Transitive partially hyperbolic diffeomorphisms on 3-manifolds

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Abstract

The known examples of transitive partially hyperbolic diffeomorphisms on 3-manifolds belong to 3 basic classes: perturbations of skew products over an Anosov map of T^2, perturbations of the time one map of a transitive Anosov flow, and certain derived from Anosov diffeomorphisms of the torus T^3. In this work we characterize the two first types by a local hypothesis associated to one closed periodic curve.

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Introduction

This paper is about transitive diffeomorphisms f of a compact Riemannian 3-manifold M that are partially hyperbolic, meaning: the tangent bundle TM admits a f_*-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ so that f_* contracts uniformly the vectors in E^s, expands uniformly the vectors in E^u, and the action of f_* on E^c is less contracting than in E^s and less expanding than in E^u.

Partially hyperbolic diffeomorphisms appear naturally in at least two contexts in the theory of dynamical systems: stable ergodicity and robust transitivity. Stably ergodic diffeomorphisms preserve volume as an ergodic measure, and remain ergodic under sufficiently smooth volume-preserving perturbations.[1] Robustly transitive diffeomorphisms are topologically transitive, and remain so under C^1 perturbations. The earliest examples of stably ergodic and of robustly transitive diffeomorphisms were partially hyperbolic, and an extensive theory of partially hyperbolic diffeomorphisms has been built up from these examples. On 3-manifolds, something close to partial hyperbolicity is in fact necessary for both stable ergodicity and robust transitivity: [DPU] show that any stably ergodic or robustly transitive difffeomorphism f has an f_*-invariant dominated splitting with at least one expanding subbundle (i.e. $TM = E^{cs} \oplus E^u$) or contracting subbundle (i.e. $TM = E^s \oplus E^{cu}$).

In contrast with an Anosov diffeomorphism, perturbing a partially hyperbolic diffeomorphism can fundamentally change its dynamics, so that on the level of topological conjugacy, the dynamics of partially hyperbolic diffeomorphisms is fairly complex, even in dimension 3. On the level of isotopy preserving the hyperbolic structure, however, the landscape of known partially hyperbolic diffeomorphisms is much simpler. Indeed, the list of known examples of partially hyperbolic robustly transitive or stably ergodic diffeomorphisms on 3 manifolds is very short and breaks into 3 categories:

1. perturbations of skew products over an Anosov map of the torus T^2,
2. perturbations of the time-1 map of a transitive Anosov flow,
3. certain derived from Anosov diffeomorphisms of the torus T^3.

At a conference on partial hyperbolicity at Northwestern University, in May 2001, E. Pujals informally conjectured that this list of examples is complete. It turns out that this conjecture does have to be modified to include finite lifts of examples in 1. and 2., as we explain in Section 4. In this work, we will give some reason to believe this modified conjecture is true.

The results in this paper arose out of attempts to find counterexamples to this conjecture. In our initial attempts to disprove the conjecture of Pujals, we tried to glue together two transitive partially hyperbolic diffeomorphisms using Dehn surgery techniques. Such techniques have been used successfully, first by Franks-Williams [FW], to construct nonstandard examples of Anosov flows on 3-manifolds (that is, Anosov flows that are not isotopic to either suspensions or geodesic flows). We asked ourselves: why not put together a skew product over a horseshoe in this region, and the time-1 map of an Anosov flow in another region of the manifold? The answer we found was: "no way!" As we shall see, if a transitive partially hyperbolic diffeomorphism f looks like the perturbation of a skew product or of the time-1 map of an Anosov flow in just a small region of a 3-manifold, then f *is* the perturbation of a skew product or of the time-1 map of an Anosov flow.

More precisely, the main theme of this work is that the dynamics of a transitive, partially hyperbolic diffeomorphism f of a 3-manifold can be completely recovered from its local behavior in the neighborhood of a periodic circle γ. Let $f\colon M \to M$ be a transitive partially hyperbolic diffeomorphism (having a splitting $E^s \oplus E^c \oplus E^u$). Assume that there is an embedded circle γ that is fixed for f; that is, $f(\gamma) = \gamma$. Such an invariant circle will always be tangent to E^c, so

[1] More precisely, the stably ergodic diffeomorphisms are defined to be the C^1-interior of the C^2, volume preserving ergodic diffeomorphisms.

that it is normally hyperbolic, and its invariant manifolds $W^s(\gamma)$ and $W^u(\gamma)$ are well-defined. Note that a skew product $\tilde{A}$ over an Anosov diffeomorphism A will always admit such a circle, corresponding to the fixed point of A, and a transitive Anosov flow φ will have a dense set of such closed orbits, each one fixed by the time-1 map φ_1. Further, by the theory of normally hyperbolic foliations in [HPS], any perturbation of $\tilde{A}$ or of φ_1 will also have such a circle γ.

