MECHANISMS FOR CHAOS

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Introduction

The term "chaos" refers to highly unpredictable behavior in a closed, deterministic system, but there is no agreed-upon mathematical definition of a chaotic dynamical system. Certainly we have a feel for what chaos means: a typical orbit fills the phase space, a small error in initial condition can have disastrous consequences in the long run, and orbits behave in some sense randomly. Without attempting to fix a definition of chaos, we will focus here on families of dynamical systems that are typically chaotic and the mechanisms behind this chaotic behavior.

A *mechanism* for a dynamical behavior has three interrelated features:

- it is based on rough, geometric features of the system and as little *a priori* information about the actual dynamics as possible;
- it is verifiable in specific examples; and
- it is robust, persistent under perturbations of the system.

After giving some background, we will give some examples of mechanisms in dynamics, some of them going back to the origins of the subject and other discovered quite recently. Some of the writing that follows is adapted from other works, some of it collaborative. The author thanks her collaborators Artur Avila, Christian Bonatti, Sylvain Crovisier and Lorenzo Díaz for permission to include some of this discussion, which appears in Sections 7.2 and 8.

1. Smooth dynamics

The field of dynamics (or *dynamical systems*) has its roots in the study of physical systems whose evolution over time is dictated by some family of ordinary differential equations:

$$\frac{dx}{dt} = F(x); \qquad x \in M, \tag{1}$$

Where x is a point in the *phase space* M, and F is a smooth vector field on M.

Under suitable uniformity assumptions on the vector field F, which hold for example when M is compact, the equation (1) can be solved uniquely for all time, for any initial condition $x_0 \in M$; or what is the same, there exists a map $\varphi\colon M \times \mathbb{R} \to M$ with the following properties:

(1) $\varphi^t := \varphi(\cdot, t)$ is a *flow*: $\varphi^0(x) = x$, for all $x \in M$, and $\varphi^{s+t} = \varphi^s \circ \varphi^t$, for all $s, t \in \mathbb{R}$.
(2) For every $x_0 \in M$ and $t_0 \in \mathbb{R}$, we have

$$\left.\frac{d\varphi^t(x)}{dt}\right|_{(x,t)=(x_0,t_0)} = F(\varphi^{t_0}(x_0)).$$

The family of diffeomorphisms $\{\varphi^t : t \in \mathbb{R}\}$ form a continuous-time dynamical system: for $x_0 \in M$, the point $\varphi^t(x_0)$ represents the location of the point x_0 at time t. The flow condition in (1) means that the rules for time evolution (given by the vector field F) do not depend on time.

A key family of examples comes from Hamiltonian mechanics. Here the phase space is a symplectic manifold (N, ω), such as $\mathbb{R}^{2n} := \{(x_1, \ldots, x_n, y_1, \ldots, y_n)\}$ with the standard symplectic form $\omega = \sum_j dx_j \wedge dy_j$, and the vector field F is generated by a smooth function $H \colon N \to \mathbb{R}$ called the *Hamiltonian*, as follows: F is the unique solution to the equation

$$\omega(F, \cdot) = dH(\cdot). \tag{2}$$

In the case where $N = \mathbb{R}^{2n}$ with the standard form ω, we recover from the definition (2) the classical Hamilton equations:

$$\frac{dx_j}{dt} = \frac{\partial H}{\partial y_j}, \quad \frac{dy_j}{dt} = -\frac{\partial H}{\partial x_j}.$$

When $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ represent position and momentum coordinates of a particle in $\mathbb{R}^n$, and H is the total (kinetic and potential) energy of the system, these equations have the following interpretation. The first equation simply expresses the fact that the velocity of a particle is the derivative of its kinetic energy with respect to its momentum, and the second expresses the fact that the force exerted on the particle is the negative gradient of potential energy.

Any Hamiltonian vector field is tangent to the regular level sets $\{H \equiv c\} \subset N$, and the restriction of this vector field to a compact regular level set $M := H^{-1}(c)$ generates a flow. This flow preserves a natural volume form on M called the Liouville volume form. Since M is compact, this volume form determines a finite measure vol on M, which one can normalize so that $\mathrm{vol}(M) = 1$. In the case where $N = \mathbb{R}^{2n}$ with the standard symplectic form, this Liouville volume form is simply the contraction of the standard volume form $dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n$ on $\mathbb{R}^n$ with the gradient vector field ∇H.

Dynamics is concerned with the properties of the *orbits* $\mathcal{O}(x_0) := \{\varphi^t(x_0) : t \in \mathbb{R}\}$ of a flow, viewed both as an ensemble and for individual values of x_0. One asks the following questions, for example:

- Are there *periodic orbits*, that is points $x_0 \in M$ such that for some $t_0 \neq 0$, we have $\varphi^{t_0}(x_0) = x_0$. For such x_0, the orbit $\mathcal{O}(x_0)$ is compact, diffeomorphic to a circle. Are there infinitely many such orbits? Are they dense in the phase space M?
- In general, what is the structure of the orbit closures $\overline{\mathcal{O}(x_0)}$? Can they equal M?
- What other compact, φ^t-invariant sets are there, and do they carry additional structures?

- What are the *statisitical* properties of the orbits? For example, if φ^t is generated by a Hamiltonian vector field, then how is the orbit of a *typical* point x_0 distributed with respect to the Liouville volume vol?

Thus far we have discussed flows, but more generally we can ask the same questions of a diffeomorphism $f\colon M \to M$. Here M denotes a closed (i.e. compact, without boundary) smooth manifold. For example we could take the time-t_0 map $f = \varphi^{t_0}$ of a flow φ^t, for some fixed $t_0 \in \mathbb{R}$. The time evolution given by f is discrete, defined by

$$f^n := \underbrace{f \circ \cdots \circ f}_{n \text{ times}},$$

for $n \in \mathbb{N}$, and $f^{-n} := (f^{-1})^n$. Consistent with the convention for flows, $f^n(x)$ then represents the location of the point x after n units of time. For the remainder of this paper, we will focus primarily on diffeomorphisms; there are parallel discussions for flows with similar conclusions.

We now define a few topological dynamical notions that will be used in the sequel; in the next section we discuss the statistical analysis of diffeomorphisms, known as smooth ergodic theory.

A diffeomorphism $f\colon M \to M$ is *(topologically) transitive* if there exists a point $x \in M$ such that

$$\overline{\mathcal{O}_f(x)} = M,$$

where $\mathcal{O}_f(x) := \{f^n(x) : n \in \mathbb{Z}\}$ is the orbit of x under f. A straightforward application of the Baire category theorem gives the following alternative criterion for topological transitivity.

Lemma 1. *$f\colon M \to M$ is topologically transitive if and only if for every pair of nonempty open sets $U, V \subset M$, there exists $n \in \mathbb{N}$ such that*

$$f^{-n}(U) \cap V \neq \emptyset.$$

A strengthening of transitivity and perhaps the most concrete hallmark of chaos in a dynamical system is *mixing*, which means what it sounds like: a given region of its phase space, evolved over time, eventually and permenantly intersects any other given region. Precisely, a diffeomorphism $f\colon M \to M$ is *(topologically) mixing* if for every pair of nonempty open sets $U, V \subset M$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$f^{-n}(U) \cap V \neq \emptyset.$$

Clearly topological mixing implies transitivity. We next define the analogues of topological transitivity and mixing in the *statistical* analysis of dynamical systems, known as ergodic theory.

2. Smooth ergodic theory

Smooth ergodic theory studies the dynamical properties of smooth maps from a statistical point of view. A natural object of study is a measure-preserving system (M, vol, f),

FIGURE 1. A paint mixer. This mixer, viewed as a dynamical system, is *mixing* if no matter where the two different paints are added, and in what quantities, the mixer will combine them to produce an even mixture.

where M is a closed manifold equipped with a Riemannian metric, vol is the volume measure of this metric, normalized so that $\mathrm{vol}(M) = 1$, and $f\colon M \to M$ is a diffeomorphism preserving vol, meaning that for every Borel set $B \subset M$, we have

$$\mathrm{vol}(f(B)) = \mathrm{vol}(B).^1 \tag{3}$$

Hamiltonian flows restricted to compact regular level sets give a rich class of examples; here we will focus on the following toy examples of measure-preserving systems, whose behaviors are illustrative of some general phenomena.

(1) **Isometries.** Let $f\colon M \to M$ be an isometry of a closed Riemannian manifold. Then the Riemannian volume is clearly f-invariant. A simple example to keep in mind is the translation $R_\alpha\colon \mathbb{T} := \mathbb{R}/\mathbb{Z} \to \mathbb{T}$ defined by

$$R_\alpha(x) := x + \alpha,$$

for some fixed $\alpha \in \mathbb{T}$, which is an isometry of the Euclidean metric on the circle $\mathbb{T}$.

(2) **Toral automorphisms.** Let $A \in \mathrm{SL}(n, \mathbb{Z})$, and consider the linear action of A on $\mathbb{R}^n$. Note that $A(\mathbb{Z}^n) = \mathbb{Z}^n$, since $\det(A) = 1$. Then A descends to a diffeomorphism $f_A\colon \mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{T}^n$ of the n-torus $\mathbb{T}^n$. The Lebesgue/Haar measure, which is normalized volume for the Euclidean metric on $\mathbb{T}^n$ is f_A-invariant, since the change of variables formula for Lebesgue measure gives

$$\mathrm{vol}(f_A^{-1}(B)) = \det(A) \cdot \mathrm{vol}(B) = \mathrm{vol}(B),$$

for all Borel sets B.

[1]For a general measurable map f, the definition of measure-preserving requires that $\mathrm{vol}(f^{-1}(B)) = \mathrm{vol}(B)$ for every Borel set B; when f has a measurable inverse, this is equivalent to (3).

Ergodicity of a measure-preserving system (M, vol, f) is a minimality condition: we say that (M, vol, f) is *ergodic* if for every Borel set $X \subset M$, if $f(X) = X$, then X has volume 0 or 1, meaning that it is trivial from the perspective of volume. An equivalent formulation of ergodicity highlights the similarities between ergodicity and transitivity: f is ergodic if and only if for every pair of Borel sets X and Y of positive volume, there exists $n \in \mathbb{Z}$ such that

$$\mathrm{vol}(f^{-n}X \cap Y) > 0.$$

By Lemma 1, ergodicity implies topological transitivity, since nonempty open sets have positive volume.

Another equivalent formulation of ergodicity is the property of having no square integrable invariants: (M, vol, f) is ergodic if and only if for every $\phi \in L^2(M, \mathrm{vol}, f)$,

$$\phi \circ f = \phi \implies \phi \equiv 0, \quad \mathrm{vol} - a.e. \tag{4}$$

From this formulation, using basic harmonic analysis, we can characterize completely which rotations and toral automorphisms are ergodic; one simply rewrites the equation $\phi \circ f = \phi$ in terms of the Fourier coefficients of ϕ (which is possible, due to the algebraic nature of the map f in these examples), and then determines whether this equation has nontrivial solutions. One obtains

- The rotation $R_\alpha \colon \mathbb{T} \to \mathbb{T}$ is ergodic (with respect to Lebesgue/Haar measure) if and only if α is irrational.
- The toral automorphism f_A is ergodic if and only if A has no eigenvalues that are roots of unity.

