Math 199 Introduction to Analysis and Linear Algebra, Section 43

Homework # 9 , due March 7

Problem 1

Read Chapter 2 in *"Foundations of Mathematical Analysis"* by Paul J. Sally

Problem 2

Let p be a prime number. Then the set $F = \mathbb{Z}_p$ of congruence classes modulo p is a field. Let $V = F \times F$ be the vector space of ordered pais of elements in F.

(1) How many vectors are there in the vector space V?
(2) What is the dimension of V?
(3) How many 0-dimensional vector spaces does V have? Justify your answer.
(4) How many 1-dimensional vector spaces does V have? Justify your answer.
(5) How many 2-dimensional vector spaces does V have? Justify your answer.
(6) Make a list of the 1-dimensional vector spaces for $p = 5$. (Hint: Consider first the subspaces $W = \text{span}\{\mathbf{v}\}$, where $\mathbf{v} \in V$ is a single vector. Then find out which of the these subspaces are the same.)

Remark: It is important that we consider a field F with finitely many elelments. If we consider the vector space $V = \mathbb{R}^2$ over $F = \mathbb{R}$ then we cannot list all the 1-dimensional subspaces, since there are infintely many. But the space of all such 1-dimensional subspaces play an important role in mathematics.

Problem 3

Consider the vector space $V_n = \mathbb{R}_{\leq n}[x]$ over $F = \mathbb{R}$ of polynomials of degree less than or equal to n. Denote an arbitrary element of V_n by $p(x) = a_0 + a_1 x + \cdots + a_n x^n$.

(1) Show that the map $T : V_n \to V_n$ which sends $p(x)$ to its derivative $T(p(x)) = p'(x) = a_1 + \cdots + n a_n x^{n-1}$ is a linear transformation.
(2) Show that T is neither surjective nor injective. (Hint find an element $p(x) \neq 0$ such that $T(p(x)) = 0$ to show that the map T is not injective. Then find an element $q(x) \in V_n$ such that there does not exists any $p(x) \in V_n$ with $T(p(x)) = p'(x) = q(x)$ to show that the map T is not surjective.)
(3) Determine the vector subspace $T(V_n) \subset V_n$.
(4) Determine the vector subspace $\ker(T) \subset V_n$.

Problem 4 Consider the vector space $V = \mathbb{R}^3$ over $\mathbb{R}$. Let $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ be the standard basis. Let $T : V \to V$ be the linear transformation determined by $T(\mathbf{e_1}) = (1, 2, 3)$, $T(\mathbf{e_2}) = (4, 5, 6)$ and $T(\mathbf{e_3}) = (7, 8, 9)$.

(1) Let $\mathbf{v} = (2, 3, 2)$. What is $T(\mathbf{v})$?
(2) Determine the vector space $T(\mathbb{R}^3) \subset \mathbb{R}^3$.
(3) Determine the vector space $\ker(T) \subset \mathbb{R}^3$.