Our first result characterizes the skew products in terms of the local dynamics near γ. Let $W^s_\delta(\gamma)$ and $W^u_\delta(\gamma)$ denote the union of the strong stable and strong unstable segments, respectively, of length δ through the point of γ. If f is a skew product over an Anosov diffeomorphism A (see Section 4 for a precise definition), then these local stable manifolds meet in a collection of circles corresponding to homoclinic points for A. We show that the existence of one such circle implies that f is a skew product:

Theorem 1 *Let f be a partially hyperbolic, transitive diffeomorphism of a compact 3-manifold M. Assume that there is an embedded circle γ such that $f(\gamma) = \gamma$.*

Suppose there exists $\delta > 0$ such that $W^s_\delta(\gamma) \cap W^u_\delta(\gamma) \setminus \gamma$ contains a connected component c that is a circle. Then:

1. *The diffeomorphism f is dynamically coherent*[2].
2. *Each center leaf is compact (so is a circle) and the center foliation defines a Seifert bundle on M.*
3. *If the center-stable and the center-unstable foliations are transversely orientable, then M is a S^1-bundle over T^2, and f is conjugate to a (topological) skew product over a linear Anosov map of T^2.*
4. *If the center-stable or the center-unstable foliations are not orientable, then covering of M corresponding to the possible tranverse orientations is an S^1-bundle and the natural lift $\tilde{f}$ of f is conjugate to a (topological) skew product over a linear Anosov map of T^2.*

In Section 4 we first discuss the definition of skew product we are using here, then we survey a few examples. We produce a transitive partially hyperbolic diffeomorphism where the center bundle on M is not trivial, so the foliation is a Seifert bundle with singular leaves. We also describe an example in which the center bundle is not orientable.

Question 0.1 *Can this result be extended to the case where γ is periodic under f? The proof of Theorem 1 extends to the case where the iterates $f^k\gamma$ are disjoint from γ, for k less than the period of γ. Since the center bundle might not be uniquely integrable, nontrivial self-intersections in the orbit of γ could, in theory, occur.*

We next turn to the perturbations of the time-1 map of an Anosov flow. Notice that, if $f = \varphi_1$ is the time-1 map of such a flow φ_t, then $W^s(\gamma) \cap W^u(\gamma)$ contains a family of non-compact homoclinic curves, corresponding to homoclinic orbits of φ_t (and the theory in [HPS] implies that such a curve exists for perturbations of f as well). The existence of such a curve is perhaps the simplest possible local characterization of a perturbation of φ_1. We do not know whether such a characterization holds, although we hope it does. Unluckily, we currently need additional hypotheses:

Theorem 2 *Let f be a partially hyperbolic dynamically coherent diffeomorphism on a compact 3-manifold M and let $\mathcal{F}^{ss}, \mathcal{F}^{cs}, \mathcal{F}^{cu}, \mathcal{F}^{uu}$ and $\mathcal{F}^c$ be the invariant foliations of f.*

[2] A partially hyperbolic diffeomorphism f is *dynamically coherent* if it admits f-invariant foliations $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$, tangent to the bundles $E^s \oplus E^c$ and $E^c \oplus E^u$, respectively. These foliations are necessarily subfoliated by the strong stable and unstable foliations $\mathcal{F}^{ss}$ and $\mathcal{F}^{uu}$, tangent to E^s and E^u. It also follows that $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ intersect along a f-invariant foliation $\mathcal{F}^c$ tangent to E^c.

Assume that there is a closed center leaf γ which is periodic under f and such that each center leaf in $W^s_{loc}(\gamma)$ is periodic for f.

Then:

1. *there is an $n \in \mathbb{N}$ such that f^n sends every center leaf to itself.*
2. *there is an $L > 0$ such that for any $x \in M$ the length $d_c(x, f^n(x))$ of the smaller center segment joining x to $f(x)$ is bounded by L*
3. *each leaf $\mathcal{L}^{cu}$ of $\mathcal{F}^{cu}$ is a cylinder or a plane (according to whether it contains a closed center leaf or not) and is trivially bi-foliated by $\mathcal{F}^c$ and $\mathcal{F}^{uu}$.*
4. *the center foliation supports a continuous flow conjugate to a transitive expansive flow.*

Expansive flows on compact 3-manifolds are very close to being Anosov flows. In fact, as Fried proved for Anosov flows, Brunella [Br] showed that expansive flows have Birkhoff sections on which the return map is pseudo-Anosov. This implies in particular that expansive flows have Markov partitions. Despite the fact that expansive flows on 3-manifolds are now well understood, the following conjecture has not been completely proven (as far as we know):

Conjecture 1 *Let X be an expansive flow on a compact* 3*-manifold. If the stable and the unstable manifolds of all the periodic orbits of X are nonsingular, then X is topologically equivalent to an Anosov flow.*

Notice that a proof of this conjecture would imply that, under the hypotheses of Theorem 2, the center foliation carries a topological flow conjugate to an Anosov flow.

