > 0$.

(1) Show that $<$ is well-defined. For this you have to show that it is independent from the sequence (a_n) which we pick to represent the equivalence class $[(a_n)]$, so you have to show that for every null sequence (k_n) we have: $(a_n + k_n) > (0)$ if and only if $(a_n) > (0)$.

(2) Show that $<$ defines an order on $\mathbb{R}$, i.e. you have to show that the rules of order (O1)-(O4) are satisfied. Showing (02)-(04) is fairly simple, to show (01) you will have to show that for every Cauchy sequence (a_n) which does not converge to 0 there exist an $N \in \mathbb{N}$ such that either $a_n > 0$ holds for all $n > N$ or $a_n < 0$ holds for all $n > N$.

Problem 4

Use that $\mathbb{Q}$ and $\mathbb{R}$ are ordered field to show the following consequences

(1) Let $a, b, c, d \in \mathbb{Q}$ with $a, b, c, d > 0$. If $\frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. In particular between any two rational numbers you find another rational.

(2) Let $a, b \in \mathbb{R}$ with $a, b > 0$. We define the *arithmetic*, *geometric* and *harmonic* mean by

$$A(a,b) := \frac{a+b}{2}$$
$$G(a,b) := \sqrt{ab}$$
$$H(a,b) := \frac{1}{A(\frac{1}{a}, \frac{1}{b})} = \frac{2ab}{a+b}.$$

Show that

$$H(a,b) \leq G(a,b) \leq A(a,b).$$

Show that equality of the means holds if an only if $a = b$.

Problem 5

(1) Consider the sequence a_n in $\mathbb{Q}$ defined by $a_1 = 1$ and $a_{n+1} = \frac{a_n^2+3}{2a_n}$.
a) **not required**: Show that (a_n) is a Cauchy sequence in $\mathbb{Q}$.
b) **required**: Give the real number $[(a_n)]$ a more familiar name.

(2) Consider the sequence (a_n) in $\mathbb{Q}$ defined by $a_n = \sum_{k=0}^{n} \frac{1}{10^k}$. Show that (a_n) is a Cauchy sequence in $\mathbb{Q}$. (Hint: Use the formula in from Problem 2). What is the real number $[(a_n)]$?

Problem 6 Let $\mathbb{C} := \{(a,b) \mid a, b \in \mathbb{R}\}$ be the *complex numbers*. We define the addition and multiplication as follows:

$$(a,b) + (c,d) := (a+c, b+d) \text{ and } (a,b) \cdot (c,d) := (ac - bd, ad + bc).$$

(1) Show that $\mathbb{C}$ is a field

(2) Show that $\mathbb{C}$ cannot be ordered. (Hint: Consider i^2, where $i = (0,1)$.)