The interest in ergodicity lies beyond the definition, and goes back to Boltzmann in the 1870's in his study of ideal gases, which can be realized as Hamiltonian systems preserving a Liouville volume. Boltzmann originally hypothesized that ergodicity should hold for such systems, and his non-rigorous formulation of ergodicity was close in spirit to the following statement of the pointwise ergodic theorem for diffeomorphisms, proved by von Neumann and Birkhoff in the 1930's:

Theorem 2. *If f is ergodic with respect to volume, then its orbits are equidistributed, in the following sense: for almost every $x \in M$ (with respect to* vol*), and for any continuous function $\phi \colon M \to \mathbb{R}$:*

$$\lim_{n\to\infty} \frac{1}{n}\left(\phi(x) + \phi(f(x)) + \cdots \phi(f^{n-1}(x))\right) = \int_M \phi \, d\mathrm{vol}. \tag{5}$$

The left hand side of (5) is the average value of ϕ along the first n iterates of x, and the right hand side is the spatial average (or expectation) of ϕ; this remarkable theorem thus establishes the fundamental property of an ergodic system:

time averages = space averages.

In Boltzmann's own words, ergodicity "characterizes the probability of a state by the average time in which the system is in this state."[2]

[2]L. Boltzmann,Wissenschaftliche Abhandlungen, vol. II, p. 582

Returning to our toy examples, as noted above, any irrational rotation R_α is ergodic. In fact such a rotation has a stronger and rather special property called unique ergodicity, which is equivalent to the property that the limit in (5) exists for *every* $x \in \mathbb{T}$.[3] While unique ergodicity is a strong property, the ergodicity of irrational rotations is fragile; the ergodic map R_α can be perturbed to obtain the non-ergodic map $f_{p/q}$, where $\alpha \approx p/q$.

While irrational rotations are ergodic, they are hardly chaotic, as there is no sensitive dependence on initial conditions: if two points are at distance δ from each other, their images under R_α^n are at distance δ, for all $n \in \mathbb{Z}$.

At the opposite extreme of the rotations, in more than one sense, is the automorphism f_A of the 2-torus $\mathbb{T}^2$ induced by multiplication by the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. The dynamics of the map f_A were first discussed by René Thom, and it was affectionately nicknamed the *cat map* by Vladimir Arnol'd. It has two eigenvalues $\lambda = (3+\sqrt{5})/2 > 1$ and $\lambda^{-1} = (3-\sqrt{5})/2 < 1$; since neither is a root of 1, it is ergodic with respect to the area vol.

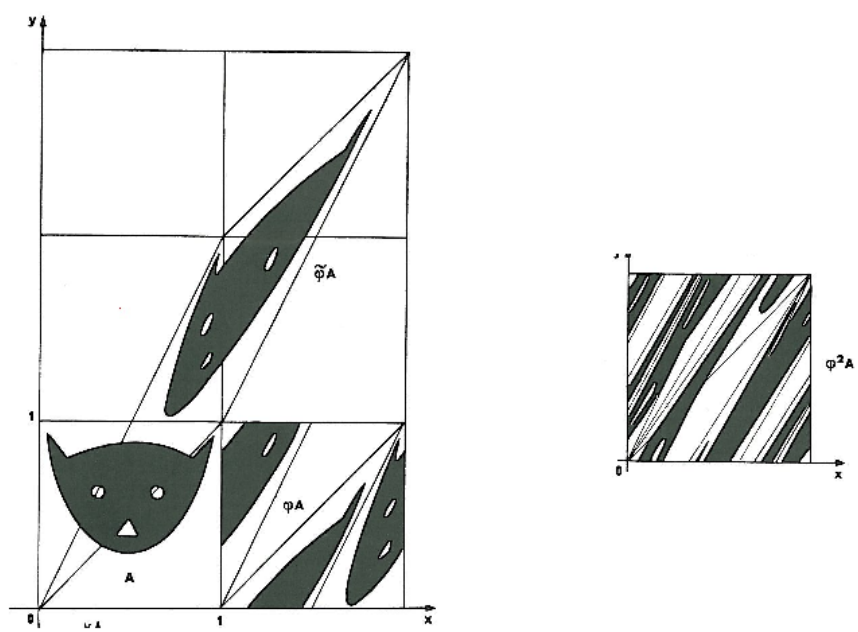


FIGURE 2. The action of f_A on a cat, from [2]. A cat is drawn in a square fundamental domain for $\mathbb{T}^2$ at the lower left. Its image under A is shown in the parallelogram, and it is reassembled into another fundamental do main to show its image under f_A. The image of the cat under f_A^2 is depicted at right.

[3]This is a consequence of Weyl's equidistribution theorem and can be proved using elementary analysis. See, e.g. [46].

This map is not only ergodic but has the stronger property of being mixing, which is the measure-theoretic analogue of topological mixing. We say that (M, vol, f) is *mixing* if for every pair of Borel sets X and Y of positive volume, we have

$$\lim_{n\to\infty} \mathrm{vol}(f^{-n}X \cap Y) = \mathrm{vol}(X)\mathrm{vol}(Y).$$

Clearly mixing implies both ergodicity and topological mixing.[4] The mixing f_A can also be proved using harmonic analysis (in particular spectral theory). Another way to prove f_A is mixing is to find a coding of almost all of the orbits by a stationary Markov process, which shows that the system is isomorphic to a Bernoulli shift, a repeated random (many-sided) coin toss.

As we shall see, f_A is also *stably ergodic*, meaning that its ergodicity cannot be destroyed by perturbations. It is even stably mixing [5], and so it is reasonable to call it robustly chaotic. While the ergodicity of f_A can be proved using Fourier analysis, this proof sheds little light on the stability of its ergodic properties: for example, there is no closed form expression for $\phi \circ f = \phi$ in terms of Fourier coefficients when f is a volume-preserving diffeomorphism close to, but not equal to, f_A. Fortunately, there is is a much more robust, geometric proof of ergodicity of f_A, which applies more generally and to perturbations, due to Anosov [1]. It is our first mechanism for chaos. This proof, a form of the *Hopf argument*, has an interesting backstory, originating in the 1930's.

2.1. **The Hopf argument.** One of the earliest arguments for proving ergodicity, still in use today, was originally employed by Eberhard Hopf in the 1930's in the context of Riemannian geometry. Hopf studied the geodesic flow over an arbitrary closed, negatively curved surface [6] and proved that it is ergodic with respect to the Liouville volume.

These geodesic flows (examples of "Anosov flows" in current terminogy) have one-dimensional invariant expanded and contracted distributions E^u and E^s (line bundles in this case), tangent to invariant foliations $\mathcal{W}^u$ and $\mathcal{W}^s$, respectively. Hopf's ergodicity argument uses Birkhoff's ergodic theorem to show that any invariant set for the flow must consist of essentially whole leaves of both $\mathcal{W}^u$ and $\mathcal{W}^s$. Invariance under the flow implies that the same is true for the foliations $\mathcal{W}^{cu}$ and $\mathcal{W}^{cs}$ formed by flowing the leaves of $\mathcal{W}^u$ and $\mathcal{W}^s$, respectively. Since the leaves of these foliations are transverse, a version of Fubini's theorem implies that every invariant set for the flow must have full measure in neighborhoods of fixed, uniform size in the manifold. Ergodicity follows from connectedness and this *local ergodicity*. The version of Fubini's theorem employed by Hopf is fairly

[4]Observe that the rotation R_α is not mixing, even when α is irrational, because it is not topologically mixing: if X and Y are small intervals, the intervals $f^{-n}(X)$ and Y will intersect for infinitely many values of n, but there are also infinitely many n with $f^{-n}(X) \cap Y = \emptyset$.

[5]In fact it is *stably Bernoulli*.

[6]Every Riemannian metric on a manifold P determines a Hamiltonian flow $\varphi^t \colon TP \to TP$ on the tangent bundle TP determined by the Hamiltonian $H(v) = \frac{1}{2}\|v\|^2$. When P is compact, the unit tangent bundle $T^1P = H^{-1}(1/2)$ is compact and invariant under this flow. The flow $\varphi^t \colon T^1P \to T^1P$ is called the *geodesic flow* because its orbits project to unit speed geodesics in P under the canonical projection $T^1P \to P$.

straightforward, since in his setting, the foliations $\mathcal{W}^{cu}$ and $\mathcal{W}^{cs}$ (and indeed, $\mathcal{W}^u$ and $\mathcal{W}^s$) are C^1.

Hopf himself foresaw the general usefulness of his methods beyond the geometric context of geodesic flows. In 1940 he wrote: "The range of applicability of the method of asymptotic geodesics extends far beyond surfaces of negative curvature. In [48], n-dimensional manifolds of negative curvature were already investigated. But the method allows itself to be applied to much more general variational problems with an independent variable, aided by the Finsler geometry of the problems. This points to a wide field of problems in differential equations in which it will now be possible to determine the complete course of the solutions in the sense of the inexactly measuring observer."[49]

Kolmogorov, too, saw the potential of the Hopf argument as a general method to prove ergodicity. In his 1954 ICM address [51], he wrote: "it is extremely likely that, for arbitrary k, there are examples of canonical systems with k degrees of freedom and with stable transitiveness [i.e. ergodicity] and mixing... I have in mind motion along geodesics on compact manifolds of constant negative curvature... " Kolmogorov's intuition was clearly guided by the robust nature of Hopf's ergodicity proof; indeed, inspected carefully with a modern eye, Hopf's original approach gives a complete proof of Kolmogorov's assertion that the geodesic flow for a hyperbolic manifold remains ergodic when perturbed within the class of Hamiltonian (or even volume-preserving) flows.[7]

Around ten years later, Anosov [1] generalized Hopf's theorem to closed manifolds of strictly negative (but far from constant) sectional curvatures, in any dimension. The key advance was to extend the Fubini part of Hopf's argument when the foliations $\mathcal{W}^u$ and $\mathcal{W}^s$ are not C^1 but still satisfy an absolute continuity property.