In Section 4 we give examples where the center leaves are all periodic but where the period depends of the leaf.

Recently, Brin, Burago and Ivanov have announced results of a similar nature to those in this paper. They study partially hyperbolic diffeomorphisms of a 3-manifold M but replace the assumption of transitivity with some knowledge of M. For instance, they show that there are no partially hyperbolic diffeomorphisms of the 3-sphere. Finally, we remark that our assumption of transitivity in the results above is a reasonably mild one in the volume preserving setting: in [DW] it is shown that transitivity is a C^1-open and dense property among partially hyperbolic, volume presrving diffeomorphisms of any compact manifold.

1 Some general tools on transitive partially hyperbolic systems

1.1 Density of the orbit of center-stable or center-unstable disks

Let $f\colon M \to M$ be a transitive partially hyperbolic diffeomorphism of a compact 3-manifold M. A *center-unstable disk* is a disk D^{cu} embedded in M and tangent to the center-unstable bundle E^{cu}.

Lemma 1.1 *Under the above hypotheses, the set of points whose positive f-orbit is dense in M contains a dense subset of any center-unstable disk D^{cu}.*

In the same way, the set of points whose negative f-orbit is dense in M contains a dense subset of any center-stable disk D^{cs}.

Proof: Since f is transitive, we can choose a point x whose ω-limit set is M. The same holds for every point in the orbit the strong stable manifold $W^{ss}(x)$. To complete the proof, note that the orbit of $W^{ss}(x)$ (where x is a point whose positive orbit is dense in M) cuts each center-unstable disk D^{cu} in a dense subset. □

Corollary 1.2 *Given any center-stable disk D^{cs} and any center-unstable disk D^{cu}, the union over $n > 0$ of the intersections $f^n(D^{cu}) \cap D^{cs}$ is dense in D^{cs}.*

Proof: Let x be a point in the interior of D^{cu} whose forward orbit is dense, and consider a unstable segment through x in D^{cu}. The length of this unstable segment increases under positive iterates of f (and in particular does not approach 0). The image of this segment will cut D^{cs} each time the orbit of x is close enough to some point in the interior of D^{cs}. □

1.2 Invariant manifolds of periodic circles

Remark 1.3 *Any embedded periodic circle γ is tangent to the center bundle. In fact, if at some point $x \in \gamma$ the tangent direction to γ had some non-zero component in the unstable (resp. stable) direction then the length of positive (resp. negative) iterates of γ would approach $+\infty$, contradicting the fact that γ is periodic.*

From the remark above, γ is a normally hyperbolic periodic circle, so that [HPS] implies that the stable and unstable manifolds of the orbit of γ (denoted $W^s(\gamma)$ and $W^u(\gamma)$, respectively) are well-defined.

Lemma 1.4 *The stable and unstable manifolds of γ are injectively immersed surfaces tangent to E^{cs} and E^{cu} respectively, and coincide with the union of the strong stable and unstable leaves, respectively, through the orbit of γ. In particular, each of them is the union of finitely many open cylinders or open Mœbius bands.*

Proof: According to [HPS], the local stable and unstable manifolds of γ are surfaces, tangent along γ to E^{cs} and E^{cu}, respectively. Moreover, the global stable and unstable manifolds of γ are the (increasing) union of the negative and positive iterates of the local ones, so that the invariant manifolds are injectively immersed surfaces.

Consider a point $x \in W^s(\gamma)$ and a vector v tangent at x to $W^s(\gamma)$. The angle between $f^n_*(v)$ and $E^{cs}(x)$ approaches zero as $n \to +\infty$ (so that $f^n(x)$ converges to γ in the local stable manifold). This implies that v has no component in the E^u direction. As a consequence, $W^s(\gamma)$ is everywhere tangent to E^{cs}. In the same way, by considering negative iterates, we see that $W^u(\gamma)$ is tangent to E^{cu}.

Recall that the strong stable and the strong unstable bundles of a partially hyperbolic diffeomorphism are uniquely integrable (see, e.g. [HPS]). It follows that $W^s(\gamma)$ and $W^u(\gamma)$ are sub-foliated by the leaves of the strong stable and strong unstable foliations, respectively. Once more using that any point of these invariant manifolds has an iterate in the local invariant manifold, we obtain that the global invariant manifolds are the union of the strong invariant leaves through the orbit of γ.

