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HÖLDER FOLIATIONS

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1. Introduction. It is the goal of this paper to estimate the regularity of the holonomy maps of certain dynamically invariant foliations. They are θ-Hölder. This Hölder regularity is a crucial component of the analysis appearing in our papers, Pugh and Shub [12] and Wilkinson [16], where we establish stable ergodicity for a wide class of dynamical systems, a class which includes Anosov diffeomorphisms, the time t-maps of many Anosov flows, and many examples defined on homogeneous spaces of Lie groups. We state here our main results, place them in context, and then go on to explain them more fully in §2. The notation $m(T)$ stands for the *conorm* (or *minimum norm*) of a linear transformation, $m(T) = \|T^{-1}\|^{-1}$.

THEOREM A. *Suppose that $f: M \to M$ is a C^2 diffeomorphism, partially hyperbolic with respect to the splitting $TM = E^u \oplus E^c \oplus E^s$. Then, for some $\theta \in (0, 1)$ and all $p \in M$, its expansion and contraction rates satisfy a θ-*pinching condition

$$\|T_p^s f\| \, \|T_p^u f\|^\theta < m(T_p^c f) \qquad \textit{and} \qquad \|T_p^c f\| < m(T_p^u f) m(T_p^s f)^\theta.$$

For any such θ, the local unstable and stable holonomy maps are uniformly θ-Hölder.

THEOREM B. *Suppose that $f: M \to M$ is a partially hyperbolic C^2 diffeomorphism and f leaves invariant a foliation $\mathscr{W}^c$ tangent to the center direction E^c. (The tangent plane to the $\mathscr{W}^c$-leaf at p is E_p^c.) If the expansion and contraction rates satisfy the* center bunching conditions

$$\|T_p^s f\| \, \|T_p^c f\| < m(T_p^c f) \qquad \textit{and} \qquad \|T_p^c f\| < m(T_p^u f) m(T_p^c f),$$

then the local unstable and local stable holonomy maps are uniformly C^1 when restricted to each center unstable and each center stable leaf, respectively.

A C^2 volume-preserving diffeomorphism of a compact, connected manifold $M \to M$ is *stably ergodic* if it and all its C^2 small volume-preserving perturbations are ergodic. In 1962 Anosov [2] proved that totally hyperbolic diffeomorphisms are stably ergodic. By contrast, the theory of Kolmogorov, Arnold, and Moser produces open sets of nonergodic diffeomorphisms that have no hyperbolicity at all. In a series of recent papers, we have been studying the mixed situation, in which the dynamical system is partially, but not totally, hyperbolic. Our main

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theme is that a little hyperbolicity goes a long way toward guaranteeing stably ergodic behavior, and that such behavior is more prevalent than one might have thought. Our analysis proceeds by examining the stable and unstable manifold structure of a partially hyperbolic dynamical system, especially the holonomy along the stable and unstable leaves. In the Anosov case (totally hyperbolic), the stable and unstable leaves have complementary dimensions and are transverse to each other, while in the partially hyperbolic case there is also a center (fairly neutral) direction, so transversality between stable and unstable leaves is lost. The regularity results we prove in this paper are used to overcome the technical difficulties caused by this lack of transversality.

J. Schmeling and Ra. Siegmund-Schultze [13] have proved Hölder holonomy results like Theorem A when the diffeomorphism f is totally hyperbolic. Even in this restricted case, our proofs are simpler. Hölder regularity results that are analogous to Theorem A, but stated in terms of the splitting $TM = E^u \oplus E^c \oplus E^s$ instead of the holonomy maps of the foliations, have been proved by Boris Hasselblatt [9]. As Hasselblatt points out, they neither imply nor are implied by holonomy results such as Theorem A.

The organization of the rest of the paper is as follows. In §2 we define the concepts relevant to Theorems A and B and discuss some examples. In §3 we show how to dynamically trivialize a vector bundle. It is a result of independent interest, especially useful in simplifying proofs that involve the invariant section technique. Also in §3 we discuss a Hölder invariant section theorem. In §4 we prove Theorem A, and in §5 we prove Theorem B. In §6 we discuss the general issue of regularity of foliations.

2. Partially hyperbolic dynamics. Recall that the *norm*, *conorm*, and *bolicity* (or *condition number*) of a linear transformation T from one normed linear space to another are

$$\|T\| = \sup\{|Tv| : |v| = 1\},$$

$$m(T) = \inf\{|Tv| : |v| = 1\}, \qquad \text{and}$$

$$\operatorname{bol}(T) = \frac{\|T\|}{m(T)}.$$

If $\|T\| < 1$, then T contracts the length of each vector by a factor <1, while if $m(T) > 1$, then T expands the length of each vector by a factor >1. When T is invertible, $m(T) = \|T^{-1}\|^{-1}$ and $\operatorname{bol}(T) = \|T\|\,\|T^{-1}\|$. Bolicity measures the extent to which a linear transformation distorts the shape of the unit ball. If E is a vector bundle over a base space X, and the fibers of E are normed linear spaces, and if $F: E \to E$ is a vector bundle morphism, then we write $\|F\| = \sup \|F_x\|$ and $m(F) = \inf m(F_x)$, where F_x is the restriction of F to the fiber E_x, and x varies over the base space X. If $\|F\| < 1$, then F *contracts* the bundle E, while if $m(F) > 1$, then F *expands* it.

Let $f: M \to M$ be a diffeomorphism of a compact, connected, boundaryless manifold M, and assume that TM splits as the sum of three continuous vector subbundles,

$$TM = E^u \oplus E^c \oplus E^s,$$

each of which is invariant under Tf, and $E^u \neq 0 \neq E^s$. We say that f is *partially hyperbolic* if, with respect to some Riemann structure on TM, Tf expands E^u, Tf contracts E^s, and for all $p \in M$

$$\|T_p^s f\| < m(T_p^c f) \quad \text{and} \quad \|T_p^c f\| < m(T_p^u f), \tag{1}$$

where $T^u f$, $T^c f$, $T^s f$ refer to the restrictions of Tf to E^u, E^c, E^s. Equation (1) means that $T^s f$ contracts more sharply than $T^c f$ does, while $T^u f$ expands more sharply than $T^c f$ does. Since M is compact, (1) can be rewritten as

$$\sup \|T_p^s f\| \, \|T_{fp}^c f^{-1}\| < 1 \quad \text{and} \quad \sup \|T_{fp}^u f^{-1}\| \, \|T_p^c f\| < 1, \tag{1$'$}$$

the suprema being taken as p varies in M. The bundles E^u, E^c, E^s are called *unstable* (or *strong unstable*), *center*, and *stable* (or *strong stable*), while the bundles $E^{cu} = E^u \oplus E^c$ and $E^{cs} = E^c \oplus E^s$ are called *center unstable* and *center stable*. The diffeomorphism f is *uniformly partially hyperbolic* if it is partially hyperbolic and if (1) can be strengthened to

$$\|T^s f\| < m(T^c f) \quad \text{and} \quad \|T^c f\| < m(T^u f). \tag{2}$$

If E^c is the zero bundle, $E^c = 0$, then (1) is vacuously true, and a partially hyperbolic diffeomorphism f is *totally hyperbolic*, or *Anosov*. Note that the θ-pinching condition

$$\|T_p^s f\| \, \|T_p^u f\|^\theta < m(T_p^c f) \quad \text{and} \quad \|T_p^c f\| < m(T_p^u f) m(T_p^s f)^\theta \tag{3}$$

in Theorem A is merely (1) with factors $\|T_p^u f\|^\theta$ and $m(T_p^s f)^\theta$ inserted. Thus every partially hyperbolic diffeomorphism satisfies the θ-pinching condition for some $\theta \in (0, 1)$; (3) is not an additional assumption, it merely expresses the extent to which (1) can be relaxed. In contrast, consider the center bunching condition in Theorem B

$$\|T_p^s f\| \, \|T_p^c f\| < m(T_p^c f) \quad \text{and} \quad \|T_p^c f\| < m(T_p^u f) m(T_p^c f). \tag{4}$$

It is (1) with factors $\|T_p^c f\|$ and $m(T_p^c f)$ inserted, and it does present an additional assumption. One can rewrite (4) in terms of the center bolicity $b_p = \|T_p^c f\| / m(T_p^c f)$ as

$$\|T_p^s f\| < \frac{1}{b_p} \quad \text{and} \quad b_p < m(T_p^u f). \tag{4$'$}$$

This concludes our discussion of the hypotheses in Theorems A and B, and we pass to the conclusions. According to Hirsch, Pugh, and Shub [10], there are unique f-invariant foliations, $\mathscr{W}^u$ and $\mathscr{W}^s$ tangent to E^u and E^s. Although the foliations $\mathscr{W}^u$ and $\mathscr{W}^s$ have leaves of class C^1, this does not make them C^1 foliations. (See §6 for a more wide-ranging discussion of these regularity issues.) What are the holonomy maps of such foliations and what does it mean that they are Hölder? In the present section, we will analyze the holonomy of a general foliation $\mathscr{F}$ of M, such that each leaf $\mathscr{F}_p$ is tangent at p to a plane $F_p \subset T_pM$, and $p \mapsto F_p$ is a continuous section of the Grassmannian of TM.

At two nearby points $p, q \in M$, we draw local transversals τ_p, τ_q to $\mathscr{F}$ and examine the effect of sending $y \in \tau_p$ to $h(y) \in \tau_q$ by sliding along the leaves of $\mathscr{F}$. The map $h\colon \tau_p \to \tau_q$ is the *holonomy* of $\mathscr{F}$.

The foliation $\mathscr{F}$ has *locally θ-Hölder holonomy* if

$$d_q(h(y), h(y')) \leqslant H d_p(y, y')^\theta,$$

where $y, y' \in \tau_p$, and d_p, d_q refer to natural metrics on τ_p, τ_q, say, path metrics with respect to a fixed Riemann structure on TM. The constant H is the Hölder constant, and it should be independent of the choice of "reasonable" transversals. We now spell out just which transversals to a foliation $\mathscr{F}$ are reasonable.

Fix a smooth Riemann structure on TM, and let exp be its exponential map. Fix a positive number L. Let $F^\perp$ be the orthogonal complement to $F = T\mathscr{F}$,

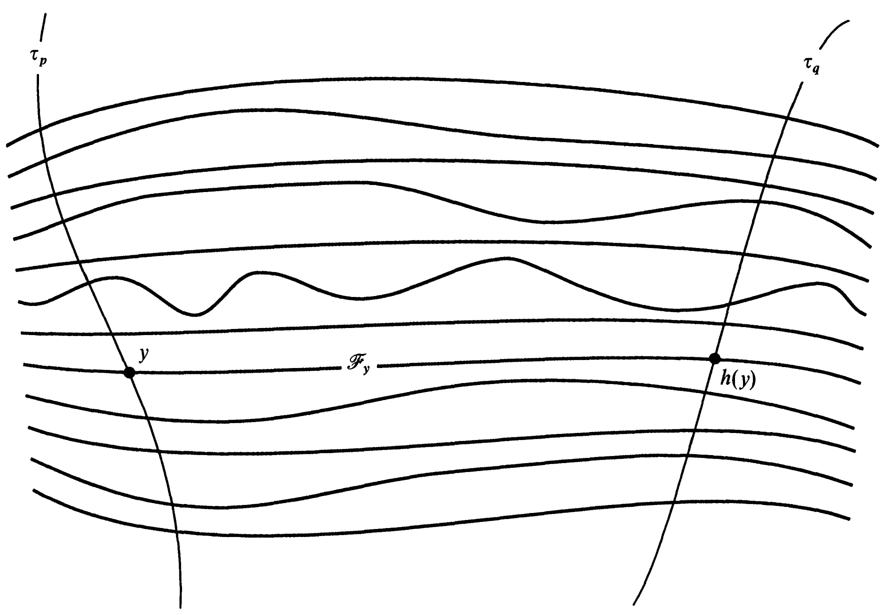

FIGURE 1. The holonomy map h of $\mathscr{F}$

$TM = F \oplus F^{\perp}$. Consider a C^1 function

$$t\colon F_p^{\perp}(\delta) \to F_p$$

such that $t(0) = 0$ and the derivative of t has norm $\leqslant L$ everywhere in $F_p^{\perp}(\delta)$. If δ is small, then the disc

$$\exp(\{v + t(v) \in T_pM\colon v \in F_p^{\perp}(\delta)\}) = \tau$$

is well defined and transverse to $\mathscr{F}$. In fact, by compactness of M and continuity of $T\mathscr{F}$, there is a $\delta > 0$ such that all such discs $\tau = \tau_p$ of size δ at all points $p \in M$ are uniformly transverse to $\mathscr{F}$. Let τ_p, τ_q be two transversals of size δ. Assume that p, q lie on a common leaf of $\mathscr{F}$ and that they can be joined by a path γ of length $\leqslant 1$ in the leaf. Because $\mathscr{F}$ is a foliation, γ lifts continuously to nearby leaves. Thus the holonomy map $h\colon \tau_p(\rho) \to \tau_q$ is well defined at all points y with $d_p(y, p) \leqslant \rho$. Compactness implies that we can choose a uniform $\rho > 0$ for all such transversals τ_p, τ_q. Here then is the conclusion of Theorem A relative to the foliation $\mathscr{W}^u$:

There are constants ρ, H such that for all points p, q which can be joined by a path of length $\leqslant 1$ in a common leaf of $\mathscr{W}^u$ and for all transversals τ_p, τ_q as above, the holonomy map $h\colon \tau_p(\rho) \to \tau_q$ is well defined and, relative to the Riemann path metrics on τ_p, τ_q, it satisfies the Hölder estimate

$$d_q(h(y), h(y')) \leqslant H d_p(y, y')^{\theta} \qquad \text{for all } y, y' \in \tau_p(\rho). \tag{5}$$

Theorem A also makes the same assertions for the stable foliation $\mathscr{W}^s$. Note that reversing p and q shows that the inverse of h is also Hölder, so Theorem A actually asserts that the holonomy maps are uniformly θ-biHölder.

In Theorem B we assume that f leaves invariant a foliation $\mathscr{W}^c$ integrating the center direction E^c. The fact that the partially hyperbolic diffeomorphism f leaves $\mathscr{W}^c$ invariant is equivalent to f being 1-normally hyperbolic at $\mathscr{W}^c$. Normal hyperbolicity is investigated at length in [10], and it is shown that through each leaf L of an invariant 1-normally hyperbolic foliation there pass a weak unstable leaf and a weak stable leaf. They are C^1 leaf immersed and meet transversally at L; they are composed of strong unstable and strong stable leaves, respectively. In our case, $L = W^c$ and the two leaves are W^{cu} and W^{cs}. Thus $\mathscr{W}^u$ foliates W^{cu} and $\mathscr{W}^s$ foliates W^{cs}. Theorem B asserts that the foliations of each W^{cu} by $\mathscr{W}^u$ and each W^{cs} by $\mathscr{W}^s$ are of class C^1. To a large extent, the proof appears in [10].

A priori, $\mathscr{W}^c$ foliates neither W^{cu} nor W^{cs}. Also, the leaves W^{cu} need not fit together to form a foliation $\mathscr{W}^{cu}$, nor need the leaves W^{cs} form a foliation $\mathscr{W}^{cs}$, for a leaf immersion can have nontrivial tangential self-intersection. This leads us to say that the partially hyperbolic diffeomorphism f is *dynamically coherent* if, tangent to E^{cu}, E^c, E^{cs}, there are f-invariant foliations $\mathscr{W}^{cu}$, $\mathscr{W}^c$, $\mathscr{W}^{cs}$ such that $\mathscr{W}^u$

and $\mathcal{W}^c$ subfoliate $\mathcal{W}^{cu}$, while $\mathcal{W}^c$ and $\mathcal{W}^s$ subfoliate $\mathcal{W}^{cs}$. (One foliation *subfoliates* a second if the leaves of the second are unions of leaves of the first.) The phrase "dynamically coherent" indicates that the unstable, center unstable, center, center stable, and stable orbit-classes fit together nicely. A restatement of Theorem B in this context is the following.

COROLLARY 2.1. *If the partially hyperbolic, C^2 diffeomorphism f is dynamically coherent and center bunched, then $\mathcal{W}^u$ C^1 subfoliates $\mathcal{W}^{cu}$ and $\mathcal{W}^s$ C^1 subfoliates $\mathcal{W}^{cs}$.*

Theorems A and B have generalizations to the case where f is partially hyperbolic on a subset of M, rather than on all of M. The natural definition of this concept is that $\Lambda \subset M$ is an f-invariant, compact subset of M, and Tf leaves invariant a continuous splitting

$$T_\Lambda M = E^u \oplus E^c \oplus E^s,$$

where $E^u \neq 0 \neq E^s$, Tf expands E^u, Tf contracts E^s, and the inequalities

$$\|T_p^s f\| < m(T_p^c f) \qquad \text{and} \qquad \|T_p^c f\| < m(T_p^u f)$$

hold for all $p \in \Lambda$. Clearly, if f is partially hyperbolic at $\Lambda = M$, then f is partially hyperbolic in the sense discussed above. Also, if $E^c = 0$, then Λ is a hyperbolic set for f. It is shown in [10] that tangent to E^u and E^s there are laminations through Λ, invariant under f. (A lamination is a partial foliation. Its leaves, or lamina, do not necessarily foliate a full neighborhood of Λ.)

THEOREM A′. *Suppose that $f: M \to M$ is a C^2 diffeomorphism that is partially hyperbolic at a compact f-invariant subset $\Lambda \subset M$. For some $\theta \in (0, 1)$, it satisfies a θ-pinching condition at the invariant set Λ. The local holonomy maps along its stable and unstable laminations through Λ are uniformly θ-Hölder.*

THEOREM B′. *Suppose that $f: M \to M$ is a C^2 diffeomorphism that is partially hyperbolic at a compact f-invariant subset $\Lambda \subset M$, and the center direction E^c integrates to an f-invariant lamination $\mathcal{L}^c$ of Λ. If f satisfies the center bunching condition in Theorem B, then the local unstable and local stable holonomy maps are uniformly C^1 when restricted to each center unstable leaf and center stable leaf, respectively.*

Dynamic coherence is defined as before: all five bundles E^u, E^{cu}, E^c, E^{cs}, E^s integrate to laminations through Λ, the center and unstable laminations subfoliate the center unstable lamination, while the center and stable laminations subfoliate the center stable lamination. Then Theorem B′ implies the following.

COROLLARY 2.2. *If f is as in Theorem B′ and is dynamically coherent, then $\mathcal{W}^u$ and $\mathcal{W}^s$ C^1 subfoliate $\mathcal{W}^{cu}$ and $\mathcal{W}^{cs}$, respectively.*

See §4 and §5 for the proofs of Theorem A′ and B′.

Here are some examples of partially hyperbolic diffeomorphisms. All are uniformly partially hyperbolic. First, of course, there are the totally hyperbolic Anosov diffeomorphisms. Second, there are time t-maps of Anosov flows. Recall that under an *Anosov flow* φ on M, the tangent bundle splits as

$$TM = E^u \oplus E^\varphi \oplus E^s,$$

and for some Riemann structure on TM and for all $t > 0$, $T\varphi_t$ expands E^u while it contracts E^s. The Anosov vector field X that generates φ is nonvanishing, so E^φ is a line field, and the Riemann structure can be chosen so that $|X| \equiv 1$. As for any smooth flow, $T_p\varphi_t\colon X(p) \mapsto X(\varphi_t p)$. Thus $T\varphi_t$ sends $T_p^\varphi M$ isometrically to $T^\varphi_{\varphi_t p}M$, and φ_t is partially hyperbolic with respect to the Anosov splitting.

Third, there are the algebraic examples analyzed by Brezin and Shub [4]. Let G be a connected Lie group, and let $\Gamma \subset G$ be a lattice. Fix any $g \in G$ that leaves Γ invariant and any automorphism $A\colon G \to G$. Set $M = G/\Gamma$ and consider the *affine diffeomorphism* f of M to itself induced by projecting $L_g \circ A\colon G \to G$ down to M. This gives an automorphism $\mathrm{ad}(g)DA(e)$ of the Lie algebra of G and splits T_eG into the sum of generalized eigenspaces corresponding to eigenvalues with magnitude >1, $=1$, and <1. Translate the splitting to the other tangent spaces T_hG, and project it down to TM. When the first and third subspaces are nonzero, this gives a partially hyperbolic splitting for f. Other partially hyperbolic splittings arise by starting with generalized eigenspaces corresponding to eigenvalues with magnitude $<\rho$, between ρ and $1/\rho$, and $>1/\rho$, where $0 < \rho < 1$ is a constant.

Fourth, there are hyperbolic basic sets for Axiom A diffeomorphisms and flows. They are partially hyperbolic, not on the whole manifold M, but only at a compact f-invariant subset $\Lambda \subset M$. Fifth, there are perturbations of the first four types of examples. For, as is shown in [10], a diffeomorphism remains partially hyperbolic under C^1 small perturbations. Sixth, there are iterates of the preceding examples.

3. Bundle dynamics. In this section we discuss two topics: trivialization of a vector bundle with prescribed dynamics in the base space, and dynamically invariant sections of a Banach bundle.

In differential topology, it is well known that a vector bundle E over a compact base X has an *inverse bundle*, a vector bundle E' over X for which there is a vector bundle automorphism

$$\begin{array}{ccc} E \oplus E' & \xrightarrow{\tau} & \varepsilon \\ \downarrow & & \downarrow \\ X & \xrightarrow{id} & X, \end{array}$$

where ε is a *trivial bundle*: $\varepsilon = X \times \mathbb{R}^N$ for some N. One says that E' *trivializes* E. Less well known and underappreciated is the following elementary result from K-theory, which was explained to us by Jeremy Kahn.

LEMMA 3.1 (Dynamic trivialization). *Given a vector bundle E over the compact base X, there exists an inverse bundle E'' over X such that each vector bundle isomorphism covering a base homeomorphism*

$$\begin{array}{ccc} E & \xrightarrow{T} & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & X \end{array}$$

extends to a vector bundle isomorphism

$$\begin{array}{ccc} E \oplus E'' & \xrightarrow{T \oplus T''} & E \oplus E'' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & X. \end{array}$$

When E and T are smooth, so are E'' and T''.

Proof #1. Let E' trivialize E, and let $\tau: E \oplus E' \to \varepsilon$ be a trivialization. Set $E'' = E' \oplus E \oplus E'$. Clearly, E'' trivializes E since $E \oplus E'' = E \oplus E' \oplus E \oplus E' \cong \varepsilon \oplus \varepsilon$ is trivial. Using the pullback notation, we have

$$\begin{aligned} f^*E'' &= f^*(E' \oplus E \oplus E') \\ &\cong f^*(E' \oplus E) \oplus f^*E' \\ &\cong f^*(\varepsilon) \oplus f^*E' = \varepsilon \oplus f^*E' \\ &\cong E' \oplus E \oplus f^*E' \\ &\cong E' \oplus f^*E \oplus f^*E' \\ &\cong E' \oplus f^*(E \oplus E') \\ &\cong E' \oplus f^*(\varepsilon) = E' \oplus \varepsilon \\ &\cong E' \oplus E \oplus E' = E''. \end{aligned}$$ □

For the reader who wants to keep track of what happens at the vector level under these bundle maps, we offer a second proof.