The Anosov/Hopf argument also gives the ergodicity of C^2, volume-preserving uniformly hyperbolic diffeomorphisms, now known as the *Anosov diffeomorphisms*. These are the diffeomorphisms $f\colon M \to M$ for which there exists a Df-invariant splitting $TM = E^u \oplus E^s$ and an integer $N \geq 1$ such that for every unit vector $v \in M$:

$$v \in E^u \implies \|Df^N v\| > 2, \quad \text{and } v \in E^s \implies \|Df^N v\| < 1/2. \tag{6}$$

An example of an Anosov diffeomorphism is the cat map f_A. The matrix A is symmetric, and so its eigenspaces E_λ and $E_{\lambda^{-1}}$ are orthogonal. Its eigenvalues and singular values coincide. Writing $\mathbb{R}^2 = E_\lambda \oplus E_{\lambda^{-1}}$, and identifying the tangent spaces $T_x\mathbb{T}^2$ with $\mathbb{R}^2$ in the natural way, we obtain a splitting of the tangent bundle

$$T\mathbb{T}^2 = \mathbb{T}^2 \times \mathbb{R}^2 = E^u \oplus E^s,$$

where $E^u(x) := \{x\} \times E_\lambda$ and $E^s(x) := \{x\} \times E_{\lambda^{-1}}$. This splitting is invariant under Df_A: for every $x \in \mathbb{T}^2$, $D_x f_A(E^u(x)) = E^u(f_A(x))$, and $D_x f_A(E^s(x)) = E^s(f_A(x))$. Finally, it is hyperbolic in the sense that for every $x \in \mathbb{T}^2$ and for every unit vector $v \in T_x\mathbb{T}^2$:

$$v \in E^u(x) \implies \|Df_A(v)\| = \lambda > 1, \quad \text{and } v \in E^s(x) \implies \|Df_A^{-1}(v)\| = \lambda > 1.$$

[7]For flows that are perturbations of constant negative curvature geodesic flows, the foliations $\mathcal{W}^{cu}$ and $\mathcal{W}^{cs}$ are C^1, which is enough to carry out the Fubini step. The same regularity of the foliations does not hold for geodesic flows on arbitrary negatively curved manifolds.

The bundle E^u is unstable, expanding by the derivative of f_A, and the bundle E^s is stable.

Let's detail the Hopf argument for the cat map f_A.

(1) The expanding and contracting subbundles E^u and E^s of the splitting $T\mathbb{T}^2 = E^u \oplus E^s$ are tangent to foliations $\mathcal{W}^u$ and $\mathcal{W}^s$ of $\mathbb{T}^2$ by immersed lines. These lines are parallel to the expanding and contracting eigendirections of A and wind densely around the torus, since they have irrational slope. The leaves of this pair of foliations are perpendicular to each other, since A is symmetric.
(2) A clever application of the pointwise ergodic theorem (presented here as a special case in Theorem 2) shows that any $\phi \in L^2(\mathbb{T}^2, \mu)$ satisfying $\phi \circ f = \phi$ is, up to a set of area 0, constant along leaves of the foliation $\mathcal{W}^u$, and (again, up to a set of area 0) constant along leaves of $\mathcal{W}^s$. This is the core of Hopf's original argument.
(3) Locally, the pair of foliations $\mathcal{W}^u$ and $\mathcal{W}^s$ are just (rotated versions of) intervals parallel to the x and y axes. In these rotated coordinates, $\phi(x, y)$ is a measurable function constant a.e. in x and constant a.e. in y. Fubini's theorem then implies that such a ϕ must be constant a.e. This conclusion holds in local charts, but since $\mathbb{T}^2$ is connected, ϕ must be constant.
(4) Since any f-invariant function $\phi \in L^2(\mathbb{T}^2, \mu)$ is constant almost everywhere with respect to μ, we conclude that f_A is ergodic with respect to μ.

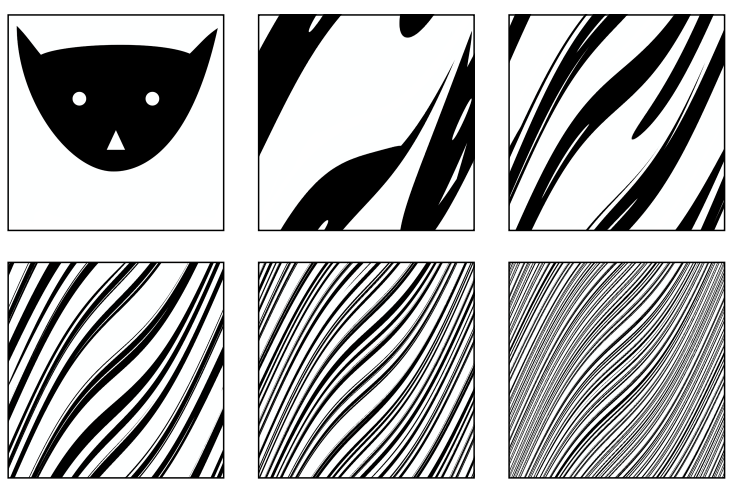


FIGURE 3. A nonlinear cat map: the first 5 iterates of

$$f_\epsilon(x, y) = (x + y, x + 2y + \epsilon \sin(2\pi(x + y))),$$

for $\epsilon = .1$. (Images by David Dumas).

These steps make it clear that the Hopf/Anosov argument is a true *mechanism* for ergodicity. It uses only the coarse information of the existence of complementary expanding and contracting dynamics on the tangent bundle and nothing *a priori* about the dynamics on M other than volume preservation. These tangent bundle dynamics are easily seen to persist under perturbations of the map that are not too large in the C^1 topology. Moreover

the existence of the complementary bundles is easily verified in examples using something called a "cone condition." For example, one can get a useful estimate on an interval of values ϵ near 0 for which the perturbation

$$f_\epsilon(x,y) = (x+y, x+2y+\epsilon\sin(2\pi(x+y)))$$

of the cat map is Anosov.

3. Stable ergodicity

We now widen our scope of vision to the space of all diffeomorphisms, a dynamical universe if you will. Let M be a closed manifold with a fixed Riemannian structure, and let vol be the Riemannian volume. Fixing $r \in [1,\infty]$, denote by $\mathrm{Diff}^r(M)$ the space of all C^r diffeomorphisms of M and by $\mathrm{Diff}^r_{\mathrm{vol}}(M)$ those diffeomorphisms in $\mathrm{Diff}^r(M)$ preserving volume. Both spaces are complete metric spaces when equipped with the C^r topology. One can also equip these spaces with the C^s topology, for any $s \in [1,r]$. If s is strictly less than r, then the C^s topology is strictly coarser than the C^r topology. For example, $\mathrm{Diff}^r(M)$ is not complete in this coarser topology. A simple rule of thumb to keep in mind is that for $s < r$, C^s open sets are larger than C^r open ones, and C^s dense sets are smaller than C^r dense ones.

We consider the dynamical properties of a C^r diffeomorphism f as it is perturbed inside of $\mathrm{Diff}^r(M)$ (or $\mathrm{Diff}^r_{\mathrm{vol}}(M)$, if f preserves volume). To this end, we say that $f \in \mathrm{Diff}^r_{\mathrm{vol}}(M)$, is *stably* (or *robustly*) ergodic if there is a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^r_{\mathrm{vol}}(M)$ such that every $g \in \mathcal{U}$ is ergodic with respect to volume. We will adopt the strongest possible convention for stability: we ask that the neighborhood $\mathcal{U}$ be C^1-open in $\mathrm{Diff}^r_{\mathrm{vol}}(M)$.

The case $r = 1$ is a bit of a mystery: there are no known examples of C^1-open sets of ergodic diffeomorphisms in $\mathrm{Diff}^1_{\mathrm{vol}}(M)$. Thus the strongest possible version of stable ergodicity where we have results is C^1-stable ergodicity inside of $\mathrm{Diff}^2_{\mathrm{vol}}(M)$, and by convention, this is what we mean when we say "stably ergodic" without qualification.

We first observe that there are no stably ergodic diffeomorphisms on the circle. Let vol be the Lebesgue measure on the circle (the analysis is similar if vol is any measure induced by a metric on $\mathbb{T}$). If f preserves vol, then f is a rotation R_α. Thus for any r, the space $\mathrm{Diff}^r_{\mathrm{vol}}(\mathbb{T})$ is just the circle $\mathbb{T}$, with the usual topology under the isomorphism sending R_α to α. As observed above, R_α is ergodic if and only if $\alpha \notin \mathbb{Q}/\mathbb{Z}$. Thus the ergodic diffeomorphisms in $\mathrm{Diff}^r_{\mathrm{vol}}(\mathbb{T})$ have empty interior (although they form a large subset in every other sense). One dimension higher, the torus $\mathbb{T}^2$ supports Anosov diffeomorphisms (such as the cat map), which are stably ergodic, the ergodicity given by the mechanism of the Hopf/Anosov argument.

Interest in stably ergodic dynamical systems dates back at least to 1954 the ICM address of Kolmogorov quoted above. In general, Anosov diffeomorphisms form a C^1-open class, and C^2, volume-preserving Anosov diffeomorphisms are ergodic, and so Anosov diffeomorphisms are stably ergodic. The general expectations of Hopf and Kolmogorov were thus met in Anosov's work. But there is more to the story: while Anosov diffeomorphisms

gave the first examples of stably ergodic systems, their existence raised the question of what other examples there might be.

Since stable ergodicity is by definition a robust property, it is natural to search for an alternate robust, geometric/topological dynamical characterization of the stably ergodic diffeomorphisms. Bonatti-Díaz-Pujals [18] proved a key result which can be slightly modified to show that stable ergodicity of $f\colon M \to M$ implies the existence of a *dominated splitting*, that is, a Df-invariant splitting $TM = E_1 \oplus E_2 \oplus \ldots \oplus E_k$, $k \geq 2$ and an integer $N \geq 1$ such that for every $x \in M$ and every pair of unit vectors $v, w \in T_xM$:

$$v \in E_i,\ w \in E_j \text{ with } i < j \implies \|Df^N v\| > 2\|Df^N w\|.$$

The splitting $TM = E^u \oplus E^s$ of an Anosov diffeomorphism is an example of a dominated splitting, and indeed Anosov diffeomorphisms are stably ergodic.

Observe that the the Bonatti-Díaz-Pujals result implies that there are no stably ergodic diffeomorphisms on surfaces other than $\mathbb{T}^2$: the existence of an invariant line bundle on a surface M forces the Euler characteristic of M to vanish. The existence of a dominated splitting turns out to be a relatively weak obstruction in higher dimensions (for example, in odd dimension, where the Euler charactistic vanishes).

4. Partial hyperbolicity

Grayson, Pugh and Shub [41] established the existence of non-Anosov stably ergodic diffeomorphisms in 1995. They considered the time-one map of the geodesic flow for a surface of constant negative curvature. These examples belong to a class of dynamical systems known as the partially hyperbolic diffeomorphisms.

The stably ergodic diffeomorphisms built by Grayson, Pugh and Shub have a dominated splitting of a special type, incorporating features of Anosov diffeomorphisms: they are *partially hyperbolic*, meaning there exists a dominated splitting $TM = E^u \oplus E^c \oplus E^s$, with both E^u and E^s nontrivial, and an integer $N \geq 1$ such that (6) holds for any unit vector v. The bundles E^u, E^c and E^s are called the *unstable, center* and *stable bundles*, respectively. Anosov diffeomorphisms are the partially hyperbolic diffeomorphisms for which E^c is trivial.