Finally, each component of the stable or unstable manifolds is an annulus or a Mœbius band according to whether the corresponding strong stable or unstable bundle is orientable or not along γ. □

Corollary 1.5 *Let γ_1 and γ_2 be two (perhaps identical) invariant circles. Then $W^s(\gamma_1) \cap W^u(\gamma_2)$ is an f-invariant, countable union of disjoint open arcs or circles tangent to E^c.*

Proof: Since $W^s(\gamma_1)$ is an injectively immersed surface tangent to E^{cs}, and $W^u(\gamma_2)$ is tangent to E^{cu}, these two surfaces are transverse and meet each other along a countable family of open curves or circles, each of which is tangent to $E^c = E^{cu} \cap E^{cs}$. □

1.3 A criterion for integrability

We first state a general argument for the integrability of a codimension 1 plane field:

Proposition 1.6 *Let M be a compact n-manifold, and let E be a continuous codimension 1 plane field (distribution of hyperplanes) on M. Assume that there is a codimension-1 submanifold S (possibly with a compact boundary) injectively immersed in M, tangent to E, dense in M and complete in the metric induced by M. Then there is a unique continuous foliation $\mathcal{F}$ whose leaves are C^1 and tangent to E and such that S is contained in a leaf.*

Remark 1.7 *This is a codimension-1 argument; the analagous statement is not true in higher codimension.*

Proof: We construct, for each $x \in M$, a foliation chart $\varphi_x : U_x \to \mathbb{R}^n$, where U_x is a neighborhood of x, and φ_x is a homeomorphism sending each connected component of $S \cap U_x$ into a horizontal hyperplane. This will define a foliation provided that the overlaps $\varphi_x \cap \varphi_y^{-1} : \varphi_y(U_x \cap U_y) \to \varphi_x(U_x \cap U_y)$, where defined, preserve the horizontal direction.

For $\delta > 0$, let S_δ denote the points in S whose distance in S to ∂S is greater than δ. Since the boundary of S is compact, the set S_δ is dense in M for every $\delta > 0$. Since S is complete, and M is compact, there exists a $\delta_0 > 0$ such that, for all $\delta \in (0, \delta_0)$, and for every $y \in S_{2\delta}$, the set of points in S whose distance (for the induced metric on S) to y is less than δ is an embedded disk $\Delta(y, \delta) \subset S$ disjoint from the boundary of S. Fix such a $\delta_0 > 0$.

Choose a smooth 1-dimensional foliation $\mathcal{F}$ transverse to E. For each $x \in M$ and $\delta > 0$ we denote by $I(x, \delta)$ the segment of leaf of $\mathcal{F}$ centered at x with length δ. Note that there is $\delta_1 \in (0, \delta_0]$ such that, for any $\delta \in (0, \delta_1]$ and for any $x \in M$, $y \in S_\delta$, the segment $I(x, \delta)$ is embedded in M (i.e., is not a circle) and $I(x, \delta) \cap \Delta(y, \delta)$ contains at most one point.

Claim 1 *Let x be a point of M, let $\delta \in (0, \delta_1]$ and let y_1 and y_2 be two different points in $S_\delta \cap I(x, \delta)$. Then the disks $\Delta(y_1, \frac{\delta}{3})$ and $\Delta(y_2, \frac{\delta}{3})$ are disjoint.*

Proof: If they are not disjoint, then $y_2 \in \Delta(y_1, \delta)$ so y_1 and y_2 are two intersection points of $\Delta(y_1, \delta)$ with $I(x, \delta)$. The choice of δ then implies that $y_1 = y_2$. □

We now fix some $\delta \in (0, \frac{\delta_1}{3}]$. There exists $\varepsilon \in (0, \delta]$ such that, if $y \in S_\delta$ and $d(x, y) < \varepsilon$, then $I(x, \delta) \cap \Delta(y, \delta)$ consists of exactly one point, and this point is contained in the intersection $I(x, \frac{\delta}{3}) \cap \Delta(y, \frac{\delta}{3})$. Now by transversality of $\mathcal{F}$ and E, the intersection point $I(x, \delta) \cap \Delta(y, \delta)$ varies continuously in the set of $x \in M$ that satisfy the inequality $d(x, y) < \varepsilon$. [3]

Now fix a point $x_0 \in M$. Note that $S_{2\delta}$ is dense in M and so contains points arbitrarily close to any point of $I(x_0, \delta)$. One deduces easily that $S_\delta \cap I(x_0, \delta)$ is dense in $I(x_0, \delta)$. The set $I(x_0, \delta) \setminus \{x_0\}$ has 2 connected components, call them I_1 and I_2. Since $S_\delta \cap I_1$ is dense in I_1, there exists $x_1 \in S_\delta \cap I_1$ such that $d(x_0, x_1) < \frac{\varepsilon}{20}$. In the same way, there exists $x_2 \in S_\delta \cap I_2$ such that $d(x_0, x_2) < \frac{\varepsilon}{20}$. Fix such x_1, x_2.