Proof #2. Define $E'' = E' \oplus E \oplus E'$ as above. We construct a bundle isomorphism $T'': E'' \to E''$ covering f. We have bundle isomorphisms S, S' defined by the commutative diagrams

$$\begin{tikzcd}
E \oplus E' \arrow[r, "\tau"] \arrow[d] \arrow[rrr, bend left, "S"] & \varepsilon \arrow[r, "f \times id"] \arrow[d] & \varepsilon \arrow[r, "\tau^{-1}"] \arrow[d] & E \oplus E' \arrow[d] \\
X \arrow[r, "id"'] & X \arrow[r, "f"'] & X \arrow[r, "id"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
E' \oplus E \arrow[r, "sw"] \arrow[d] \arrow[rrr, bend left, "S'"] & E \oplus E' \arrow[r, "S"] \arrow[d] & E \oplus E' \arrow[r, "sw"] \arrow[d] & E' \oplus E \arrow[d] \\
X \arrow[r, "id"'] & X \arrow[r, "f"'] & X \arrow[r, "id"'] & X
\end{tikzcd}$$

where *sw* switches the order of E and E'. This gives isomorphisms of the fibers

$$\begin{array}{ccc}
E''_x = (E'_x \oplus E_x) \oplus E'_x & \xrightarrow{S'_x \oplus id'_x} & (E'_{fx} \oplus E_{fx}) \oplus E'_x \\
 & & \| \\
E'_{fx} \oplus E_x \oplus E'_x & \xleftarrow{id'_{fx} \oplus T_x^{-1} \oplus id'_x} & E'_{fx} \oplus E_{fx} \oplus E'_x \\
\| & & \\
E'_{fx} \oplus (E_x \oplus E'_x) & \xrightarrow{id'_{fx} \oplus S_x} & E'_{fx} \oplus E_{fx} \oplus E'_{fx} = E''_{fx}.
\end{array}$$

The composition is $T''_x: E''_x \to E''_{fx}$, and T''_x is a continuous function of x since all the maps S_x, S'_x, T_x are. If T is smooth, we can choose τ smooth. Then T'' and E'' are smooth. □

The proof that the local unstable manifold at p of a partially hyperbolic diffeomorphism is a Hölder function of p relies on the invariant section theorem in [10], improved to the Hölder world in Shub [14] and Wilkinson [16]. The invariant section theorem concerns the existence, uniqueness, and regularity of a section of a bundle, when the section is required to be invariant under a fiber contraction. Let us recall the definitions. Commutativity of the diagram

$$\begin{tikzcd}
E \arrow[r, "F"] \arrow[d, "\pi"'] & E_1 \supset E \arrow[d, "\pi"] \\
X \arrow[r, "h"] & X_1 \supset X
\end{tikzcd}$$

defines a bundle map F. The fibers $E_x = \pi^{-1}(x)$ are assumed to be complete, nonempty, uniformly bounded metric spaces, and the *base map* h is assumed to biject X onto a set that contains X. (We say that h *overflows* X.) Finally, F is assumed to contract fibers in the sense that for all $y, y' \in E_x$, and for all $x \in X$,

$$d_{hx}(F(y), F(y')) \leqslant k_x d_x(y, y'),$$

where $\sup k_x = k < 1$, and the notation d_x refers to the metric on the fiber E_x. The constant k_x is the *fiber constant*. Under these assumptions F is a *fiber contraction*, and there exists a unique section $\sigma: X \to E$ which is *invariant* under F in the sense that for all $x \in X$,

$$\sigma \circ h(x) = F \circ \sigma(x).$$

The proof is natural and straightforward. Merely consider the space Σ of all sections $\sigma: X \to E$, furnished with the sup metric

$$d(\sigma, \sigma') = \sup\{d_x(\sigma(x), \sigma'(x)) \colon x \in X\}.$$

Σ is complete, and the natural F-action $F_\#: \Sigma \to \Sigma$ defined by

$$F_\#(\sigma)(x) = F \circ \sigma \circ h^{-1}(x) \qquad \text{for all } x \in X$$

is a contraction of Σ. The unique fixed point of $F_\#$ is the unique F-invariant section of E.

The unique F-invariant section of E is denoted as σ_F. If we start with any section σ of E and iterate it under $F^n_\#$, it converges to σ_F; $F^n_\#(\sigma) \to \sigma_F$ as $n \to \infty$. Thus if F leaves invariant a closed subbundle P of E, then σ_F is a section of P.

Existence and uniqueness were essentially trivial, but regularity of σ_F is somewhat subtler. For example, if F and h^{-1} are continuous, then so is σ_F. For $F_\#$

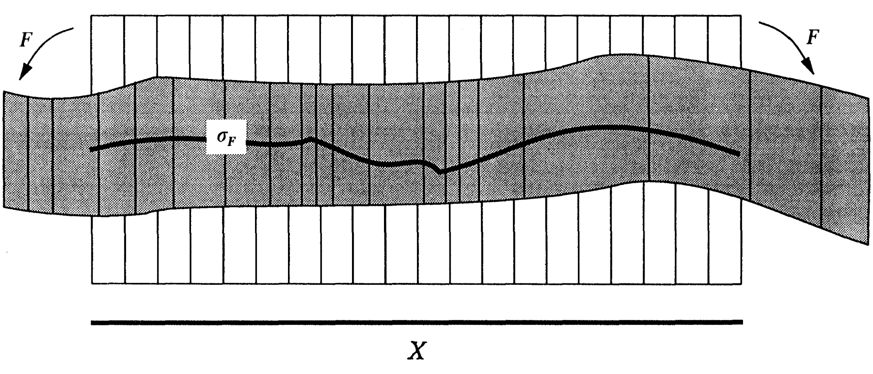

FIGURE 2. A fiber contraction and its invariant section

sends the closed subspace of continuous sections $\Sigma^c \subset \Sigma$ into itself, and so the fixed point σ_F of $F_\#$ must lie in Σ^c. Conditions that guarantee that σ_F is C^r, $r \geqslant 1$, are at the heart of stable manifold theory; see also the proof of Theorem B. Here are sufficient conditions on a fiber contraction which imply that σ_F is θ-Hölder:

(a) $E = X \times Y$, where X is a compact metric space and Y is a closed, bounded subset of a Banach space;

(b) there exists a $\delta > 0$ such that for each $x \in X$,

$$\inf\left\{\frac{d(h(x), h(x'))}{d(x, x')} : x' \in X \text{ and } d(x, x') < \delta\right\} = \omega_x > 0,$$

and $\inf \omega_x = \omega > 0$;

(c) $\sup k_x \omega_x^{-\theta} < 1$;

(d) there exists a constant $L \geqslant 1$ such that for all x, $x' \in X$, and all $y \in Y$, $|F_Y(x, y) - F_Y(x', y)| \leqslant L d(x, x')^\theta$.

The constant ω_x is the *base constant.* It describes how sharply h contracts the base space at x. Condition (c) says that F contracts the fiber at x more sharply at Hölder scale θ than it contracts the base at x. The fiber constant θ-dominates the base constant. In condition (d), F_Y is the Y-component of F, and the inequality in (d) amounts to the assumption that F is θ-Hölder. To summarize, we have the following.

THEOREM 3.2. (Pointwise Hölder section). *Under hypotheses (a)–(d), the unique F-invariant section of $X \times Y$ is θ-Hölder; i.e., if we write $\sigma_F(x) = (x, s(x)) \in X \times Y$, then*

$$|s(x) - s(x')| \leqslant H d(x, x')^\theta \qquad \text{for all } x, x' \in X. \tag{6}$$

Moreover, if R bounds the diameter of Y, then the Hölder constant H is no greater than

$$\frac{LR}{\omega\delta^\theta(1 - \sup k_x \omega_x^{-\theta})}. \tag{7}$$

The proof of Theorem 3.2 appears in Wilkinson [16, pp. 29–36]. Under the uniform assumption $k\omega^{-\theta} < 1$, instead of (c), and a Lipschitz assumption, rather than (d), the proof appears in Shub [14, pp. 44–48]. Like the proof of every regularity result for invariant sections, the idea is to show that the natural map $F_\#$ on the space of sections carries a subspace Σ^θ into itself, where Σ^θ is the set of θ-Hölder sections as in (6), with Hölder constant as in (7). Clearly, Σ^θ is a closed subset of Σ^c, and hence the unique fixed point of $F_\#$, σ_F, lies in Σ^θ.

4. Proof of Theorem A. Recall from §2 that Theorem A becomes Theorem A′ when the diffeomorphism f is assumed to be partially hyperbolic at a compact invariant subset $\Lambda \subset M$, instead of all of M, so it suffices to prove Theorem A′. The proof has three steps. First, using Theorem 3.2, it is shown that the function taking a point to its local unstable manifold is θ-Hölder. This is Corollary 4.2. Then it is shown that this conclusion implies that the holonomy maps are θ-Hölder at sufficiently small scale. This is Corollary 4.4. After the proof of Corollary 4.4, we deduce Theorem A′.

To apply Theorem 3.2, the Hölder section theorem, we use some simple facts from differential topology. First, recall that a function defined on a closed set Λ in a manifold M is said to be smooth if it extends to a smooth function defined on a neighborhood of Λ. Next recall what this means in terms of vector bundles. Let E be a smooth vector bundle over M, and let C be a continuous vector subbundle of $E|_\Lambda$ where Λ is a closed subset of M. Using the Tietze extension theorem and a partition of unity, such a C always extends to a continuous subbundle $\hat{C}$ of $E|_U$ where U is a neighborhood of Λ in M. If one such extension $\hat{C}$ is smooth, then C itself is said to be smooth. A bundle map $T: C \to C$ is said to be smooth if it extends to a smooth bundle map of a smooth local extension of C. Any C can be approximated by a smooth $\tilde{C} \subset E|_\Lambda$, and any bundle map $T: C \to C$ can be approximated by a smooth bundle map $\tilde{T}: \tilde{C} \to \tilde{C}$. Finally, if C is trivial, then any extension $\hat{C}$ is trivial, at least when $\hat{C}$ is restricted to a small enough neighborhood of Λ; an approximation to a trivial bundle is trivial; and a smooth approximation to a trivial bundle is smoothly trivial.

We assume that f is partially hyperbolic at a compact invariant subset $\Lambda \subset M$, with respect to the splitting $T_\Lambda M = E^u \oplus E^c \oplus E^s$. We lift f to TM using the exponential of the fixed smooth Riemann structure. Commutativity of the diagram

$$\begin{tikzcd} T_pM(\delta) \arrow[r, "\bar{f}_p"] \arrow[d, "\exp_p"'] & T_{fp}M \arrow[d, "\exp_{fp}"] \\ M \arrow[r, "f"] & M \end{tikzcd}$$

defines a C^2 map $\bar{f}: TM(\delta) \to TM$ that covers f. Given $g: E^u_p(\delta) \to E^{cs}_p$, define the special norm

$$|g|_* = \sup \frac{|g(x)|}{|x|},$$

where x, of course, ranges over the nonzero vectors in $E^u_p(\delta)$. The set of continuous function g with $|g|_* < \infty$ forms a Banach space $\mathscr{G}^*_p$. Define

$$\mathscr{G}_p = \{g \in \mathscr{G}^*_p\colon \operatorname{Lip} g \leqslant 1\}.$$

This gives a bundle $\mathscr{G}$ over Λ with fiber $\mathscr{G}_p$ at p, on which $\bar{f}$ acts naturally according to the graph transform:

$$\operatorname{graph} \bar{f}_{\# g} = \bar{f}(\operatorname{graph} g) \cap (E^u_{fp}(\delta) \times E^{cs}_{fp}) \qquad \begin{array}{ccc} \mathscr{G} & \xrightarrow{\bar{f}_\#} & \mathscr{G} \\ \downarrow & & \downarrow \\ \Lambda & \xrightarrow{f} & \Lambda. \end{array}$$

It is a standard calculation (see, for instance, [10, p. 57], [14, p. 64], or [16, pp. 35–36]) that with respect to the special norm, $\bar{f}_\#$ is a fiber contraction and

(8) $\qquad \bar{f}_\#$ has fiber constant $\doteq \|T^{cs}_p f\|/m(T^u_p f)$ and

$\qquad \bar{f}_\#$ has base constant $\doteq m(T^{cs}_p f)$.

We would like to use the Hölder Section Theorem 3.2 to show that the unique $\bar{f}_\#$-invariant section $\sigma_{\bar{f}_\#}$ of $\mathscr{G}$ is θ-Hölder. For $\sigma_{\bar{f}_\#}$ describes the local unstable manifolds of f, and its Hölderness would imply that the local unstable manifold through p is a θ-Hölder function of p. A priori, $\mathscr{G}$ is neither trivial nor a Hölder bundle, so Theorem 3.2 cannot be applied as it stands; it does not even make sense to assert that $\sigma_{\bar{f}_\#}$ is Hölder. We must modify $\mathscr{G}$ and $\bar{f}_\#$.