Brin and Pesin [27] and independently Pugh and Shub [58] first examined the ergodic properties of partially hyperbolic systems soon after the work of Anosov and Sinai on hyperbolic systems.

A simple example of a partially hyperbolic diffeomorphism is the product $f_A \times R_\alpha\colon \mathbb{T}^3 = \mathbb{T}^2 \times \mathbb{T} \to \mathbb{T}^3$, where f_A is the cat map on $\mathbb{T}^2$, and R_α is rotation by α on the circle $\mathbb{T}$. Writing $T\mathbb{T}^3 = T\mathbb{T}^2 \oplus T\mathbb{T}$, the partially hyperbolic splitting $T\mathbb{T}^3 = E^u \oplus E^c \oplus E^s$ is given by $E^u = E^u_{f_A} \oplus \{0\}$, $E^s = E^s_{f_A} \oplus \{0\}$, and $E^c = \{0\} \oplus T\mathbb{T}$, where $T\mathbb{T}^2 = E^u_{f_A} \oplus E^s_{f_A}$ is the Anosov splitting for f_A. One can check that this example is ergodic if and only α is irrational.

Partial hyperbolicity is a C^1-open condition: any diffeomorphism sufficiently C^1-close to a partially hyperbolic diffeomorphism is itself partially hyperbolic. For an extensive

discussion of examples of partially hyperbolic dynamical systems, see the survey article [31] and the book [56]. Among these examples are: the time-1 map of an Anosov flow, the frame flow for a compact manifold of negative sectional curvature, and many affine transformations of compact homogeneous spaces. All of these examples preserve the volume induced by a Riemannian metric on M.

As with Anosov diffeomorphisms and flows, the unstable bundle E^u and the stable bundle E^s of a partially hyperbolic diffeomorphism are uniquely integrable, tangent to the unstable and stable foliations, $\mathcal{W}^u$ and $\mathcal{W}^s$, respectively. However, they do not span TM, and thus are not transverse (except in the Anosov case). The Hopf argument for local ergodicity may fail without further assumptions: there *do* exist non-ergodic partially hyperbolic diffeomorphisms, for example the map $f_A \times \mathbf{R}_{p/q}$.

5. Accessibility

As a substitute for transversality in the Hopf argument, Grayson-Pugh-Shub [41] introduced an additional assumption, a property called "accessibility," which is a term borrowed from the control theory literature.[8] A partially hyperbolic diffeomorphism is *accessible* if any two points in the manifold can be connected by a continuous path that is piecewise contained in leaves of $\mathcal{W}^u$ and $\mathcal{W}^s$. We say that f is *stably accessible* if any (volume preserving) C^1 perturbation of f is accessible.

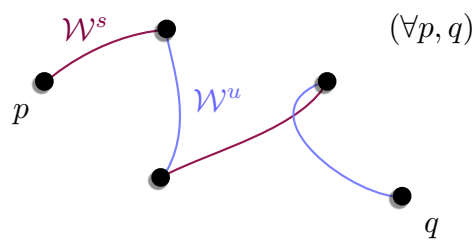


Figure 4. Accessibility: any point $q \in M$ can be reached from any point $p \in M$ by a path alternately tangent to leaves of $\mathcal{W}^s$ and $\mathcal{W}^u$.

Note that in the examples $f_A \times \mathbf{R}_\alpha$ on $\mathbb{T}^3$ discussed above, accessibility does *not* hold. Indeed the foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ tangent to E^s and E^u are also tangent to the horizontal foliation $\{T\mathbb{T}^2 \times \{\theta\} : \theta \in \mathbb{T}\}$ of $\mathbb{T}^3$ by 2-tori: starting at p, one can access only the points on the same horizontal torus as p. None of these examples is stably ergodic, either. If $\alpha = p/q$ is rational, then $f_A \times \mathbf{R}_\alpha$ is not ergodic, and if α is irrational, then $f_A \times \mathbf{R}_\alpha$, while ergodic, can be approximated arbitrarily well by a nonergodic map of the form $f_A \times \mathbf{R}_{p/q}$.

Heuristically, there is a connection between accessibility and ergodicity. The Hopf argument can be carried out in the partially hyperbolic setting, and the ergodic (and even mixing) properties of the system can be reduced to the ergodic properties of the measurable

[8] Brin and Pesin had earlier studied the accessibility property and its stability properties in the 1970's and used it to prove topological properties of conservative partially hyperbolic systems such as transitivity [26, 27].

equivalence relation generated by the pair of foliations $(\mathcal{W}^u, \mathcal{W}^s)$. However, in trying to show that these ergodic properties follow from the assumption of accessibility, one quickly encounters substantial issues involving sets of measure zero and delicate measure-theoretic and geometric properties of the foliations.

6. The Pugh-Shub conjectures

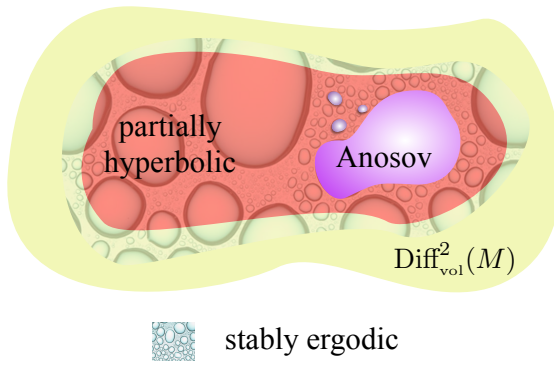


Figure 5. A graphical representation of Conjecture 1. The stably ergodic diffeomorphisms are represented by the interior of bubbles.

Motivated by their breakthrough result with Grayson [41], Pugh and Shub conjectured that accessibility implies ergodicity, for a C^2, partially hyperbolic conservative diffeomorphism [59].

Conjecture 1 (Pugh-Shub [59])**.** *On any compact manifold, ergodicity holds for an open and dense set of C^2, volume preserving partially hyperbolic diffeomorphisms.*

In light of the fact that partial hyperbolicity does not imply ergodicity, and there exist ergodic partially hyperbolic diffeomorphisms that are not stably ergodic, this is perhaps an audacious conjecture. But when accessibility is introduced as a mechanism for ergodicity, it becomes more tractable. Pugh and Shub indeed split Conjecture 1 into two parts using the concept of accessibility.

Conjecture 2 (Pugh-Shub [59])**.** *Accessibility holds for an open and dense subset of C^2 partially hyperbolic diffeomorphisms, volume preserving or not.*

Conjecture 3 (Pugh-Shub [59])**.** *A partially hyperbolic C^2 volume preserving diffeomorphism with the accessibility property is ergodic.*

6.1. Status of these conjectures.

The C^1 topology allows for enough flexibility in perturbations that Conjecture 2 has been completely verified in this context:

Theorem 3 (Dolgopyat-Wilkinson [38])**.** *For any $r \geq 1$, accessibility holds for a C^1 open and dense subset of the partially hyperbolic diffeomorphisms in $\mathrm{Diff}^r(M)$, volume-preserving or not.*

Theorem 3 also applies inside the space of partially hyperbolic symplectomorphisms.

The complete version of Conjecture 3 has been verified for systems with 1-dimensional center bundle.

Theorem 4 (Rodríguez Hertz-Rodríguez Hertz-Ures [63])**.** *For any $r \geq 1$, accessibility is C^1 open and C^r dense among the partially hyperbolic diffeomorphisms in* $\mathrm{Diff}^r_m(M)$ *with one-dimensional center bundle.*

This theorem was proved earlier in a much more restricted context by Niţică-Török [55]. The C^1 openness of accesssibility was shown in [37]. A version of Theorem 4 for non-volume preserving diffeomorphisms was later proved in [30].

The reason that it is possible to improve Theorem 3 from C^1 density to C^r density in this context is that the global structure of accessibility classes is well-understood. By *accessibility class* we mean an equivalence class with respect to the relation generated by the pair of foliations $\mathcal{W}^s, \mathcal{W}^u$. When the dimension of E^c is 1, accessibility classes are (C^1 immersed) submanifolds. Whether this is always true when $\dim(E^c) > 1$ is unknown and is an important obstacle to attacking the general case of Conjecture 2.

In this vein, a preliminary question is whether accessibility implies stable accessibility. This is the case when the center bundle E^c is 1 or 2 dimensional [37, 8] and has been announced by Burns- M. A. Rodriguez Hertz -Ures in the case of 3 dimensional center.

Conjecture 3 has been proved under the hypothesis of center bunching, which is a mild spectral condition on the restriction of Df to the center bundle E^c.

Theorem 5 (Burns-Wilkinson [32])**.** *A C^2, partially hyperbolic volume preserving diffeomorphism with the accessibility property is ergodic (in fact mixing) if it is center bunched.*

Center bunching is satisfied by most examples of interest, including all partially hyperbolic diffeomorphisms with $\dim(E^c) = 1$. The proof in [32] is a modification of the Hopf Argument using Lebesgue density points and a delicate analysis of the geometric and measure-theoretic properties of the stable and unstable foliations, using a type of dynamically-constructed set called a *julienne*. The validity of Conjecture 3 in the absence of center bunching is currently an open question.

On the other hand, if we relax the requirement on the strength of the density from C^r to C^1, we have the following rather general recent result.

Theorem 6 (Avila-Crovisier-Wilkinson [6])**.** *Stable ergodicity (mixing) is C^1-dense in the space of C^r partially hyperbolic volume-preserving diffeomorphisms on a compact connected manifold, for any $r > 1$.*

Among the class of partially hyperbolic diffeomorphisms where E^c has dimension one or two, Theorem A has been established earlier in [20] and [61]. The proof of Theorem 6 takes a different approach, using blenders, which was first employed by Hertz-Hertz-Tahzibi-Ures [61]. We will describe this proof in Section 8, but first we give some background on blenders and their role as a mechanism for robust transitivity.

7. Robust transitivity and mixing

Around the same time Grayson, Pugh and Shub found the first examples of non-Anosov, stably ergodic (mixing) diffeomorphisms, Bonatti and Díaz proved a result in the topological category of a very similar flavor. We say that a diffeomorphism $f \in \mathrm{Diff}^r(M)$ is *robustly (topologically) transitive* if there exists a (C^1-open) neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^r(M)$ such that every $g \in \mathcal{U}$ is transitive. Robust (topological) mixing is defined analogously. It is possible to be robustly transitive without preserving volume, and while ergodicity with respect to volume implies transitivity, stable ergodicity does not imply robust transitivity, as for the latter condition to hold, one must have transitivity of *all* perturbations, not just those that preserve volume. Transitive Anosov diffeomorphisms are robustly mixing (it is not know whether there exist non-transitive Anosov diffeomorphisms). This follows from the structural hyperbolic theory developed by Smale, but we shall give an idea of a different proof in the subsection below.