Observe that for any $z \in \Delta(x_1, \frac{\varepsilon}{10})$ the distance $d(z, x_2)$ is bounded by $\frac{2\varepsilon}{10} < \varepsilon$ so that $I(z, \delta) \cap \Delta(y, \delta)$ consists of a single point $\psi(z) \in I(z, \frac{\delta}{3}) \cap \Delta(y, \frac{\delta}{3})$. Let $J_z \subset I(z, \delta)$ be the segment of $I(z, \delta)$ joining z to $\psi(z)$. The segment J_z varies continuously with z, since $\psi(z)$ varies continuously with z. Then there is $\varepsilon_1 < \frac{\varepsilon}{10}$ such that for any $z \in \Delta(x_1, \varepsilon_1)$ the length of J_z is less than $\frac{\varepsilon}{9}$. Let $\Delta_{x_1} = \Delta(x_1, \varepsilon_1)$, and let $\Delta_{x_2} = \psi(\Delta_1) \subset \Delta(x_2, \frac{\delta}{3})$. Finally, let $B = \bigcup_{z \in \Delta_{x_1}} J_z$; then B is a tubelike object bounded on top and bottom by Δ_{x_1} and Δ_{x_2}, and

[3] As we don't know at this stage that the disks $\Delta(y, \delta)$ are disks on leaves of a foliation, we cannot ensure that they vary continuously with y; for this reason, the continuity in y of the intersection point $I(x, \delta) \cap \Delta(y, \delta)$ is not guaranteed.

on the sides by the union of segments in $\mathcal{F}$. Denote by $\pi: B \to \Delta_{x_1}$ the natural projection. Note that the diameter of B is less than $\frac{2\varepsilon}{9} + 2\varepsilon_1 < \varepsilon$.

Lemma 1.8 *For any $p \in B \cap S_{3\delta}$ the connected component of $S \cap B$ containing p is a disk Δ_p and the projection $\pi: B \to \Delta_{x_1}$ induces an homeomorphism from Δ_p to Δ_{x_1}.*

If p and p' are two points in $B \cap S_{4\delta}$, then the disks Δ_p and $\Delta_{p'}$ are either disjoint or equal. Finally, the union of these disks is dense in B and cuts any segment J_z in a dense subset.

Proof: For any $p \in S_{4\delta} \cap B$ and any $z \in \Delta_{x_1}$ the distance $d(p, z)$ is less than the diameter of B, which is less than ε. It follows that the disk $\Delta(p, \delta)$ cuts any segment $I(z, \delta)$ in exactly one point $a(z, p) \in \Delta(p, \frac{\delta}{3}) \cap I(z, \frac{\delta}{3})$, varying continuously with z. Moreover, by the claim above, if $p \notin \Delta_{x_1} \cup \Delta_{x_2}$, then the disk $\Delta(p, \frac{\delta}{3})$ is disjoint from the two disks Δ_{x_1} and Δ_{x_2}, which contain by construction all of the endpoints of the segments J_z. As a consequence, the point $a(z, p)$ belongs to the interior of J_z. It follows that $\pi: \Delta(p, \frac{\delta}{3}) \cap B \to \Delta_{x_1}$ is a homeomorphism. For $p \in J_{x_1} \cap S_{3\delta}$, we denote by Δ_p the intersection: $\Delta_p = \Delta(p, \frac{\delta}{3}) \cap B$.

By construction, the union of the disks Δ_p contains $S_\delta \cap J_z$ and so is dense in J_z.

It remains to show that, for $p, p' \in S_{4\delta}$ the two disks Δ_p and $\Delta_{p'}$ are either equal or disjoint. To see this, just note that $\Delta_{p'}$ contains a point q on the segment J_z containing p. We will apply the claim with $y_1 = p$ and $y_2 = q$, by verifying that $p, q \in S_{3\delta}$ and $\Delta_p \subset \Delta(p, \delta), \Delta_{p'} \subset \Delta(q, \delta)$.