We first approximate E^u and E^{cs} by smooth subbundles $\tilde{E}^u$ and $\tilde{E}^{cs}$ in $T_\Lambda M$. Respecting the splitting $T_\Lambda M = \tilde{E}^u \oplus \tilde{E}^{cs}$, the derivative of $\bar{f}_p$ at $v \in T_p M(\delta)$ is

$$\begin{bmatrix} A & B \\ C & K \end{bmatrix} \quad \text{where} \quad \begin{array}{ll} A_v\colon \tilde{E}^u \to \tilde{E}^u & B_v\colon \tilde{E}^u \to \tilde{E}^{cs} \\ C_v\colon \tilde{E}^{cs} \to \tilde{E}^u & K_v\colon \tilde{E}^{cs} \to \tilde{E}^{cs}. \end{array}$$

When δ is small and $\tilde{E}^u$, $\tilde{E}^{cs}$ closely approximate E^u, E^{cs}, the linear maps A_v, B_v, C_v, K_v closely approximate $T^u_p f$, 0, 0, $T^{cs}_p f$, respectively. The set of continuous maps $g\colon \tilde{E}^u_p(\delta) \to \tilde{E}^{cs}_p$ whose special norm $|g|_* = \sup |g(x)|/|x|$ is finite forms a Banach space $\tilde{\mathscr{G}}^*_p$. The subset $\tilde{\mathscr{G}}_p = \{g \in \tilde{\mathscr{G}}^*_p\colon \operatorname{Lip} g \leqslant 1\}$ is closed and bounded. This gives a bundle $\tilde{\mathscr{G}}$ over Λ with fiber $\tilde{\mathscr{G}}_p$ at p, which is a smoothing of $\mathscr{G}$, and on which $\bar{f}$ also acts naturally according to the graph transform:

$$\operatorname{graph} \bar{f}_{\# g} = \bar{f}(\operatorname{graph} g) \cap (\tilde{E}^u_{fp}(\delta) \times \tilde{E}^{cs}_{fp}) \qquad \begin{array}{ccc} \tilde{\mathscr{G}} & \xrightarrow{\bar{f}_\#} & \tilde{\mathscr{G}} \\ \downarrow & & \downarrow \\ \Lambda & \xrightarrow{f} & \Lambda. \end{array}$$

Although not trivial, $\tilde{\mathscr{G}}$ is smooth, and it has a unique $\bar{f}_\#$-invariant section γ.

THEOREM 4.1. *The unique $\bar{f}_{\#}$-invariant section $\gamma\colon \Lambda \to \tilde{\mathcal{G}}$ is θ-Hölder.*

Proof. The dynamic trivialization Lemma 3.1 implies that there are smooth bundles H_1 and H_2 over Λ that trivialize $\tilde{E}^u$ and $\tilde{E}^{cs}$, and further that there are C^2 bundle isomorphisms $S_1\colon H_1 \to H_1$ and $S_2\colon H_2 \to H_2$ covering f. Fix a smooth inner product structure on $H = H_1 \oplus H_2$. After multiplying S_1 and S_2 by appropriate positive constants, we can assume that

S_1 expands H_1 much more sharply than Tf expands E^u, and
S_2 contracts H_2 much more sharply than Tf contracts E^{cs}.

TM carries a smooth Riemann structure with respect to which hypotheses (1), (3) of Theorem A hold. It is said to be *adapted* to f. Together with the chosen inner product structure on H, this gives preferred inner product structures on $\tilde{E}^u \oplus H_1$, $\tilde{E}^{cs} \oplus H_2$, and $T_\Lambda M \oplus H$. The trivial bundles $\Lambda \times \mathbb{R}^{m_1}$, $\Lambda \times \mathbb{R}^{m_2}$, $\Lambda \times \mathbb{R}^{m_1+m_2}$ carry constant, Euclidean inner product structures, but the trivializing bundle automorphisms

$$\tilde{E}^u \oplus H_1 \cong \Lambda \times \mathbb{R}^{m_1},$$

$$\tilde{E}^{cs} \oplus H_2 \cong \Lambda \times \mathbb{R}^{m_2},$$

$$T_\Lambda M \oplus H \cong \Lambda \times \mathbb{R}^{m_1+m_2}$$

need not be isometric.

To cope with this lack of isometry, we recall a fact from linear algebra. If $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ are inner products on the same finite-dimensional vector space V, then there is a canonical automorphism $Q\colon V \to V$ that sends the first inner product structure to the second, in the sense that for all $v, w \in V$,

$$\langle Qv, Qw\rangle_2 = \langle v, w\rangle_1 .$$

To find Q, note that for each $v \in V$, there is a unique $v' \in V$ such that for all $w \in V$,

$$\langle v', w\rangle_2 = \langle v, w\rangle_1 .$$

The mapping $T\colon v \mapsto v'$ is an automorphism of V, which is easily seen to be positive definite symmetric with respect to the inner product $\langle\ ,\ \rangle_2$. Set $Q = \sqrt{T}$, the unique positive definite symmetric square root of T. Then, for all v, $w \in V$,

$$\langle Qv, Qw\rangle_2 = \langle Q^2 v, w\rangle_2 = \langle Tv, w\rangle_2 = \langle v, w\rangle_1 .$$

Applying this fact from linear algebra fiber-by-fiber gives bundle automorphisms

$$\begin{array}{ccc} \Lambda \times \mathbb{R}^{m_1} & \xrightarrow{a_1} & \widetilde{E}^u \oplus H_1 \\ \downarrow & & \downarrow \\ \Lambda & \xrightarrow{id} & \Lambda \end{array} \qquad \begin{array}{ccc} \Lambda \times \mathbb{R}^{m_2} & \xrightarrow{a_2} & \widetilde{E}^{cs} \oplus H_2 \\ \downarrow & & \downarrow \\ \Lambda & \xrightarrow{id} & \Lambda \end{array}$$

that carry the Euclidean inner product structures to the preferred inner product structures. Since all the inner product structures are smooth, so are the automorphisms a_1, a_2. Set $a = a_1 \oplus a_2$, and define F by commutativity of

$$\begin{array}{ccc} \Lambda \times \mathbb{R}^{m_1+m_2}(\delta) & \xrightarrow{F} & \Lambda \times \mathbb{R}^{m_1+m_2} \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\ T_\Lambda M(\delta) \oplus H & \xrightarrow{\bar{f} \oplus S} & T_\Lambda M \oplus H \\ \downarrow & & \downarrow \\ \Lambda & \xrightarrow{f} & \Lambda . \end{array}$$

F is C^2 and has the same properties with respect to the Euclidean inner product structure on $\Lambda \times \mathbb{R}^{m_1+m_2}$ that $\bar{f} \oplus S$ has with respect to the preferred inner product structure on $T_\Lambda M \oplus H$.

Thus writing $F_p = F(p, \ )$ with respect to $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ gives the Taylor expression

$$F_p(z) = \begin{bmatrix} \bar{A}_p & 0 \\ 0 & \bar{K}_p \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + r_p(z), \tag{9}$$

where $z = (x, y)$, $\bar{A}_p = A_p \oplus S_1$, $\bar{K}_p = K_p \oplus S_2$, and r_p is the remainder. Since F is C^2 and sends the zero section $\Lambda \times 0$ to itself, we get an estimate

$$\frac{|F_p(z) - F_q(z)|}{|z|} \leqslant \operatorname{Lip}(DF)\, d(p, q). \tag{10}$$

The C^1 mean value theorem gives $F_p(z) = F_p(z) - F_p(0) = \int_0^1 (DF_p)_{tz}\, dt\, z$, and so

$$\frac{|F_p(z) - F_q(z)|}{|z|} \leqslant \frac{\int_0^1 \|(DF_p)_{tz} - (DF_q)_{tz}\|\, dt\, |z|}{|z|} \leqslant \operatorname{Lip}(DF)\, d(p, q).$$

When the point $z \in \mathbb{R}^{m_1+m_2}$ has norm $<\delta$, δ is small, and $\tilde{E}^u$, $\tilde{E}^{cs}$ closely approximate E^u, E^{cs}, we see that the C^1-size of the remainder is small, say $\|Dr_p\| < \varepsilon$. (The remainder absorbs the off-diagonal linear terms B_p, C_p, but these terms are C^1-small when $\tilde{E}^u$, $\tilde{E}^{cs}$ closely approximate E^u, E^{cs}. On the other hand, although finite, the Lipschitz constant of DF grows large as $\tilde{E}^u$, $\tilde{E}^{cs}$ more and more closely approximate E^u, E^{cs}.)

The Banach space $G = \{g \in C^0(\mathbb{R}^{m_1}(\delta), \mathbb{R}^{m_2}): |g|_* < \infty\}$, equipped with the special norm $|g|_* = \sup |g(x)/|x||$ as above, contains the closed, bounded set

$$Y = \{g \in G: \operatorname{Lip} g \leqslant 1\}.$$

By construction, $\bar{A}_p \oplus \bar{K}_p$ is partially hyperbolic with approximately the same pinching that $T_p f$ has. Thus F acts naturally on maps $g \in Y$, and we get a fiber contraction

$$\begin{array}{ccc} \Lambda \times Y & \xrightarrow{F_\#} & \Lambda \times Y \\ \downarrow & & \downarrow \\ \Lambda & \xrightarrow{f} & \Lambda \end{array}$$

where $F_\#(p, g) = (fp, g_p)$ and graph $g_p = F_p(\text{graph } g) \cap (\mathbb{R}^{m_1}(\delta) \times \mathbb{R}^{m_2})$.

We claim that the Hölder Section Theorem 3.2 applies to $F_\#$ and that consequently its unique invariant section is θ-Hölder. By construction, the bundle space $\Lambda \times Y$ is trivial and Y is a closed, bounded subset of the Banach space G. (In fact, Y is compact.) This verifies hypothesis (a) of Theorem 3.2. Since the remainder in the Taylor expansion (9) is C^1-small, (8) becomes

(11) $$F_\# \text{ has fiber constant } \doteq \|T_p^{cs} f\|/m(T_p^u f) \quad \text{and}$$
$$F_\# \text{ has base constant } \doteq m(T_p^{cs} f).$$

Together with the θ-pinching condition (3), (11) verifies hypotheses (b), (c) of Theorem 3.2. It remains to verify hypothesis (d). We actually prove more: not only is $F_\#$ Hölder, it is Lipschitz. Given $p, q \in \Lambda$ and $g \in Y$, we claim that for some constant L,

(12) $$|(F_\#)_Y(p, g) - (F_\#)_Y(q, g)|_* \leqslant L\, d(p, q).$$

The Y-component of $F_\#(p, g)$ is the function g_p referred to above. Its graph is contained in the image under F_p of the graph of g. We must show that

$$\frac{|g_p(x) - g_q(x)|}{|x|} \leqslant L\, d(p, q).$$

The formula for g_p is

$$g_p(x) = F_{2p}(h_p^{-1}(x), g(h_p^{-1}(x))) \qquad \text{and} \qquad h_p(x) = F_{1p}(x, g(x)),$$

where $F_p(z) = (F_{1p}(z), F_{2p}(z))$ with respect to $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$. The map $h_p\colon \mathbb{R}^{m_1}(\delta) \to \mathbb{R}^{m_1}$ overflows and is an expansion since $h_p(x) = \bar{A}_p x + r_{1p}(x, g(x))$ and r_{1p} has small C^1-size. Thus $\operatorname{Lip}(h_p^{-1}) < 1$. Set $z_p = (h_p^{-1}(x), g(h_p^{-1}(x)))$ and $z_q = (h_q^{-1}(x), g(h_q^{-1}(x)))$. Then, by (10),

$$\begin{aligned} |g_p(x) - g_q(x)| &\leqslant |F_{2p}(z_p) - F_{2q}(z_p)| + |F_{2q}(z_p) - F_{2q}(z_q)| \\ &\leqslant \operatorname{Lip}(DF)\, d(p, q)|z_p| + \operatorname{Lip}(F_q)|z_p - z_q| \\ &\leqslant \operatorname{Lip}(DF)\, d(p, q)|x| + \operatorname{Lip}(F_q)|z_p - z_q|. \end{aligned}$$

Now

$$\begin{aligned} |h_p^{-1}(x) - h_q^{-1}(x)| &= |h_p^{-1} \circ h_q \circ h_q^{-1}(x) - h_p^{-1} \circ h_p \circ h_q^{-1}(x)| \\ &\leqslant \operatorname{Lip}(h_p^{-1})|h_q \circ h_q^{-1}(x) - h_p \circ h_q^{-1}(x)| \\ &\leqslant |F_{1q}(z_q) - F_{1p}(z_q)| \\ &\leqslant \operatorname{Lip}(DF)\, d(p, q)|z_q| \\ &\leqslant \operatorname{Lip}(DF)\, d(p, q)|x|. \end{aligned}$$

Since $\operatorname{Lip} g \leqslant 1$, this implies that $|z_p - z_q| \leqslant 2 \operatorname{Lip}(DF)\, d(p, q)|x|$, and

$$\frac{|g_p(x) - g_q(x)|}{|x|} \leqslant \left\{ \operatorname{Lip}(DF) + 2\left[\sup_q \operatorname{Lip}(F_q)\right] \operatorname{Lip}(DF) \right\} d(p, q).$$

This verifies (12) and hypothesis (d) of Theorem 3.2, so we conclude that the unique $F_\#$-invariant section of $\Lambda \times Y$ is θ-Hölder. The bundle $\Lambda \times Y$ contains a subbundle P consisting of pairs (p, g) such that g sends $\tilde{E}_p^u(\delta) \subset \mathbb{R}^{m_1}(\delta)$ into $\tilde{E}_p^{cs} \subset \mathbb{R}^{m_2}$. Since F leaves $T_\Lambda M$ invariant, $F_\#$ leaves P invariant. The subbundle P is closed in $\Lambda \times Y$. Thus the unique $F_\#$-invariant section of $\Lambda \times Y$ actually is a section of P.