Until the early 1990's examples of non-Anosov, robustly transitive diffeomorphisms were few and far between. Those that had been constructed, by Shub and Mañé, were homotopic to Anosov diffeomorphisms. In a breakthrough result, Bonatti-Díaz [17] constructed stably transitive diffeomorphisms homotopic to (and aribitrarily C^∞ close to) the identity. Like the Grayson-Pugh-Shub stably ergodic examples in [41], the robustly transitive diffeomorphisms in [17] arise as perturbations of the time-t_0 map of the geodesic flow over a hyperbolic surface. Hence they are partially hyperbolic. But the mechanism for robust transitivity is entirely different from the mechanism for stable ergodicity and relies on an object they call a *blender*. To understand the role of a blender, we first prove the robust mixing of the cat map, which does not use blenders.

7.1. Robust mixing of the cat map.

The cat map f_A is mixing, as is any C^1-small perturbation (not necessarily volume-preserving) of f_A. This follows from the general theory of Anosov diffeomorphisms on tori, but we present here an elementary proof that is manifestly robust.

Here is an outline. Let E^u and E^s be the unstable and stable line fields for f_A described in Section 2.1. They are tangent to affine foliations of $\mathbb{T}^2$ by lines of irrational slope. We say that $\gamma_u\colon [0,1] \to \mathbb{T}^2$ is an *unstable curve* if $\gamma_u'(t)$ makes angle less than $\pi/4$ with $E^u(\gamma_u(t))$, for all $t \in [0,1]$. Similarly, $\gamma_s\colon [0,1] \to \mathbb{T}^2$ is a *stable curve* if $\gamma_s'(t)$ makes angle less than $\pi/4$ with $E^s(\gamma_s(t))$, for all $t \in [0,1]$.

One verifies the following facts for the map $f = f_A$:

(1) If γ_u is an unstable curve, then $f \circ \gamma_u$ an unstable curve.
(2) Similarly, if γ_s is a stable curve, then $f^{-1} \circ \gamma_s$ is a stable curve.
(3) There exists a constant $\kappa > 1$ such that if γ_u is an unstable curve, then length of the curve $f \circ \gamma_u$ is at least κ times the length of γ_u:
$$\ell(f \circ \gamma_u) \geq \kappa \ell(\gamma_u).$$
(4) Similarly, if γ_s is a stable curve, then
$$\ell(f^{-1} \circ \gamma_s) \geq \kappa \ell(\gamma_s).$$

(5) There exists $D > 1$ such that if γ_s and γ_u are stable and unstable curves with $\ell(\gamma_s), \ell(\gamma_u) > D$, then $\gamma_u[0,1] \cap \gamma_s[0,1] \neq \emptyset$.

There facts are easily seen to hold for C^1-small perturbations of f_A, and so they give the ingredients for a *mechanism*. Let f be any map satisfying properties (1)-(5). To prove that f is mixing, we fix two nonempty open sets U and V in $\mathbb{T}^2$. The set U is open and nonempty and so contains the image of a stable curve γ_s, and the set V similarly contains the image of an unstable curve γ_u. Let D and κ be given as above, and fix $N_0 \in \mathbb{N}$ such that

$$\min\{\ell(\gamma_u), \ell(\gamma_s)\} > D\kappa^{-N_0}.$$

Fix $n \geq 2N_0$, and consider the intersection $f^{-n}(U) \cap V$. This intersection is nonempty if and only if its image $f^{N_0}(f^{-n}(U) \cap V) = f^{N_0-n}(U) \cap f^{N_0}(V)$ is nonempty. The set $f^{N_0-n}(U)$ contains the image of the curve $f^{N_0-n} \circ \gamma_s$ and the set $f^{N_0}(V)$ contains the image of the curve $f^{N_0} \circ \gamma_u$. Applying properties (2) and (4) inductively, we obtain that the length of the curve $f^{N_0-n} \circ \gamma_s$ is at least $\kappa^{n-N_0}\ell(\gamma_s)$, which is at least $\kappa^{N_0}\ell(\gamma_s)$, which is $> D$. Similarly the length of $f^{N_0} \circ \gamma_u$ is $> D$. Property (5) then implies that

$$f^{N_0-n} \circ \gamma_s[0,1] \cap f^{N_0} \circ \gamma_u[0,1] \neq \emptyset.$$

Since this intersection is contained in $f^{N_0-n}(U) \cap f^{N_0}(V)$, the latter intersection is nonempty as well, and so

$$f^{-n}(U) \cap V \neq \emptyset,$$

for all $n \geq N = 2N_0$, which proves that f is mixing. Thus f_A is robustly mixing.

What happens when we try to apply this type argument to a partially hyperbolic diffeomorphism, for example the map $f_A \times R_\alpha$ on $\mathbb{T}^3$? Instead of an Anosov splitting, we have a partially hyperbolic splitting $T\mathbb{T}^3 = E^u \oplus E^c \oplus E^s$. We can define stable curves and unstable curves in a similar way, and properties (1)-(4) will continue to hold. But as we are in dimension 3, there is no hope for condition (5) to hold: two curves in general do not intersect in dimension 3.

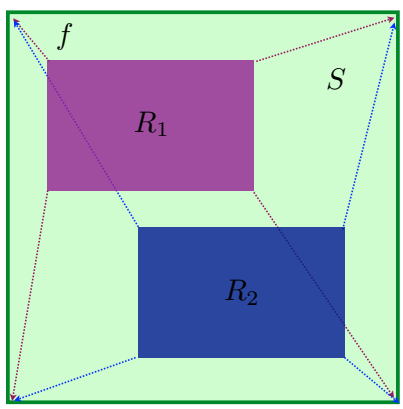
We can remedy the dimension problem by replacing stable and unstable curves with center-stable and center-unstable disks, which make small angle with $E^s \oplus E^c$ and $E^u \oplus E^c$, respectively. Then the analogues of conditions (1) and (2) hold. Moreover the images of center unstable disks get long in the unstable direction and the preimages of center stable disks get long in the stable direction. But this is not enough to ensure that sufficiently large iterates intersect. Clearly this is not the case, as the map $f_A \times R_\alpha$ is not mixing![9]

So to obtain a mechanism for robust mixing in a partially hyperbolic map, we need some condition, similar in spirit to stable accessibility, to ensure that two center stable and center unstable disks, sufficiently long in the stable and unstable directions, respectively, must intersect. This is the role of a blender.

[9]For the essentially the same reason that R_α is not mixing: two sets of the form $X = \mathbb{T}^2 \times I$, $Y = \mathbb{T}^2 \times J$, with I and J sufficiently small intervals in $\mathbb{T}$ will not mix.

7.2. **Blenders.** A blender is a compact invariant set on which a diffeomorphism has a certain behavior. This behavior forces topologically "thin" sets to intersect in a robust way, producing rich dynamics. The term "blender" describes its function: to blend together stable and unstable manifolds. Blenders have been used to construct diffeomorphisms with surprising properties and have played an important role in the classification of smooth dynamical systems. An elementary description of blenders can be found in [16], from which some of the discussion below is taken.

One of the original applications of blenders is also one of the more striking. Recall that a diffeomorphism g of a compact manifold is *robustly transitive* if there exists a point x whose orbit $\{g^n(x) : n \geq 0\}$ is dense in the manifold, and moreover this property persists when g is slightly perturbed. Until the 1990's there were no known robustly transitive diffeomorphisms in the isotopy class of the identity map on any manifold. Bonatti and Díaz[10]) used blenders to construct robustly transitive diffeomorphisms as perturbations of the identity map on certain 3-manifolds.



$f\colon R_1 \cup R_2 \to S$

FIGURE 6. An example of a proto-blender. The map f is defined on the union of the two rectangles R_1 and R_2 in the square S; f sends each R_i onto the entire square S affinely, respecting the horizontal and vertical directions, with the horizontal expansion factor less than 2. Note that f fixes a unique point in each rectangle R_i.

To construct a blender one typically starts with a *proto-blender*; an example is the map f pictured in Figure 6. The function f maps each of the two rectangles R_1 and R_2 affinely onto the square S and has the property that the vertical projections of R_1 and R_2 onto the horizontal direction overlap. Each rectangle contains a unique fixed point for f.

The compact set $\Omega = \bigcap_{n\geq 0} f^{-n}S$ is f-invariant, meaning $f(\Omega) = \Omega$, and is characterized as the set of points in S on which f can be iterated infinitely many times: $x \in \Omega$ if and only if $f^n(x) \in S$ for all $n \geq 0$. Ω is a Cantor set, obtained by intersecting all preimages $f^{-i}(S)$ of the square, which nest in a regular pattern as in Figure 7.

[10]This is also where the term "blender" was coined.

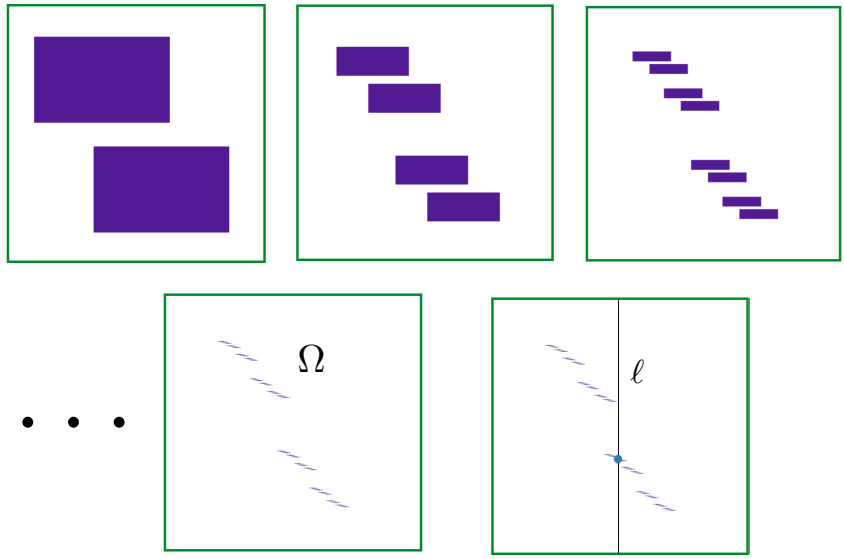


FIGURE 7. The invariant Cantor set Ω produced by the proto-blender f is the nested intersection of preimages of S under f. Any vertical line segment ℓ close to the center of the square intersects Ω in at least one point. The line segment can be replaced by segment with nearly vertical slope, or even a smooth curve nearly tangent to the vertical direction.

Any vertical line ℓ between the fixed points in R_1 and in R_2 will meet Ω. To prove this, it is enough to see that for every i the vertical projection of the set $f^{-i}(S)$ (consisting of 2^i horizontal rectangles) onto the horizontal is an interval. This can be checked inductively, observing that the projection of $f^{-i-1}(S)$ is the union of two re-scaled copies of the projection of $f^{-i}(S)$, which overlap.