The point p' belongs to $S_{4\delta}$ by hypothesis, and $q \in \Delta(p', \frac{\delta}{3})$, so q belongs to $S_{4\delta - \frac{\delta}{3}} \subset S_{3\delta}$. By hypothesis p belongs to $S_{4\delta} \subset S_{3\delta}$. Clearly Δ_p is contained in $\Delta(p, \frac{\delta}{3}) \subset \Delta(p, \delta)$; since $q \in \Delta(p', \frac{\delta}{3})$ and $\Delta_{p'} \subset \Delta(p', \frac{\delta}{3})$, we also have that $\Delta_{p'}$ is contained in the disk $\Delta(q, \delta)$.

As δ has been fixed less than $\frac{\delta_1}{3}$, the claim shows that, if $p \neq q$, then the two disks are disjoint. $\square$

Lemma 1.9 *There is a unique homeomorphism $\varphi: B \to \Delta_{x_1} \times J_{x_1}$ such that the image of any segment J_z is the "vertical segment" $\{z\} \times J_{x_1}$ and such that, for any connected component C of $S \cap B$, there exists $t \in J_{x_1}$ such that $\varphi(C) \subset \Delta(x_1, \frac{\varepsilon}{10}) \times \{t\}$.*

Proof: For any $z \in \delta_{x_1}$ we fix an orientation of the segment J_{x_1}, "from Δ_{x_1} to Δ_{x_2}"; let $A_z = J_z \cap \bigcup_{p \in S_{4\delta} \cap B} \Delta_p$. As any disk Δ_p cuts any segment J_z in exactly one point, and as the disk Δ_p are pairwise disjoint or equal, there is a natural bijection $h_z: A_z \to A_{x_1}$; this bijection associates to any point in A_z the unique point of A_x belonging to the same disk Δ_p. This bijection is increasing for the orientation fixed on these segments, and A_{x_1} and A_z are dense in J_{x_1} and J_z, respectively. Recall that any increasing map between two dense subsets of an interval extends in a unique way to a continuous map of the interval. So h_z extends in a unique way to a homeomorphism $h_z: J_z \to J_{x_1}$.

Now the desired map $\varphi: B \to \Delta_{x_1} \times J_{x_1}$ is defined by $\varphi(p) = (\pi(p), h_{\pi(p)}(p))$. Notice that any connected component of $B \setminus \bigcup_{p \in S_{4\delta} \cap B} \Delta_p$ is now a disk, because its image under φ is a horizontal disk. In particular, any component C of $S \cap B$ is either a disk Δ_p, or is contained in a component of $B \setminus \bigcup_{p \in S_{4\delta} \cap B} \Delta_p$. In both cases $\varphi(C)$ is contained in an horizontal disk. $\square$

Lemma 1.9 above proves the existence of foliated boxes B around any point x_0. The fact that any connected component of the intersection of S with B is mapped to a horizontal disk assures that the chart maps the connected component of the intersections of the leaves of another box B' with B in horizontal disks. This ensures that the boxes B form the atlas of a foliation on M, whose leaves contains the connected components of S.

$\square$

Remark 1.10 - Proposition 1.6 remains true if we replace the hypothesis:

"*S is a dense, complete hypersurface tangent to E and with compact boundary*"

by

"*S contains a family of disks tangent to E of fixed radius, whose centers form a dense subset in M*".

- However, the Proposition does not hold under the following hypothesis:

"*S contains a family of disjoint disks of fixed radius, tangent to E, whose union is a dense subset of M*". Indeed, there are line fields on surfaces admitting a dense family of pairwise disjoint segments of length 1, but such that no foliation contains these segments in its leaves. For this reason, we have taken some care in explaining the proof of Proposition 1.6.

1.4 A criterion for dynamical coherence

If γ is a periodic circle, then cutting along the component of $W^s(\gamma)$ containing γ, we obtain one or two semi-open cylinders (depending on whether $W^s(\gamma)$ is union of Mœbius bands or cylinders) called the *stable separatrices* of γ. Each separatrix is an immersion of $S^1 \times [0, +\infty)$ that is injective on $S^1 \times (0, +\infty)$ (in the Mœbius case, the immersion is not injective on the boundary).

Proposition 1.11 *Let f be a transitive, partially hyperbolic diffeomorphism of a compact 3-manifold, with a periodic circle γ. Assume that at least one stable and one unstable separatrix of γ is complete (for the induced metric).*

Then there is an invariant foliation $\mathcal{F}^{cs}$ tangent to E^{cs}, and a unique center unstable foliation $\mathcal{F}^{cu}$ tangent to E^{cu}. Intersecting the leaves of these foliations gives an invariant foliation $\mathcal{F}^c$ tangent to E^c. In other words, f is dynamically coherent.

Proof: The orbit of the complete stable separatrix satisfies all the hypotheses of Proposition 1.6: the orbit of the separatrix is an f-invariant, complete injectively immersed surface with compact boundary (the orbit of γ) tangent to E^{cs}, and finally is dense according to Lemma 1.1. So it extends in a unique way to a foliation $\mathcal{F}^{cs}$ tangent to E^{cs}.