For each $p \in \Lambda$ let $\gamma_p\colon \tilde{E}_p^u(\delta) \to \tilde{E}_p^{cs}$ be the restriction of $\sigma_{F_\#}(p)$ to $\tilde{E}_p^u(\delta)$. The map $p \mapsto \gamma_p$ is a section of bundle $\tilde{\mathscr{G}}$, and it is invariant under $\bar{f}_\#$. Hence it is the unique $\bar{f}_\#$-invariant section of $\tilde{\mathscr{G}}$. Since the restriction of a Hölder function is Hölder, γ is θ-Hölder. □

COROLLARY 4.2. *If f is partially hyperbolic at Λ and Tf satisfies the θ-pinching* *condition (3), then the local unstable manifold of f at $p \in \Lambda$ can be represented as the exponential image of the graph of a function in T_pM that depends in a θ-Hölder fashion on p.*

Proof. The local unstable manifold of f at p is $\exp_p$ graph γ_p. □

THEOREM 4.3. *If $\mathscr{L}$ is a lamination and locally the leaf of $\mathscr{L}$ can be represented as the exponential image of the graph of a function in T_pM that depends in a θ-Hölder continuous fashion on p, then the holonomy of $\mathscr{L}$ is locally uniformly θ-Hölder at small scale.*

Proof. For simplicity we first give the proof when $\mathscr{L}$ laminates all of M, i.e., when we have a foliation $\mathscr{F}$ of M by k-dimensional leaves. Fix a smooth Riemann structure on TM, fix a point $p \in M$, and use $\exp_p^{-1}$ to lift $\mathscr{F}$ to a foliation $\overline{\mathscr{F}}$ of a small neighborhood of the origin in T_pM. The leaves of $\overline{\mathscr{F}}$ are *plaques*: they are small, fairly flat, k-dimensional discs embedded in T_pM. It suffices to show that $\overline{\mathscr{F}}$ has uniformly θ-Hölder holonomy at small scale near the origin of T_pM. Split T_pM as $F \oplus F^\perp$ where F is the tangent plane to the $\overline{\mathscr{F}}$-leaf at p. Fix any $x_0 \in F(\delta)$ with $\delta > 0$ small, and express the $\overline{\mathscr{F}}$-plaque through (x_0, y) as the graph of a C^1 function, $g_y: F(\delta) \to F^\perp$. Call $q = \exp_p(x_0, y)$. Then $\exp_p$ graph g_y is a neighborhood of q in $\mathscr{F}_q$, $g_y(x_0) = y$, and $y \mapsto g_y$ is a θ-Hölder function $F^\perp(\delta) \to C^0(F(\delta), F^\perp)$.

If $x_0, x_1 \in F(\delta)$, set $V_0 = x_0 \times F^\perp(\delta)$ and $V_1 = x_1 \times F^\perp$. They are vertical transversals to $\overline{\mathscr{F}}$, and the $\overline{\mathscr{F}}$-holonomy from V_0 to V_1 is given as $h_V: y \mapsto g_y(x_1)$. By construction h_V is θ-Hölder. We also want to show that the $\overline{\mathscr{F}}$-holonomy between other transversals is θ-Hölder.

Fix a number $L > 0$. When $\delta > 0$ is small, $\sup_{x,y} \|(Dg_y)_x\| = l < 1/L$. For if $x_0 = x = y = 0$, then $Dg_y = 0$. Let $t_0, t_1: F^\perp \to F$ be C^1 functions such that $t_0(0) = x_0$, $t_1(0) = x_1$, and the norms of Dt_0 and Dt_1 are no greater than L. The graphs τ_0, τ_1 of t_0, t_1 have *coslope* $\leqslant L$. Near the origin, τ_0 and τ_1 are $\overline{\mathscr{F}}$-

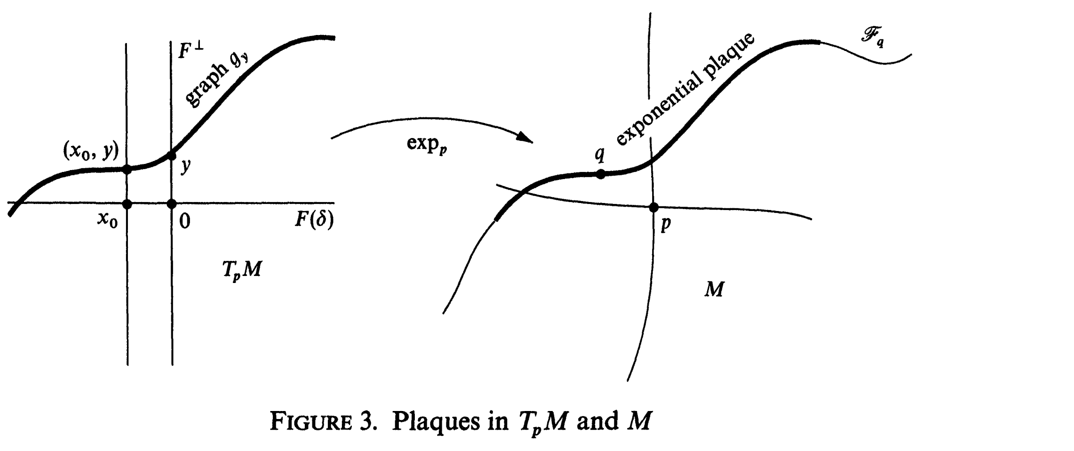

FIGURE 3. Plaques in T_pM and M

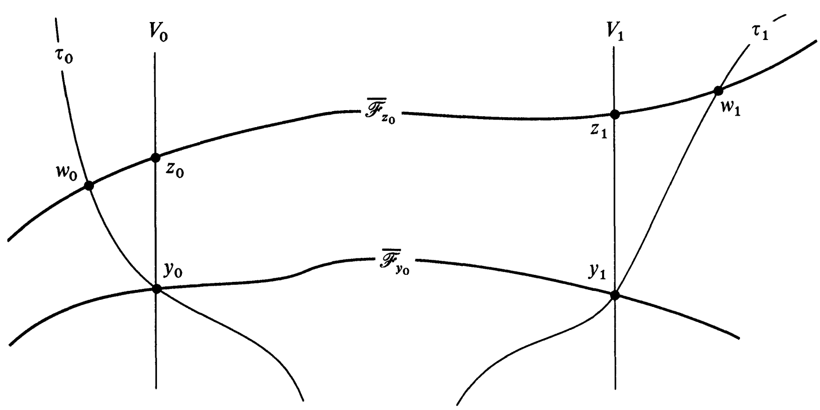

FIGURE 4. Comparing the holonomy in T_pM between vertical transversals and between transversals of coslope $\leqslant L$

transversals, and we have the $\overline{\mathscr{F}}$-holonomy map $h: \tau_0 \to \tau_1$. Let $h(y_0) = y_1$ and $h(w_0) = w_1$ for $y_0, w_0 \in \tau_0$. Let z_0, z_1 be the points where $\overline{\mathscr{F}}_{w_0}$ strikes V_0, V_1.

We know that $|z_1 - y_1| \leqslant H|z_0 - y_0|^\theta$. The ratio $|w_1 - y_1|/|z_1 - y_1|$ is bounded since the slopes of the $\overline{\mathscr{F}}$-leaves are uniformly less than the slopes of the transversals, $l < 1/L$. Similarly, $|w_0 - y_0|/|z_0 - y_0|$ is bounded away from zero. Thus,

$$|h(w_0) - h(y_0)| \leqslant H'|w_0 - y_0|^\theta$$

where H' is a multiple of H. In fact, by trigonometry, one easily shows that

$$H' = H\left(\frac{2(1 + L^2)}{1 - lL}\right)$$

suffices. Hence the $\overline{\mathscr{F}}$-holonomy from τ_0 to τ_1 is uniformly θ-Hölder.

This completes the proof of Theorem 4.3 for foliations, and we turn to the case of laminations. Let $\mathscr{L}$ be a lamination through a compact subset $\Lambda \subset M$. As before, we fix a smooth Riemann structure on TM, fix a point $p \in M$, and use $\exp_p^{-1}$ to lift $\mathscr{L}$ to a lamination $\overline{\mathscr{L}}$ through $\overline{\Lambda} = \exp_p^{-1}(\Lambda)$ in T_pM. Then we try to show that the $\overline{\mathscr{L}}$-holonomy is uniformly θ-Hölder. Split T_pM as $F \oplus F^\perp$, where F is tangent to $\mathscr{L}_p$. Just as for foliations, the natural holonomy map between vertical transversals, $V_0 \cap \overline{\Lambda} \to V_1$, is θ-Hölder, for it merely expresses the fact that (in the exponential coordinate system at p) the plaque of $\mathscr{L}$ at q is a θ-Hölder continuous function of $q \in \Lambda$.

Unfortunately, this is not completely satisfactory. For even when Λ is nontrivial, $\overline{\Lambda}$ may fail to meet V_0 except at p. Then θ-Hölderness of the $\overline{\mathscr{L}}$-holonomy $V_0 \cap \overline{\Lambda} \to V_1$ is vacuous and implies nothing about the $\overline{\mathscr{L}}$-holonomy between nonvertical transversals. Instead of Λ, consider the set $\Lambda(\rho) = \bigcup_{p \in \Lambda} \mathscr{L}_p(\rho)$, where $\mathscr{L}_p(\rho)$ is the plaque of radius ρ at p. Thus, $\mathscr{L}_p(\rho)$ is the exponential image of the graph of a function $F_p(\rho) \to F_p^\perp$, and $\Lambda(\rho)$ is a compact subset of M that contains Λ. If the leaves of $\mathscr{L}$ happen to lie in Λ (i.e., $\mathscr{L}$ laminates Λ), then $\Lambda(\rho) = \Lambda$, but in general $\Lambda(\rho)$ is strictly larger than Λ. In the hypothesis of Theorem 4.3, we assumed that "locally the leaf of $\mathscr{L}$ can be represented as the exponential image of the graph of a function in T_pM that depends in a θ-Hölder continuous fashion on p." We shall interpret this to mean that for some $\delta > 0$, $\mathscr{L}_q(\rho)$ is a θ-Hölder function of $q \in \Lambda(\delta)$. Reducing the size of ρ lets us assume that $\rho = \delta$; i.e., $q \mapsto \mathscr{L}_q(\rho)$ is a Hölder continuous function of $q \in \Lambda(\rho)$. Note that these assumptions are stronger than $q \mapsto \mathscr{L}_q(\rho)$ being θ-Hölder as q varies in Λ, and we must pay attention to this in the proofs of Corollary 4.4 and Theorem A′. For, as we show by example after the proof of Corollary 4.4, Hölder continuity of $q \mapsto \mathscr{L}_q(\rho)$ as q varies in Λ does not imply Hölder local $\mathscr{L}$-holonomy.