A more careful inspection of this proof reveals that the intersection is *robust* in two senses: first, the line ℓ can be replaced by a line whose slope is close to vertical, or even by a C^1 curve whose tangent vectors are close to vertical; secondly, the map f can be replaced by any C^1 map $\hat{f}$ whose derivative is close to that of f. Such an $\hat{f}$ is called a *perturbation* of f.

The (topological) dimension of the Cantor set Ω is 0, the dimension of ℓ is 1, the dimension of the square is 2, and $0 + 1 < 2$. From a topological point of view, one would not expect these sets to intersect each other. But from a *metric* point of view, the fractal set Ω, when viewed along nearly vertical directions, appears to be 1-dimensional, allowing Ω to intersect a vertical line, robustly. If the rectangles R_1 and R_2 had disjoint projections, the proto-blender property would be destroyed.

We use the proto-blender to produce a blender, which is local diffeomorphism of a cube Q containing a special type of compact invariant set Λ. To produce Λ, we use the proto-blender $f\colon R_1 \cup R_2 \to S$ of Figure 1. The map f has only expanding directions and is not injective; indeed, it has precisely two inverse branches $f_1^{-1}\colon S \to R_1$ and $f_2^{-1}\colon S \to R_2$.

In dimension three, we can embed these inverse branches into a local diffeomorphism by adding a third, expanded direction, as detailed in Figure 5, where the cube Q is stretched and folded across itself by a local diffeomorphism g.

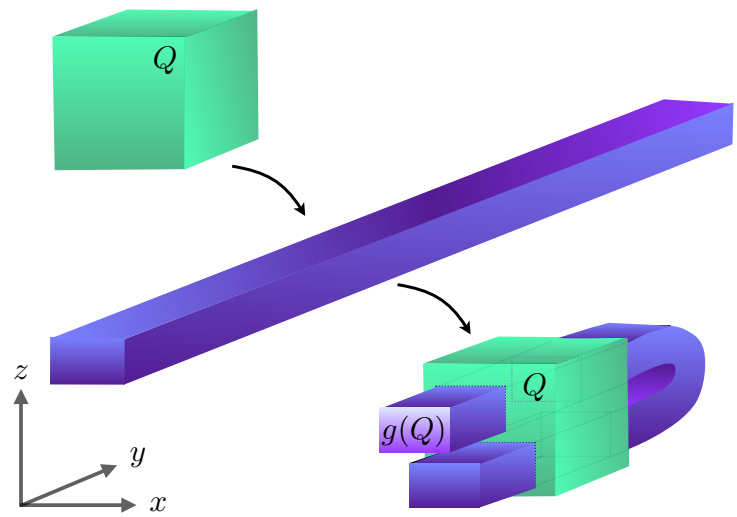


FIGURE 8. Constructing a blender, a type of horseshoe with a proto-blender built into its contracting directions. In the cube Q the local diffeomorphism g contracts the segments in the axial directions parallel to the front face (the xz-plane), elongates the cube into the third axial direction (the y-axis), and then folds this elongated piece across the original cube Q, as pictured. Each slice of $Q \cap g(Q)$ parallel to the xz-plane resembles exactly the picture of $R_1 \cup R_2$ in the square S. The restriction of g^{-1} to these rectangles in this slice just is a copy of the proto-blender f from Figure 6, whose image is another xz-slice of Q.

The set Λ in Figure is precisely the set of points whose orbits remain in the future and in the past in Q. It is a type of invariant set called a horseshoe. The set $W^u(\Lambda)$ of points in the cube that accumulate on Λ in the past is the cartesian product of the Cantor set Ω with segments parallel to the third, expanded direction. $W^u(\Lambda)$ is the analogue of the unstable manifold of a saddle, but it is a fractal object rather than a smooth submanifold.

The set Λ is an example of a blender, and its main geometric property is that any vertical, "strong stable" curve crossing Q close enough to the center intersects $W^u(\Lambda)$. In other words, this blender is a horseshoe whose unstable set behaves likes a surface even though its topological dimension is one. This property is robust. While the definition of blender is still evolving as new constructions arise, a working definition is: *A blender is a compact hyperbolic set whose unstable set has dimension strictly less than one would predict by looking at its intersection with families of submanifolds.*

The Bonatti-Díaz construction of a robustly mixing diffeomorphisms builds two such blenders in a partially hyperbolic map, starting for example with the map $f_A \times R_\alpha$ on

$\mathbb{T}^3$[11] and perturbing. If this construction is done carefully enough then every disk that is nearly tangent to $E^c \oplus E^s$ and contains a sufficiently long stable curve (that is, nearly tangent to E^s) will intersect every disk that is nearly tangent to $E^c \oplus E^u$ containing a sufficiently long unstable curve (nearly tangent to E^u). Because of the robust property of blenders, this intersection property is also robust. The proof of topological mixing, with some modification, proceeds as in Section 7.1.

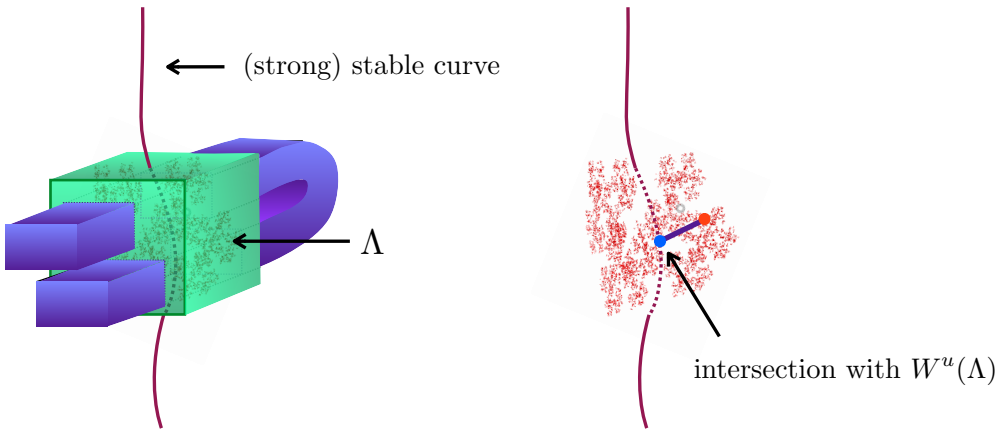


FIGURE 9. A blender in action.

8. A different approach to Conjecture 1: the proof of Theorem 6

F. Rodriguez Hertz, J. Rodriguez Hertz, Tahzibi and Ures made an important breakthrough in [61]: they found a new mechanism for ergodicity that married a topological method (blenders) with one from smooth ergodic theory (Pesin theory, described below). They used this to show Theorem 6 in the case where $\dim(E^c) = 2$. We describe this method and the full proof of Theorem 6 in this section. The material in this section is of a more technical nature.

While it is not clear how to derive ergodicity from accessibility alone, a relatively simple argument (due to Brin) shows that accessibility in a volume-preserving diffeomorphism implies *metric transitivity:* the property that almost every orbit (with respect to vol) is dense. This motivates the idea of combining accessibility with some "local" ergodicity mechanism in order to achieve full ergodicity. The approach in [5], following [61], is in a certain sense closer to the original applications of the Hopf argument than the Pugh-Shub approach, but with several crucial new ingredients added.

This modified Hopf argument uses a measurable version of stable and unstable foliations for a nonuniformly hyperbolic set, called the Pesin unstable and stable disk families. Nonuniform hyperbolicity is defined using quantities called Lyapunov exponents. A real

[11] Or the time-t_0 map of the geodesic flow over a hyperbolic surface

number χ is a *Lyapunov exponent* of a diffeomorphism $f : M \to M$ at $x \in M$ if there exists a nonzero vector $v \in T_xM$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df^n(v)\| = \chi. \tag{7}$$

If f preserves the volume vol, then Kingman's ergodic theorem implies that the limit $\chi = \chi(x, v)$ in (7) exists for for vol-almost every $x \in M$ and every nonzero $v \in T_xM$. Furthermore, Oseledets's theorem gives that $\chi(x, \cdot)$ can assume at most $\dim(M)$ distinct values $\chi_1(x), \ldots, \chi_{\ell(x)}(x)$, and that there exists a *measurable*, Df-invariant splitting of the tangent bundle (defined over a full vol-measure subset of M)

$$TM = E_1 \oplus E_2 \oplus \cdots \oplus E_\ell, \tag{8}$$

such that $\chi(x, v) = \chi_i(x)$ for $v \in E_i(x) \setminus \{0\}$. The set of $x \in M$ where these conditions are satisfied is called the set of *Oseledec regular points (for* vol *and f).* Volume preservation implies that the sum of the Lyapunov exponents is zero .

By separately summing the Lyapunov subspaces E_i corresponding to $\chi_i > 0$ and $\chi_i < 0$ we obtain a measurable splitting $TM = E^+ \oplus E^0 \oplus E^-$, where E^0 is the Lyapunov subspace (possibly trivial) for the exponent 0. We say that a positive vol-measure invariant set $X \subset M$ (not necessarily compact) is *nonuniformly hyperbolic* if for almost every $x \in X$, the exponents $\chi_i(x)$ exist and are nonzero; that is, E^0 is trivial over X.

Nonuniform hyperbolicity is a natural property for studying ergodicity: Bochi-Fayad-Pujals [12] have shown that among stably ergodic diffeomorphisms there exists a C^1-dense and open subset of systems that are *nonuniformly Anosov*: they have a dominated splitting $TM = E \oplus F$ and at vol-almost every point x we have $E^+ = E$ and $E^- = F$.

Tangent to E^+ and E^- for a nonuniformly hyperbolic set are two measurable families of invariant, smooth disks called the *Pesin unstable and stable disk families*, respectively. The dimension and diameter of these Pesin stable disks vary *measurably* over the manifold M. If we want to implement the Hopf argument as Hopf and Anosov did, the main issue is to ensure the transverse intersection of stable and unstable disks. Several difficulties arise:

(1) The Pesin disks should have "enough dimension" so that transversality is even possible.[12]
(2) The spaces E^+ and E^- along which they align should display some definite transversality (i.e. uniform boundedness of angle between them).
(3) The stable and unstable disks should be "long enough" so that they have the opportunity to intersect.

It turns out that for an accessible, partially hyperbolic diffeomorphism, difficulties (1) and (2) can be addressed globally if there exists a single nonuniformly hyperbolic set where E^+, E^- have constant dimensions and $E^+ \oplus E^-$ is dominated over this set. Metric

[12]Note that in the Pugh-Shub/Burns-Wilkinson approach, the Hopf argument is applied in a context where the dimensions of the strong stable and unstable manifolds are not enough to allow for transversality, but the analysis is much more involved than the simple implementation we are discussing, and it ends up depending on the center bunching condition.

transitivity, implied by accessibility, gives that the nonuniformly hyperbolic set must be dense, so that this dominated splitting extends to the whole manifold.