The f-invariance of the orbit of the separatrix and the uniqueness of its extension to a foliation imply that $\mathcal{F}^{cs}$ is an invariant foliation. □

The hypothesis that the separatrices of γ are complete in Proposition 1.11 does not come for free, although we don't know of an example of partially hyperbolic diffeomorphism having a non-complete separatrix. Since the issues at hand are perhaps not immediately obvious, we will discuss this hypothesis in some detail in the following section.

1.5 The center foliation and the strong unstable foliation in the center-unstable leaves

In an Anosov flow, the center unstable leaves are the union of the strong unstable leaves through the orbits of the flow. This is due to the fact that the strong unstable foliation is invariant under the flow. In other words, the union of the strong unstable leaves through an orbit γ is saturated by the orbits of the flow. This implies that this union of leaves is complete (as an injectively immersed submanifold), and therefore is a whole leaf of the center unstable foliation. One can then easily prove that the lifts of the center unstable leaves to the universal cover are diffeomorphic to planes, trivially bifoliated by the center and the strong unstable foliations.

The same argument does not work for partially hyperbolic diffeomorphisms. While the strong unstable foliations are still invariant under the dynamics, this invariance no longer implies "invariance under translation along center leaves." In theory, at least, the union of the strong stable leaves through a given center leaf might not be complete, and so might not be equal to an entire leaf of the center unstable foliation.

In this section, we consider a dynamically coherent, partially hyperbolic diffeomorphism f of a 3 manifold M, endowed with its invariant foliations $\mathcal{F}^{cs}$, $\mathcal{F}^{cu}$, $\mathcal{F}^{ss}$, $\mathcal{F}^{uu}$, and $\mathcal{F}^{c}$. For any center leaf C, we will denote by $W^s(C)$ and $W^u(C)$ the union of the strong stable and strong unstable leaves, respectively, through the points of C. We describe criteria for completeness of $W^u(C)$ and $W^s(C)$.

The *accessible boundary* of an injectively inmmersed surface $\mathcal{I}\colon S \to M$ is the set of points $x \in M$ such that there is a path $\sigma\colon [0,+\infty) \to S$ which is proper (i.e. $\sigma(t)$ converges to an end of S when $t \to \infty$), and such that its projection $\mathcal{I}\circ\sigma$ to M has finite length and $\lim_{t\to\infty} \mathcal{I}\circ\sigma(t) = x$.

Remark 1.12 *1. The completeness of S in the (pullback of the) induced Riemannian metric is equivalent to the condition that the accessible boundary be empty.*

2. If S is contained in the leaf $\mathcal{L}$ of a foliation on M, then the accessible boundary of S in M coincides with its accessible boundary in $\mathcal{L}$; since $\mathcal{L}$ is complete in the induced metric, any path in S of finite length has its endpoints in $\mathcal{L}$. As a consequence, if S is open in the leaf $\mathcal{L}$, its accessible boundary is disjoint from S.

Proposition 1.13 *For any center leaf C the accessible boundary of $W^u(C)$ is saturated by the strong unstable foliation and disjoint from $W^u(C)$. Moreover, if $W^u(C)$ is saturated by the center foliation, then it has no accessible boundary, and so it is complete.*

The analogous statement holds for $W^s(C)$.

Proof: Notice that $W^u(C)$ is an open set in the center unstable leaf $\mathcal{L}^{cu}(C)$ containing C. By Remark 1.12 (2) the accessible boundary of S is disjoint from S and coincides with its accessible boundary in $\mathcal{L}^{cu}(C)$.

Assume that the accessible boundary of $W^u(C)$ is not empty, and let x be a point of the accessible boundary of $W^u(C)$. Let $\mathcal{L}^{uu}(x)$ be the strong unstable leaf through x. Since $W^u(C)$ is saturated for $\mathcal{F}^{uu}$ and $x \notin S$, the leaf $\mathcal{L}^{uu}(x)$ is disjoint from $W^u(C)$. Let σ be a proper path in $W^u(C)$ with $\lim_{t\to\infty} \mathcal{I}\circ\sigma(t) = x$ and such that $\mathcal{I}\circ\sigma$ has finite length ℓ. Through each point of $\mathcal{I}\circ\sigma$ there is a strong unstable leaf that lies entirely in $W^u(C)$. These unstable leaves vary continuously with the point in $\mathcal{I}\circ\sigma$ and accumulate on $\mathcal{F}^{uu}(x)$, the strong unstable leaf through x. Let U be a local bi-foliated chart at x for the restriction of the foliations $\mathcal{F}^c$ and $\mathcal{F}^{uu}$ to $\mathcal{L}^{cu}(x)$. More precisely, let U be a rectangle with (continuous) coordinates (s,t), $s,t \in [-1,1]$, centered at $x=(0,0)$ whose horizontal segments $[-1,1]\times\{t\}$ are center segments, and whose vertical segments $\{s\}\times[-1,1]$ are strong unstable segments. Then there are $i \in \{1,2\}$ and $s_0 \in (0,1]$ such that $\mathcal{I}\circ\sigma$ crosses any vertical segment $\{(-1)^i s\}\times[-1,1]$, $s \in (0,s_0]$.