The set $\overline{\Lambda}(p, \nu) = \{v \in T_pM(\nu) \colon \exp_p(v) \in \Lambda\}$ is a neighborhood of p in $\overline{\Lambda}$. Its *plaque saturate* is $\overline{\Lambda}(p, \nu, \rho) = \bigcup \overline{\mathscr{L}}_v(\rho)$, where v varies in $\overline{\Lambda}(p, \nu)$. As above, let $L > 0$ be fixed, and choose

$$0 < \delta \ll \nu \ll \rho \ll 1.$$

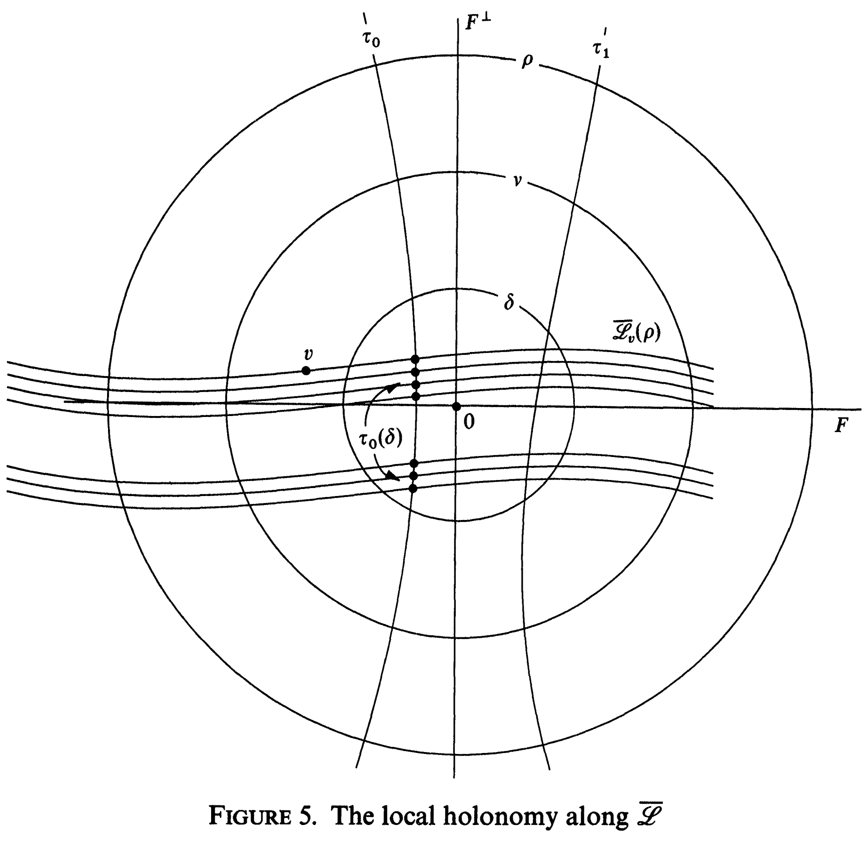

FIGURE 5. The local holonomy along $\overline{\mathscr{L}}$

Let τ_0, τ_1 be transversals to $\overline{\mathscr{L}}$ in $T_pM(v)$ that are graphs of C^1 functions t_0, $t_1: F^{\perp}(v) \to F$ whose derivatives have norm $\leqslant L$. These transversals have coslope $\leqslant L$. The set $\tau_0(\delta) = \tau_0 \cap T_pM(\delta) \cap \overline{\Lambda}(p, v, \rho)$ is a natural domain of definition of the local $\overline{\mathscr{L}}$-holonomy map h, and in fact h sends $\tau_0(\delta)$ homeomorphically into $\tau_1(v)$. For the $\overline{\mathscr{L}}$-plaques through $\tau_0 \cap T_pM(\delta)$ are part of the plaque saturate, and these plaques stretch all the way from τ_0 across τ_1. We must prove that h is uniformly θ-Hölder.

As in the foliation case, we first consider vertical transversals

$$V_0(\delta) = (x_0 \times F^{\perp}(\delta)) \cap \overline{\Lambda}(p, v, \rho)$$

$$V_1 = (x_1 \times F^{\perp}) \cap \overline{\Lambda}(p, v, \rho),$$

where $x_0, x_1 \in F(\delta)$. The $\overline{\mathscr{L}}$-plaque through $(x_0, y) \in V_0(\delta)$ contains the graph of a unique function $g_y: F(\delta) \to F^{\perp}$, and g_y is C^1 with $g_y(x_0) = y$ and $\|(Dg_y)\|$ small. Since $q \mapsto \mathscr{L}_q(\rho)$ is θ-Hölder, so is the map $y \mapsto g_y$ that sends $V_0(\delta)$ into $C^0(F(\delta), F^{\perp})$. Thus the holonomy $h: y \mapsto g_y(x_1)$ is a θ-Hölder map $V_0(\delta) \to V_1$. Then the same trigonometry argument as in the foliation case lets us pass from vertical transversals to general transversals τ_0, τ_1 in $T_pM(\delta)$ having coslope $\leqslant L$. Note that the argument is valid because the saturating plaques crossing from τ_0 to τ_1 also cross vertical transversals. □

COROLLARY 4.4. *If f is partially hyperbolic at Λ and Tf satisfies the θ-pinching condition (3), then the holonomy along the unstable manifold lamination $\mathscr{W}^u$ through Λ is θ-Hölder at small scale.*

Proof. If $\Lambda = M$ or $\mathscr{W}^u$ laminates Λ, the corollary is immediate from the theorem, for, by Corollary 4.2, $p \mapsto W^u(p, \rho)$ is a θ-Hölder function of $p \in \Lambda$. On the other hand, if Λ is a proper subset of M, we must show that $q \mapsto W^u(q, \rho)$ is a θ-Hölder function of $q \in \Lambda(\rho)$. This is not hard. The set $\Lambda(\rho) = \bigcup_{p \in \Lambda} W^u(p, \rho)$ is compact and f overflows it, $f(\Lambda(\rho)) \supset \Lambda(\rho)$. In the proof of Theorem 4.1, we could just as well have worked with an overflowing base map, instead of a base homeomorphism. For Theorem 3.2 is valid in this generality. Thus Corollary 4.2 is true on $\Lambda(\rho)$: the local unstable manifold of f at $q \in \Lambda(\rho)$ can be represented as the exponential image of the graph of a function in T_qM that depends in a θ-Hölder fashion on $q \in \Lambda(\rho)$. Corollary 4.4 then follows from Theorem 4.3. □

Example. If we only assume that $\mathscr{L}$ is C^θ at Λ, rather than on $\Lambda(\rho)$, the holonomy maps may fail to be C^θ. Let $I = (-1/2, 3/2)$, $J = (-3/5, 3/5)$, and $K = (-1/5, 1/5)$. Set

$$g(x, y) = [4x^2 + (1 - 4x^2)e^{-1/y^2}]y,$$

and define $G: I \times J \to \mathbb{R}^2$ by $G(x, y) = (x, g(x, y))$. We observe that G is a homeomorphism to its image U, and U includes the rectangle $I \times K$. Consider the lamination $\mathscr{L}$ of U whose leaves are the G-images of the horizontal lines

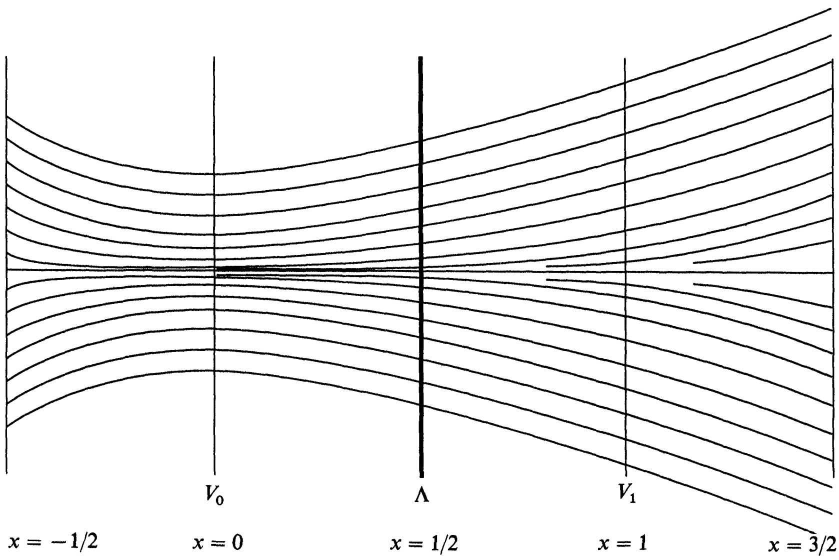

FIGURE 6. Overpinched holonomy

$y = \text{const}$. In fact, $\mathscr{L}$ is a foliation of U. Its properties are

(i) the leaf of $\mathscr{L}$ through $(1/2, y)$ is the graph of $x \mapsto g(x, y)$;
(ii) $y \mapsto g(., y)$ is a C^θ (in fact, C^∞) map $K \to C^0(I, \mathbb{R})$;
(iii) $x \mapsto g(x, y)$ is uniformly smooth with respect to x;
(iv) the holonomy map h: $V_0 \to V_1$ fails to be C^θ for all $\theta > 0$.

The transversals are $V_0 = 0 \times K$ and $V_1 = 1 \times \mathbb{R}$. The holonomy map

$$h_0: 1/2 \times K \to V_0$$

is $y \mapsto ye^{-1/y^2}$, a smooth homeomorphism which is infinitely flat at $y = 0$. The holonomy map $h_1: 1/2 \times K \to V_1$ is $y \mapsto (4 - 3e^{-1/y^2})y$, a diffeomorphism. Thus, the holonomy map h: $V_0 \to V_1$ is $h = h_1 \circ h_0^{-1}$. It has infinitely steep graph at $y = 0$, and for all $\theta > 0$, it fails to be θ-Hölder.

Proof of Theorem A'. It only remains to show that small scale θ-Hölder holonomy implies θ-Hölder holonomy at unit scale. For general foliations, this may well be false, since the composition of an α-Hölder map and a β-Hölder map is only $\alpha\beta$-Hölder. In the case at hand, the holonomy is invariant under the partially hyperbolic diffeomorphism f. A high iterate f^N transforms plaques $W^u(p, \rho)$ to unit plaques, and the small scale θ-Hölder holonomy becomes unit scale θ-Hölder holonomy. □

5. Proof of Theorem B. From §2 we know that Theorem B becomes Theorem B' when the diffeomorphism f is assumed to be partially hyperbolic at a compact invariant subset $\Lambda \subset M$, instead on all of M, so it suffices to prove Theorem B'. A hypothesis of Theorem B' is that the center plane field E^c integrates to an

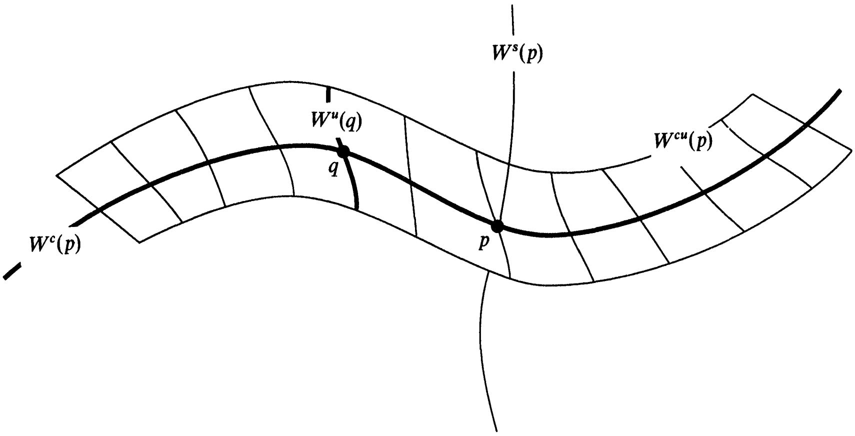

FIGURE 7. The foliation of $W^{cu}(p)$ by strong unstable manifolds $W^u(q)$

invariant lamination $\mathcal{L}^c$ of Λ. The leaves of $\mathcal{L}^c$ lie in Λ, and f is 1-normally-hyperbolic at $\mathcal{L}^c$. In [10, §§6, 7] it is shown that through the leaves of a 1-normally-hyperbolic lamination there pass unique, f-invariant C^1 leaf-immersed submanifolds W^{cu}. Existence of this family of center unstable manifolds W^{cu} is true regardless of whether E^{cu} integrates to a lamination $\mathcal{L}^{cu}$. Besides, each W^{cu} is foliated by strong unstable manifolds $W^u(q)$, $q \in W^c(p)$, and W^{cu} is tangent to E^{cu}_q at q.

