Math 199 Introduction to Analysis and Linear Algebra, Section 43

Homework # 7 , due February 21

Problem 1

Read Chapter 3.6, 3.8., 3.9 and Chapter 2.1 in *"Foundations of Mathematical Analysis"* by Paul J. Sally

Problem 2

Remember that we defined open sets $U \subset \mathbb{R}$. Closed sets $C \subset \mathbb{R}$ are then defined as those set for which the complement $(\mathbb{R} \setminus C) \subset \mathbb{R}$ is an open set.

(1) Show that given a collection of closed sets $C_i \subset \mathbb{R}$, $i \in I$, not necessarily finite, their intersection $C = \bigcap_{i \in I} C_i \subset \mathbb{R}$ is a closed set.
(2) Show that if $C_1, \cdots, C_n \subset \mathbb{R}$ are finitely many closed sets, then their union $C = C_1 \cup \cdots \cup C_n \subset \mathbb{R}$ is also a closed set.
(3) Give an example of a infinite collection of closed sets $C_i \subset \mathbb{R}$, $i \in I$ whose union $C = \bigcup_{i \in I} C_i \subset \mathbb{R}$ is not a closed set.

Problem 3

Let $S \subset \mathbb{R}$ be any set. Recall that we defined the closure of S as $\overline{S} = S \cup \{\text{accumulations points of } S\}$.

(1) Show that $\overline{\overline{S}} = \overline{S}$ (This implies that $\overline{S}$ is closed by the characterization of closed sets we proved in class).
(2) Show that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.
(3) Give an example of two set $A, B \subset \mathbb{R}$ such that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.
(4) Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
(5) **Bonus - not required** Show that if S is a closed bounded set. Then the greates lower bound $\inf(S)$ and the least upper bound $\sup(S)$ are contained in S.

Problem 4

Determine the accumulation points of the following sets - with proof. Decide whether or not the sets are closed.

(1) $S = \mathbb{Z}$.
(2) $S = \mathbb{Q}$.
(3) $S = \mathbb{R} \setminus \mathbb{Q}$.
(4) $S = \{\frac{1}{n} + \frac{1}{m} \mid n, m \in \mathbb{N}\}$.

Problem 5

Let $(z_n)_{n \in \mathbb{N}}$ be a sequence of complex numbers. Show that (z_n) converges in $\mathbb{C}$ if and only if (z_n) is a Cauchy sequence. Note that we define Cauchy sequences as in $\mathbb{R}$: A sequence $(z_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, if for every $\varepsilon > 0$ there exists some $N_\varepsilon \in \mathbb{N}$ such that, if $n, m > N_\varepsilon$, then $|z_n - z_m| < \varepsilon$. Here the absolute value is the one we defined on the take home quiz: $|z| = \sqrt{z\overline{z}}$.