Such a set is constructed in [61] for an open set of partially hyperbolic diffeomorphims, dense among those with $\dim(E^c) = 2$. This follows almost directly from results in [13] and [10]. For the partially hyperbolic diffeomorphisms with $\dim(E^c) > 2$, more delicate arguments are required. These are contained in a previous work of Avila-Crovisier-Wilkinson [5], where we prove:

Theorem 7. [5] *For a generic*[13] *map* $f \in \mathrm{Diff}^1_{\mathrm{vol}}(M)$*, either*

(1) the Lyapunov exponents of f *vanish almost everywhere, or*
(2) f *is* non-uniformly Anosov*, meaning there exists a dominated splitting* $TM = E^+ \oplus E^-$ *and* $\chi_0 > 0$ *such that for* vol-*a.e.* $x \in M$*, and for every unit vector* $v \in T_xM$

$$v \in E^+ \implies \chi(x,v) > \chi_0, \quad \text{and } v \in E^- \implies \chi(x,v) < -\chi_0.$$

Moreover, f *is ergodic.*

Partial hyperbolicity clearly forbids the first case, and thus for a C^1-residual set of C^1, partially hyperbolic diffeomorphisms there is a globally dominated, nonuniformly hyperbolic splitting $E^+ \oplus E^-$. Using the density of C^1 volume preserving diffeomorphisms among the C^2 [3] and a semicontinuity argument, one obtains a C^1-open and -dense set of C^2, volume-preserving partially hyperbolic diffeomorphisms with a non-uniformly hyperbolic set whose splitting $TM = E^+ \oplus E^-$ is dominated.

Among these diffeomorphisms, partial hyperbolicity gives the strong unstable and stable foliations $\mathcal{W}^u$ and $\mathcal{W}^s$ whose leaves subfoliate the unstable and stable Pesin disks, respectively. Thus the Pesin stable and unstable disks are *definitely large* in the strong directions E^u and E^s. To address difficulty (3), we need a way to increase the size in the non-strong directions in E^+ and E^- to a definite scale as well. This can be achieved using blenders.

In [61], it is shown how so-called stable and unstable *blenders* can be used to resolve this third difficulty in a partially hyperbolic context. Recall that an unstable blender is a "robustly thick" part of a hyperbolic set, in the sense that its stable manifold meets every strong unstable manifold $\mathcal{W}^u(x)$ that comes near it, and moreover this property is still satisfied (by its hyperbolic continuation) after any C^1-small perturbation of the dynamics. The key point is that this property may be satisfied even if the dimensions of the strong stable manifolds and strong unstable manifolds are not large: the (thick) fractal geometry of the blender will be responsible for yielding the "missing dimensions" and fix the lack of transversality.

Since the Pesin unstable disks contain the strong unstable manifolds, we conclude that any Pesin unstable manifold near the blender has a part that is trapped by the blender

[13] We say that a property in $\mathrm{Diff}^r(M)$ (respectively $\mathrm{Diff}^r_{\mathrm{vol}}(M)$) is *generic* if the property holds on a set $\mathcal{R}$ containing a countable intersection of open and dense sets in $\mathrm{Diff}^r(M)$ (respectively $\mathrm{Diff}^r_{\mathrm{vol}}(M)$). The set $\mathcal{R}$ is said to be *residual*, and it is dense, since both $\mathrm{Diff}^r(M)$ and $\mathrm{Diff}^r_{\mathrm{vol}}(M)$ are Baire spaces.

dynamics. Under iteration, this trapped part evolves according to the hyperbolic dynamics of the blender, which enlarges even the non-strong directions to a definite size. Analogously defined stable blenders play a similar role of enlarging the Pesin stable manifolds to a definite size. If the unstable and stable blenders are contained in a larger transitive hyperbolic set, then those long pieces of unstable and stable manifolds do get close to one another and will thus intersect as desired.

8.1. **Blenders in partially hyperbolic systems.** How often do blenders arise in the context of partially hyperbolic dynamics? Originally, blenders were constructed using a very concrete geometric model, which was then shown to arise in the presence of a specific arrangement of two periodic points (the unfolding of heterodimensional cycles between periodic orbits whose stable dimension differ by one) [17]. This construction is used in [61] and accounts in part for the low dimensionality assumption on E^c in their result.

The fractal geometry of such a blender effectively yields one additional dimension in the above argument, so in order to obtain multiple additional dimensions, one would need to use several such blenders. Unfortunately, there are robust obstructions to the construction of some of the heterodimensional cycles needed to produce such blenders.

A rather different approach to the construction of blenders was introduced by Moreira and Silva [53]. The basic idea is that, starting from a hyperbolic set (i.e., horseshoe) whose fractal dimension is large enough to provide the desired additional dimensions, a blender will arise after a generic perturbation (using a "fractal transversality" argument). In their work, they succeeded in implementing this idea to obtain a blender yielding a single additional dimension.

In [5], we show that if the dimension of a hyperbolic set is "very large", close to the dimension of the entire ambient manifold, then a *superblender* (a blender capable of yielding all desired additional dimensions), can be produced by a suitable C^1-small perturbation. As it turns out, any C^2 map suificiently C^1-close to an ergodic, nonuniformly Anosov map admits such very large hyperbolic sets. Using Theorem 7, we can then conclude that superblenders appear C^1 densely among partially hyperbolic dynamical systems, and Theorem 6 follows. In fact we have:

Theorem 8 (Avila-Crovisier-Wilkinson, [6])**.** *For any $r > 1$, the space of C^r partially hyperbolic volume-preserving diffeomorphisms on a compact connected manifold contains a C^1-open and -dense subset of diffeomorphisms that are nonuniformly Anosov, ergodic and in fact Bernoulli.*

As mentioned above, the Bernoulli property is the most chaotic of all: after a recoding, the dynamics behave like a random coin toss. Thus Theorem 8 implies that the typical (in the sense of C^1 density) partially hyperbolic diffeomorphism is robustly chaotic. In the presence of nonuniform hyperbolicity on a large set (guaranteed densely by Theorem 7), the mechanisms behind this phenomenon are stable accessibility, the Hopf argument and blenders.

9. Open questions, future directions

9.1. **Symplectomorphisms.** We have largely omitted a discussion of symplectomorphisms, to which many of the above considerations apply. If (M^{2n}, ω) is a symplectic manifold, and $f\colon M \to M$ preserves ω, then it preserves the normalized volume vol induced by the form ω^n. Denote by $\mathrm{Diff}^r_\omega(M)$ the space of such C^r symplectomorphisms of M. A symplectomorphism $f \in \mathrm{Diff}^r_\omega(M)$ is *stably ergodic* if there is a C^1-open neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^r_\omega(M)$ such that every $g \in \mathcal{U}$ is ergodic with respect to vol. In the symplectic category, *stable ergodicity implies partial hyperbolicity* [50].

The analogue of Theorem 7 was recently proved in [7] (using a completely different set of techniques), but the analogue of Theorem 6 remains open.

Question: *Is stable ergodicity C^1-dense among the partially hyperbolic diffeomorphisms in* $\mathrm{Diff}^r_\omega(M)$*, $r > 1$?*

Any strategy for addressing this question should be completely different from the volume-preserving case since the "non-uniform Anosov property" does not exist for non-Anosov symplectomorphisms.

9.2. **Classification problem.** A basic question is to understand which manifolds support partially hyperbolic diffeomorphisms. As the problem remains open in the classical Anosov case (in which E^c is zero-dimensional), it is surely exremely difficult in general. There has been significant progress in dimension 3, however; for example, using techniques in the theory of codimension-1 foliations, Burago and Ivanov proved that there are no partially hyperbolic diffeomorphisms of the 3-sphere [29].

Modifying this question slightly, one can ask whether the partially hyperbolic diffeomorphisms in low dimension must belong to certain "classes" (up to homotopy, for example) – such as time-t maps of flows, skew products, algebraic systems, and so on. Pujals has proposed such a program in dimension 3, which has spurred quite a few papers on the subject [28, 24, 62, 42, 43]. See the surveys [44, 57] for a detailed discussion. In particular, there is a complete classification when $\pi_1(M)$ is solvable [43] or M is a hyperbolic manifold, and something close to a classification when f is homotopic to the identity [11].

Exciting recent results show that the class of partially hyperbolic diffeomorphisms, even in dimension 3, is much richer than previously anticipated: in particular the conjecture of Pujals is false. The paper [21] (see also [23, 22]) is the state-of-the art in dimension 3, giving a highly flexible construction of new partially hyperbolic diffeomorphisms in dimension 3 on any manifold admitting an Anosov flow and disproving another conjecture in dimension 3 due to F. Rodriguez Hertz, M.A. Rodriguez Hertz and R. Ures [62]. In higher dimension, there are partially hyperbolic diffeomorphisms of simply connected manifolds and other surprising constructions [22, 23, 40, 39]. This is currently a hot area, as classification results and new constructions emerge side-by-side.

With additional hypotheses on the dynamics, one gets closer to a classification result. The first such attempt, due to Bonatti-Wilkinson [24] explored the role of dynamical

coherence (an integrability assumption on the bundles $E^u \oplus E^c$ and $E^s \oplus E^c$) and other dynamical hypotheses in Pujals's conjecture. Under the assumption that there is a center foliation (tangent to E^c) with compact leaves, there are several strong classification results [14, 33, 15]. Under the hypotheses of dynamical coherence and absolute continuity of the center foliation, there are some classification results [9]. See also [34], in which a partial classification is given assuming smoothness of the partially hyperbolic splitting, and [25], in which the partially hyperbolic diffeomorphisms with 1-dimensional, topologically neutral, center foliation are classified.

9.3. **New criteria for ergodicity.** Conjecture 1 remains open. Techniques using the Hopf argument and density points (as in Theorem 5) might have reached their limits in this problem (at least in this is the case in the absence of a significantly new idea). Theorem 7 is currently limited to the C^1 topology, but there is some hope that blenders and nonuniform hyperbolicity can be obtained via C^r-small perturbations. Perhaps a new approach will find a satisfying conclusion to this part of the story.

Except in the symplectic setting, stable ergodicity does not imply partial hyperbolicity [64], but, as mentioned above, it does imply the existence of a dominated splitting [18]. Beyond the Pugh-Shub stable ergodicity conjectures lies a question motivated by Theorem 8:

Question: *Under what additional (open? dense?) conditions is a C^2, nonuniformly Anosov diffeomorphism ergodic?*

References

[1] D. Anosov, Geodesic flows on closed Riemannian manifolds of negative curvature. *Trudy Mat. Inst. Steklov.* **90** (1967).

[2] Arnol'd, V. I.; Avez, A., *Ergodic problems of classical mechanics.* Translated from the French by A. Avez. W. A. Benjamin, Inc., New York-Amsterdam 1968.

[3] A. Avila, On the regularization of conservative maps. *Acta Math.* **205** (2010), 5–18.