We are trying to show that the subfoliation of W^{cu} by the strong unstable manifolds is uniformly C^1. The tangent bundle of W^{cu} is only C^0, so it is foolish to hope for a C^1 section of this bundle that is tangent to the strong unstable manifolds.[1] Instead of focusing attention on the tangent bundle to $\mathcal{W}^u$, we employ the method of §4, and construct directly the leaves of $\mathcal{W}^u$ via the C^1 invariant section theorem. This theorem, proved in [10] and [14], is more standard than the Hölder Invariant Section Theorem 3.2.

Think of $W^{cu}(p)$ as the base space, and the fiber at $z \in W^{cu}(p)$ as $\tilde{\mathcal{G}}_z$ where $\tilde{\mathcal{G}}_z$ is the set of functions $\tilde{E}^u_z(\delta) \to \tilde{E}^{cs}_z$ described in the proof of Theorem 4.1. The base is contracted at worst by $m(T^c_z f)$, f overflows the base, and the fiber is contracted by $(m(T^u_z f))^{-1} \| T^c_z f \|$. Center bunching implies that the fiber contraction dominates the first power of the base contraction, and so the resulting invariant section is a C^1 function of $z \in W^{cu}$. Thus the leaves of $\mathcal{W}^u|_{W^{cu}}$ are uniformly C^1, and they depend in a C^1 fashion on their centerpoint z. According to the following theorem, which is merely Theorem 4.3 in the C^1 world, this implies that the subfoliation of W^{cu} by $\mathcal{W}^u$ is C^1.

THEOREM 5.1. *If $\mathcal{F}$ is a foliation and locally the leaf of $\mathcal{F}$ can be represented*

[1] If f were 2-normally-hyperbolic at $\mathcal{L}^c$ instead of 1-normally-hyperbolic, then W^{cu} would be C^2 and its tangent bundle could support a C^1 subbundle. In fact, in this case, the restriction of E^u to W^{cu} is indeed C^1, and we get a second proof that $\mathcal{W}^u$ is a C^1 subfoliation of W^{cu}.

as the exponential image of the graph of a function in T_pM that depends in a C^1 fashion on p, then the holonomy of $\mathscr{F}$ is locally uniformly C^1.

Proof. This is the same as the proof of Theorem 4.3. See also Theorem 6.1. □

6. Regularity of foliations. It is widely agreed that a *topological foliation* $\mathscr{F}$ is a division of a manifold M into disjoint subsets called *leaves* of the foliation and denoted $\mathscr{F}_p$, such that

Each $\mathscr{F}_p$ is an injectively immersed k-dimensional (connected) manifold and $p \in \mathscr{F}_p$.

$\mathscr{F}$ is *locally trivial* in the sense that each $p \in M$ has a neighborhood U homeomorphic to the product of open discs by a map ϕ: $D^k \times D^{m-k} \to U$, such that $\phi(D^k \times y) \subset \mathscr{F}_q$, where $q = \phi(0, y)$ and $y \in D^{m-k}$.

The map ϕ is a *foliation chart* of $\mathscr{F}$. Local triviality amounts to saying that near a point p, $\mathscr{F}$ looks like a stack of pancakes. By invariance of domain, the set $\phi(D^k \times y)$ is a neighborhood of q in $\mathscr{F}_q$, and k is independent of p.

A simple example of a foliation is the product foliation of the cylinder $S^1 \times W$ by the copies of W, $\theta \times W$, for $\theta \in S^1$ and W compact. The next simplest example is the irrational foliation of the 2-torus by lines whose slope is a fixed irrational number. Each leaf in the first foliation is compact. Each leaf in the second is dense. It is not hard to see, even at the topological level, that the leaves are as complete as M. By local triviality, you cannot travel to the edge of a leaf without getting to the edge of the manifold.

Now we turn to the question of regularity. When is a foliation $\mathscr{F}$ of class C^r, $r \in \mathbb{N}$? Here general agreement is harder to come by. Three natural variants of the definition exist:

(a) the leaves are tangent to a C^r plane field;

(b) the foliation charts are C^r diffeomorphisms;

(c) the leaves and the local holonomy maps along them are uniformly C^r.

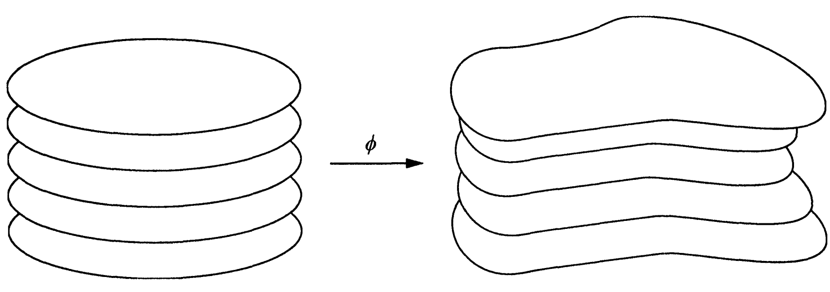

FIGURE 8. The foliation chart ϕ

TABLE 1

general r, $r \geqslant 1$	(a) $\underset{\not\Leftarrow}{\Rightarrow}$ (b) $\underset{\not\Leftarrow}{\Rightarrow}$ (c)
general $r + \theta$, $r \geqslant 1$ and $0 < \theta \leqslant \mathrm{Lip}$	(a) $\underset{\not\Leftarrow}{\Rightarrow}$ (b) $\Leftrightarrow$ (c)
$r = 1$	(a) $\underset{\not\Leftarrow}{\Rightarrow}$ (b) $\Leftrightarrow$ (c)
$r = \infty$	(a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c)

Note that (a), (b), (c) continue to make sense when the positive integer r is replaced by $r + \theta$ and $0 \leqslant \theta \leqslant \mathrm{Lip}$. The relations among (a), (b), (c) are summarized in Table 1. After discussing these implications, we go on to analyze leaf uniqueness and foliations with mixed differentiability.

To fix terminology, we focus on (b) as the natural concept of a C^r foliation: by definition the foliation $\mathscr{F}$ is of *class* C^r if and only if M can be covered by C^r foliation charts.

The origin of (a) is Frobenius's theorem; see [1, pp. 93–95]. It states that if $p \mapsto F_p$ is a C^r k-plane field on M (i.e., a C^r section of the Grassmannian G^kM), if $r \geqslant 1$, and if F is *involutive* in the sense that it is closed under Lie brackets, then through each point p there passes a unique integral manifold, and together the integral manifolds C^r foliate M. That is, (a) $\Rightarrow$ (b). An *integral manifold* of a k-plane field is an injectively immersed k-dimensional submanifold $V \subset M$ everywhere tangent to F. For each $p \in V$, $T_pV = F_p$. The submanifold V must be maximal in the sense that it is part of no larger (connected) submanifold tangent to F. The topology of V is permitted to differ from its induced topology as a subset of M, as is the case for the irrational foliation of the 2-torus.

The *tangent bundle* to $\mathscr{F}$, $T\mathscr{F}$, is the plane field F. It certainly exists when $\mathscr{F}$ is C^r, $r \geqslant 1$, and it also exists if the leaves of $\mathscr{F}$ are differentiable. If $\mathscr{F}$ is a foliation with differentiable leaves and its tangent bundle is continuous, we call $\mathscr{F}$ an *integral foliation.* All foliations in hyperbolic dynamics are integral foliations, as are the foliations discussed in §§1 through 5.

Clearly, each integral manifold of a C^r k-plane field is of class C^{r+1}, $r \geqslant 0$, and so (b) implies (a) if and only if $r = \infty$. According to Hart's smoothing theorem, however, if $r \geqslant 1$, then a foliation $\mathscr{F}$ with C^r foliation charts is diffeomorphic, by an ambient C^r diffeomorphism $M \to M$, to a foliation $\tilde{\mathscr{F}}$ with a C^r tangent bundle. The foliation $\tilde{\mathscr{F}}$ is just slightly smoother than $\mathscr{F}$. (See [7].) Thus, modulo a C^r change of variables, (b) does imply (a) when $r \geqslant 1$. The corresponding question in the topological category has a negative answer. There are topological foliations of smooth manifolds which are not homeomorphic to integral foliations. Take any topological manifold V such that V has no smooth structure but $S^1 \times V$ does have a smooth structure. (The existence of such manifolds in dimension four is a consequence of the work of Freedman, Donaldson, and others, as was explained to us by Andrew Casson.) The product foliation of $M = S^1 \times V$ cannot be homeomorphic to an integral foliation since the homeomorphic image of V would then have a continuous tangent bundle; hence V would have a C^1 structure, and hence V would have a smooth structure.

Next we discuss (c), uniformly C^r leaves and uniformly C^r holonomy. Intuitively, the plaque of $\mathscr{F}$ at p should be a C^r embedded disc whose r-jet depends continuously on p, and the local $\mathscr{F}$-holonomy should be a C^r diffeomorphism whose r-jet depends continuously on the transversals.

As in §4, we analyze $\mathscr{F}$ by lifting it to TM. We fix a smooth Riemann structure on TM, we fix a point $p \in M$, and we consider $\overline{\mathscr{F}} = \exp_p^{-1}(\mathscr{F})$. It foliates a neighborhood of the origin in T_pM and has the same local regularity properties as $\mathscr{F}$. The plaques of $\mathscr{F}$ lift to plaques of $\overline{\mathscr{F}}$, and the latter are represented as graphs of functions $g(.,y)\colon F_p(\delta) \to F_p^{\perp}$, where $g(0, y) = y$. The map

$$\phi\colon (x, y) \mapsto \exp_p \circ (x, g(x, y))$$

is a natural foliation chart for $\mathscr{F}$. We say that $\mathscr{F}$ has *uniformly C^r leaves* and *uniformly C^r holonomy* if the plaques of $\overline{\mathscr{F}}$ have these properties in a neighborhood of the origin in T_pM. That is,

> The plaque map $x \mapsto g(x, y)$ is C^r and its derivatives of order $\leqslant r$ with respect to x depend continuously on (x, y).
>
> The holonomy map $h\colon y \mapsto g(x, y)$ is C^r and its derivatives of order $\leqslant r$ with respect to y depend continuously on (x, y).

This shows that if $r \geqslant 1$ then (b) $\Rightarrow$ (c). For if $\mathscr{F}$ is a C^r foliation, then $g(x, y)$ is a C^r function of (x, y). Also, if $r = 1$, then (c) $\Rightarrow$ (b).

Note that the foregoing discussion of uniformly C^r holonomy makes perfect sense when $r = 0$. The r-jet of the holonomy map h is just h itself. Note too that when $r = 0$, (c) $\Rightarrow$ (b) is vacuous, for, by definition, every foliation has foliation charts that cover M.

What happens when $r = 2$? The plaques of $\overline{\mathscr{F}}$ are graphs of functions $g(x, y)$ that have jointly continuous first- and second-order partials with respect to x and jointly continuous first- and second-order partials with respect to y. This, however, does not imply that g is C^2. Mixed partials may fail to exist. That is, existence and joint continuity of g, g_x, g_{xx}, g_y, g_{yy} does not in general imply existence of g_{xy}. A counterexample is the function $\mathbb{R}^2 \to \mathbb{R}$ defined by

$$g(x, y) = \begin{cases} y + xy \log|\log(|x| + |y|)| & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0), \end{cases}$$

as can be checked by several applications of L'Hospital's rule. The foliation of $\mathbb{R}^2$ whose leaf through $(0, y)$ is the graph of $x \mapsto g(x, y)$ has uniformly C^2 leaves and uniformly C^2 local holonomy, but it is not a C^2 foliation. It has no C^2 foliation chart at the origin.

The amazing thing is that if r is replaced by $r + \theta$ where $0 < \theta < 1$, then the difficulties disappear. According to Lemma 2.3 of de la Llave, Marco, and Moriyon [5], as improved by Journé [11], if a function $g(x, y)$ is uniformly $C^{r+\theta}$ with respect to x and also uniformly $C^{r+\theta}$ with respect to y, then it is jointly $C^{r+\theta}$. Pure x and y derivatives do give rise automatically to mixed derivatives! Consequently, we have the following theorem.

THEOREM 6.1. *About uniformly regular foliations we know the following:*

(i) *if $r \geqslant 1$ is an integer and $0 < \theta < 1$, then a foliation that has uniformly $C^{r+\theta}$ leaves and uniformly $C^{r+\theta}$ local holonomy is a $C^{r+\theta}$ foliation;*
(ii) *a foliation with uniformly C^∞ leaves and uniformly C^∞ local holonomy is C^∞;*
(iii) *a foliation with uniformly C^1 leaves and uniformly C^1 local holonomy is C^1;*
(iv) *a foliation with uniformly C^2 leaves and uniformly C^2 local holonomy is not necessarily C^2.*

The assumption that $\mathscr{F}$ has *uniformly C^r* local holonomy is important. It is quite possible for a foliation to have uniformly C^∞ leaves and C^∞ local holonomy, yet fail to be C^1. Just draw a foliation in $\mathbb{R}^2$ whose leaves are pictured below. The bumps tend to 0 in each C^r norm, so the leaves are uniformly C^∞. The local holonomy maps between vertical transversals are all C^∞. Indeed, every holonomy between C^∞ transversals, not necessarily vertical transversals, is C^∞. But the holonomy maps are not uniformly C^1, and there is no C^1 foliation chart for $\mathscr{F}$ at the origin.

This completes our analysis of the relationships between the three possible definitions of the C^r regularity of a foliation. Next we discuss the extent to which the leaves of a foliation are unique, in analogy to the uniqueness property of solutions to ordinary differential equations.

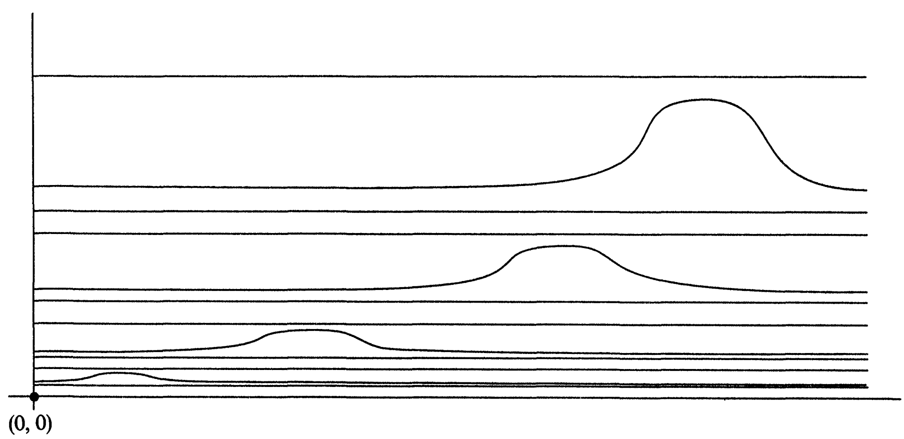

FIGURE 9. A foliation whose leaves are uniformly C^∞ and whose holonomy is C^∞, but which does not have uniformly C^1 holonomy and is therefore not a C^1 foliation

Let $\mathscr{F}$ be an integral foliation. We say that $T\mathscr{F}$ is *uniquely integrable* if each differentiable curve γ everywhere tangent to $T\mathscr{F}$ lies wholly in a leaf of $\mathscr{F}$. It cannot travel from leaf to leaf.

PROPOSITION 6.2. *If $r \geqslant 1$ and $\mathscr{F}$ is C^r, then $T\mathscr{F}$ is uniquely integrable.*

Proof. The issue is local, and we can examine γ in a foliation chart, since the chart is C^r, $r \geqslant 1$. The path γ will still be tangent to $T\mathscr{F}$, but in the chart $T\mathscr{F}$ is constantly the horizontal plane field, so γ too is horizontal. It stays in its leaf. □

As Anosov showed, the stable and unstable foliations of a smooth totally hyperbolic system are uniquely integrable despite the fact that they are not usually C^r, $r \geqslant 1$. Unique integrability is also valid for the strong stable and strong unstable foliations of a partially hyperbolic diffeomorphism, as can be shown by similar dynamical means. Unique integrability of the center foliation $\mathscr{W}^c$, when $\mathscr{W}^c$ exists, is an open question.

An equally natural definition of unique integrability might seem to be that any injectively immersed k-dimensional manifold $V \subset M$ that is everywhere tangent to the k-plane field $T\mathscr{F}$ is contained in a leaf of $\mathscr{F}$. Unique integrability in the first sense clearly implies unique integrability in the second, but the converse fails. An example is constructed as follows. Let $\mathscr{D}$ be the diagonal foliation in $\mathbb{R}^3$ whose leaves are the planes $z = x - x_0$, and let $\mathscr{F}$ be the image of $\mathscr{D}$ under the smooth homeomorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^3$, defined by

$$h(x, y, z) = (x, y, z^3 + y^2 z).$$

Under h, all points of the (x, y)-plane stay fixed and

$$Dh = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2yz & 3z^2 + y^2 \end{bmatrix}.$$

Except at the x-axis, Dh is nonsingular. Tangent to the $\mathscr{D}$-leaf at (x, y, z) is the span of $e_1 + e_3$ and e_2, so tangent to the $\mathscr{F}$-leaf at $p = h(x, y, z)$ is the span of $e_1 + (3z^2 + y^2)e_3$ and $e_2 + 2yze_3$. This plane depends continuously on p, so $T\mathscr{F}$ exists and is continuous. Off the x-axis, $\mathscr{F}$ is smooth. Since each leaf is tangent to the x-axis, $\mathscr{F}$ fails to be uniquely integrable in the first sense. The x-axis travels from leaf to leaf. On the other hand, suppose that V is an injectively immersed open 2-disc in $\mathbb{R}^3$ that is tangent to $T\mathscr{F}$. Since $T\mathscr{F}$ is never vertical (i.e., it never contains the vector e_3), V is transverse to the (x, z)-plane Π. The intersection $V \cap \Pi$ consists of at most countably many curves γ, and each lies in the closure of two connected components of $V_0 = V \backslash (V \cap \Pi)$. Each component of V_0 is everywhere tangent to $T\mathscr{F}$ and is disjoint from the x-axis, so it lies wholly in a leaf of $\mathscr{F}$. The same is true of its closure, so each γ lies wholly in an $\mathscr{F}$-leaf. Thus V lies wholly in an $\mathscr{F}$-leaf. Any injectively immersed surface is built from overlapping injectively immersed 2-discs, so $\mathscr{F}$ is uniquely integrable in the second sense, but not the first. Therefore, it is natural to define unique integrability of a foliation to mean that a curve tangent to leaves lies wholly in a leaf.

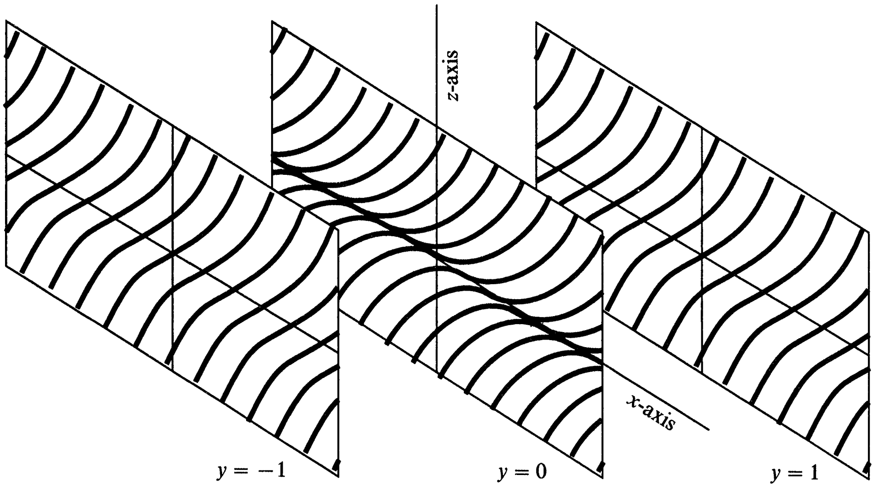

FIGURE 10. The trace of a nonunique foliation on three (x, z)-planes

Finally, we consider foliations with mixed differentiability. If a foliation has uniformly C^r leaves and uniformly C^s holonomy, we say that it is of *class* $C^{r\wedge s}$. We always assume $r \geqslant 1$; that is, $\mathscr{F}$ is an integral foliation. The symbol $r \wedge s$ is meant to suggest "C^r in x *and* C^s in y." We can restate the conclusions of Theorems A and B as follows:

(A) $\mathscr{W}^u$ and $\mathscr{W}^s$ are foliations of class $C^{1\wedge\theta}$;
(B) $\mathscr{W}^u$ $C^{1\wedge 1}$ subfoliates each $\mathscr{W}^{cu}$-leaf W^{cu}, and
$\mathscr{W}^s$ $C^{1\wedge 1}$ subfoliates each $\mathscr{W}^{cs}$-leaf W^{cs}.

In fact, if the partially hyperbolic diffeomorphism f is C^r, $r \geqslant 2$, then it is not hard to see that (A), (B) can be improved to

($\tilde{\mathrm{A}}$) $\mathscr{W}^u$ and $\mathscr{W}^s$ are foliations of class $C^{r\wedge\theta}$;
($\tilde{\mathrm{B}}$) $\mathscr{W}^u$ $C^{r\wedge 1}$ subfoliates each $\mathscr{W}^{cu}$-leaf W^{cu}, and
$\mathscr{W}^s$ $C^{r\wedge 1}$ subfoliates each $\mathscr{W}^{cs}$-leaf W^{cs}.

That is, the leaves of $\mathscr{W}^u$ and $\mathscr{W}^s$ are uniformly C^r. If, in addition, f satisfies lth-order center bunching conditions, $1 \leqslant l \leqslant r$ (i.e., f is l-normally-hyperbolic at the center foliation), then ($\tilde{\mathrm{B}}$) can be further improved to

($\tilde{\tilde{\mathrm{B}}}$) $\mathscr{W}^u$ $C^{r\wedge l}$ subfoliates each $\mathscr{W}^{cu}$-leaf W^{cu} and
$\mathscr{W}^s$ $C^{r\wedge l}$ subfoliates each $\mathscr{W}^{cs}$-leaf W^{cs}.

Not only do foliations with a low degree of regularity occur naturally in smooth nonlinear dynamics, they have basic features distinct from smooth foliations. An interesting case in point is the contrast between results of Bill Thurston [15] and Raoul Bott [3]. If E is a continuous k-plane field contained in the tangent bundle of a compact manifold, then one can ask whether E is homotopic to a plane field tangent to the leaves of a foliation. Is E integrable modulo a homotopy? If $k = 1$, the answer is obviously "yes": since E is a line field, it can be approximated by a smooth line field, and the latter integrates to a smooth 1-dimensional foliation.

Also, any approximation to E is homotopic to E. On the other hand, if $2 \leqslant k \leqslant \dim M - 2$, then Thurston's answer is "always," while Bott's is "not unless certain Pontrjagin classes vanish"—the difference being that Bott's foliation is of class C^2, while Thurston's has smooth leaves but is not transversally smooth. An outstanding question in hyperbolic dynamics is whether every Anosov diffeomorphism of a compact manifold is conjugate to one of the known linear examples. The stable and unstable foliations of these linear examples are C^∞. A first step in finding a new example might be in constructing a pair of transverse foliations to serve as its stable and unstable manifold foliations, and doing so in a way that Thurston's criteria are met, but Bott's are not.

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