[4] A. Avila, J. Bochi, A. Wilkinson, Nonuniform center bunching and the genericity of ergodicity among C^1 partially hyperbolic symplectomorphisms. *Ann. Sci. Éc. Norm. Supér.* **42** (2009), 931–979.

[5] A. Avila, S. Crovisier, A. Wilkinson, Diffeomorphisms with positive metric entropy, *Publ. Math. IHES*, **124** (2016), 589–602.

[6] A. Avila, S. Crovisier and A. Wilkinson, C^1 density of stable ergodicity. *Adv. Math.*, to appear. (58 pages).

[7] A. Avila, S. Crovisier and A. Wilkinson, Symplectomorphisms with positive metric entropy. March, 2019 preprint (21 pages).

[8] A. Avila and M. Viana, Stable accessibility with 2-dimensional center. Preprint (2018), 14 pages.

[9] A. Avila, M. Viana, and A. Wilkinson. Absolute continuity, Lyapunov exponents and rigidity II : systems with compact center leaves Preprint (2019), 55 pages.

[10] A. Baraviera, C. Bonatti, Removing zero Lyapunov exponents. *Ergodic Theory Dynam. Systems* **23** (2003), no. 6, 1655–1670.

[11] T. Barthelmé, S. Fenley, S. Frankel, R. Potrie, Partially hyperbolic diffeomorphisms homotopic to the identity on 3-manifolds, preprint.

[12] J. Bochi, B. Fayad, E. Pujals, A remark on conservative diffeomorphisms. *C. R. Math. Acad. Sci. Paris* **342** (2006), no. 10, 763–766.

[13] J. Bochi, M. Viana, The Lyapunov exponents of generic volume-preserving and symplectic maps. *Ann. of Math.* **161** (2005), no. 3, 1423–1485.

[14] D. Bohnet, Codimension-1 partially hyperbolic diffeomorphisms with a uniformly compact center foliation, *Journal of Modern Dynamics*, **6** (2013), 565–604.

[15] D. Bohnet, Ch. Bonatti, Partially hyperbolic diffeomorphisms with a uniformly compact center foliation: The quotient dynamics, **Ergodic Theory and Dynamical Systems**, **36** (2016), 1067–1105.

[16] Ch. Bonatti, S. Crovisier, L. Díaz, A. Wilkinson What is ... a blender? *Notices of the AMS*, **63** (2016), 1175–1178.

[17] Ch. Bonatti, L. J. Díaz, Persistent nonhyperbolic transitive diffeomorphisms. *Ann. of Math.* **143** (1996), no. 2, 357–396.

[18] Ch. Bonatti, L. J. Díaz, E. Pujals, A C^1-generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources. *Ann. of Math.* **158** (2003), no. 2, 355–418.

[19] Ch. Bonatti, L. J. Díaz, M. Viana, *Dynamics beyond uniform hyperbolicity. A global geometric and probabilistic perspective.* Encyclopaedia of Mathematical Sciences, 102. Mathematical Physics, III. Springer-Verlag, Berlin, 2005.

[20] Ch. Bonatti, C. Matheus, M. Viana, A. Wilkinson, Abundance of stable ergodicity. *Comment. Math. Helv.* **79** (2004), no. 4, 753–757.

[21] Ch. Bonatti, A. Gogolev, A. Hammerlindl, R. Potrie, Anomalous partially hyperbolic diffeomorphisms III: abundance and incoherence. 2017 preprint.

[22] Ch. Bonatti, Ch., A. Gogolev, R. Potrie, Anomalous partially hyperbolic diffeomorphisms II: stably ergodic examples.*Invent. Math.* **206** (2016), 801–836.

[23] Ch. Bonatti, K. Parwani, R. Potrie, Anomalous partially hyperbolic diffeomorphisms I: Dynamically coherent examples. *Ann. Sci. Éc. Norm. Supér.* **49** (2016), 1387–1402.

[24] Ch. Bonatti and A. Wilkinson. Transitive partially hyperbolic diffeomorphisms on 3-manifolds. *Topology*, 44:475–508, 2005.

[25] Ch. Bonatti, J. Zhang, Transitive partially hyperbolic diffeomorphisms with one-dimensional neutral center, preprint.

[26] M. Brin, Topological transitivity of a certain class of dynamical systems, and flows of frames on manifolds of negative curvature. *Functional Anal. Appl.* **9** (1975), 8–16.

[27] M. I. Brin and Ja. B. Pesin, *Partially hyperbolic dynamical systems.* (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974), 170–212.

[28] M. Brin, D. Burago, and S. Ivanov. On partially hyperbolic diffeomorphisms on 3-manifolds with commutative fundamental group. In *Advances in Dynamical Systems*. Cambridge Univ. Press.

[29] D. Burago and S. Ivanov. Partially hyperbolic diffeomorphisms of 3-manifolds with abelian fundamental groups. *J. Mod. Dyn.*, 2:541–580, 2008.

[30] K. Burns, F. Rodriguez Hertz, M. A. Rodriguez Hertz, A. Talitskaya, and R. Ures. Density of accessibility for partially hyperbolic diffeomorphisms with one-dimensional center. *Discrete Contin. Dyn. Syst.*, 22:75–88, 2008.

[31] K. Burns, C. Pugh, M. Shub, A. Wilkinson, *Recent results about stable ergodicity.* Smooth ergodic theory and its applications (Seattle, WA, 1999), 327–366, Proc. Sympos. Pure Math., 69, Amer. Math. Soc., Providence, RI, 2001.

[32] K. Burns, A. Wilkinson, On the ergodicity of partially hyperbolic systems. *Ann. of Math.* (2) **171** (2010), no. 1, 451–489.

[33] P. Carrasco, Compact dynamical foliations, *Ergodic Theory and Dynamical Systems*, **35** (2015), 2474–2498.

[34] P. D. Carrasco, E. Pujals, F. Rodriguez Hertz, Classification of partially hyperbolic diffeomorphisms under some rigid conditions, preprint.

[35] V. De Martino and S. Martinchich, Codimension one compact center foliations are uniformly compact, 2018 preprint.

[36] L. J. Díaz, E. Pujals, and R. Ures. Partial hyperbolicity and robust transitivity. *Acta Math.*, 183:1–43, 1999.

[37] P. Didier. Stability of accessibility. *Ergod. Th. & Dynam. Sys.*, 23:1717–1731, 2003.

[38] D. Dolgopyat, A. Wilkinson, Stable accessibility is C^1 dense. *Geometric methods in dynamics. II. Astérisque* **287** (2003), 33–60.
[39] A. Gogolev, Surgery for partially hyperbolic dynamical systems, I: Blow-ups of invariant submanifolds. *Geom. Topol.* **22** (2018), 2219–2252.
[40] A. Gogolev, P. Ontaneda, F. Rodriguez Hertz, New partially hyperbolic dynamical systems I. *Acta Math.* **215** (2015), 363–393.
[41] M. Grayson, C. Pugh, Charles, M. Shub, Stably ergodic diffeomorphisms. *Ann. of Math.* (2) **140** (1994), no. 2, 295–329.
[42] A. Hammerlindl, Partial hyperbolicity on 3-dimensional nilmanifolds. *Discrete Contin. Dyn. Syst.* **33** (2013), 3641–3669.
[43] A. Hammerlindl, R. Potrie, Classification of partially hyperbolic diffeomorphisms in 3-manifolds with solvable fundamental group. *J. Topol.* **8** (2015), 842–870.
[44] A. Hammerlindl, R. Potrie, Partial hyperbolicity and classification: a survey. *Ergodic Theory Dynam. Systems* **38** (2018), 401–443.
[45] A. Hammerlindl R. Potrie, M. Shannon, Seifert manifolds admitting partially hyperbolic diffeomorphisms. *J. Mod. Dyn.* **12** (2018), 193–222.
[46] H. Helson, *Harmonic analysis.* Addison-Wesley, 1983.
[47] M. W. Hirsch, C. C. Pugh and M. Shub, Invariant manifolds (Vol. 583). Springer.
[48] E. Hopf, Statistik der geodätischen Linien in Mannigfaltigkeiten negativer Krümung . *Ber. Verh. Sächs. Akad. Wiss. Leipzig* **91** (1939), 261–304.
[49] E. Hopf, Statistik der Lösungen geodätischer Probleme vom unstabilen Typus. II. *Math. Ann.* **117** (1940). 590–608.
[50] V. Horita and A. Tahzibi. Partial hyperbolicity for symplectic diffeomorphisms. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 23:641–661, 2006.
[51] A. Kolmogorov, Théorie générale des systèmes dynamiques et mécanique classique. *Proceedings of the International Congress of Mathematicians* (Amsterdam 1954) Vol. 1, 315–333
[52] R. Mañé, Ergodic theory and differentiable dynamics. *Ergebnisse der Mathematik und ihrer Grenzgebiete* Springer-Verlag, Berlin, 1987.
[53] C.G. Moreira and W. Silva, On the geometry of horseshoes in higher dimensions. Preprint (arXiv:1210.2623).
[54] C.G. Moreira and J.C. Yoccoz, Stable Intersections of Regular Cantor Sets with Large Hausdorff Dimensions. *Ann. of Math.* (2) **154** (2001), 45–96.
[55] V. Niţică and A. Török. An open dense set of stably ergodic diffeomorphisms in a neighborhood of a non-ergodic one. *Topology*, 40:259–278, 2001.
[56] Y. Pesin, *Lectures on partial hyperbolicity and stable ergodicity.* Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zurich, 2004.
[57] R. Potrie, Robust dynamics, invariant structures and topological classification. 2018 preprint.
[58] C. Pugh and M. Shub, Ergodicity of Anosov actions. *Invent. Math.* **15** (1972), 1–23.
[59] C. Pugh, M. Shub, Stably ergodic dynamical systems and partial hyperbolicity. *J. Complexity* **13** (1997), no. 1, 125–179.
[60] F. Rodriguez Hertz. Stable ergodicity of certain linear automorphisms of the torus. Annals of mathematics, (2005), 65-107.
[61] F. Rodriguez-Hertz, M.A. Rodriguez-Hertz, A. Tahzibi, R. Ures, New criteria for ergodicity and nonuniform hyperbolicity. *Duke Math. J.* **160** (2011), no. 3, 599–629.
[62] F. Rodriguez Hertz, M. A. Rodriguez Hertz, and R. Ures. Partial hyperbolicity and ergodicity in dimension three. *J. Mod. Dyn.*, 2:187–208, 2008.
[63] F. Rodriguez-Hertz, M.A. Rodriguez-Hertz, R. Ures, Accessibility and stable ergodicity for partially hyperbolic diffeomorphisms with 1D-center bundle. *Invent. Math.* **172** (2008), no. 2, 353–381.
[64] A. Tahzibi, *Stably ergodic diffeomorphisms which are not partially hyperbolic.* Israel J. Math. **142** (2004), 315–344.

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