Any horizontal segment $[0,(-1)^i s)\times\{t\}$ has finite length, because it is a center segment. Moreover $(0,(-1)^i s)\times\{t\} \subset W^u(C)$ and $(0,t) \notin W^u(C)$. This means that every point $(0,t)$ for this chart is an accessible point of the boundary of $W^u(C)$, and that the accessibility path through this point can be c hosen in a center leaf.

We have shown that $\mathcal{L}^{uu}(x)$ is contained in the accessible boundary of $W^u(C)$; that is, the accessible boundary is saturated by $\mathcal{F}^{uu}$. Furthermore, if the accessible boundary is not empty then $W^u(C)$ is not saturated for $\mathcal{F}^c$. This concludes the proof of the proposition. □

Problem 1 *1. Let f be a (transitive) partially hyperbolic diffeomorphism of a compact* 3-*manifold. Are the stable manifolds of the periodic circles for f all complete?* The question remains open even assuming that f is dynamically coherent.

2. *In addition, suppose that f is dynamically coherent (and the manifold has arbitrary dimension). Is it true that every center-stable leaf of f is the union of the strong stable leaves through a center-leaf?*

2 Skew products

In this section we prove Theorem 1. Throughout this section f denotes a transitive, partially hyperbolic diffeomorphism of a compact 3-manifold admitting a circle γ embedded in M that is fixed under f. We have seen that γ is tangent to the center bundle and that its invariant manifolds $W^s(\gamma)$ and $W^u(\gamma)$ are surfaces tangent to E^{cs} and E^{cu}, respectively.

Denote by $W^s_\delta(\gamma)$ and $W^u_\delta(\gamma)$ the local stable manifolds of γ. These are obtained by taking the union of the strong stable and strong unstable segments, respectively, of length δ centered at points of γ. Assume that there is $\delta > 0$ such that $W^s_\delta(\gamma) \cap W^u_\delta(\gamma) \setminus \gamma$ contains a connected component which is a circle γ_1.

2.1 f is dynamically coherent

As remarked above, γ_1 is tangent to the center bundle E^c; in particular, it is transverse to the strong stable foliation inside the cylinder or Mœbius band $W^s_\delta(\gamma)$. It follows that either γ_1 bounds a Mœbius band containing γ, or γ and γ_1 bound an annulus in $W^s_\delta(\gamma)$. Denote by $C^s \subset W^s(\gamma)$ this annulus or Mœbius band. In the same way, denote by $C^u \subset W^u(\gamma)$ the annulus (or Mœbius band) joining γ to γ_1.

Lemma 2.1 *There exists $k > 0$ such that, for any $n, m \in \mathbb{Z}$ satisfying $n - m > k$, every connected component of $f^n(C^u) \cap f^m(C^s)$ is a circle tangent to E^c.*

Proof: Note that each separatrix of γ is either fixed by f or has period 2, depending on whether f preserves the transverse orientation of γ in the corresponding invariant manifold.

Since γ_1 is contained in $W^s_\delta(\gamma)$, we have that, for $i > 0$ sufficiently large, $f^i(\gamma_1)$ is either disjoint from or contained in the open annulus $int(C^s)$, and similarly $f^{-i}(\gamma_1)$ is either disjoint from or contained in $int(C^u)$. Let k be the smallest integer such that any $i \geq k$ has this property.

It now suffices to consider the case where $m = 0$ and $n \geq k$. Let c be any component $f^n(C^u) \cap C^s$: it is compact, since it is the intersection of compact sets. Moreover, as C^s and C^u are tangent to E^{cs} and E^{cu} (and in particular are transverse compact embedded surfaces with boundary), c is a compact curve tangent to E^c. To prove that c is a circle, it suffices to show that ∂c is empty.

Because $f^n(C^u)$ is transverse to C^s, the boundary ∂c must be contained in $\partial C^s \cup \partial f^n(C^u) = \gamma \cup \gamma_1 \cup f^n(\gamma_1)$. For if $x \in c$ belongs to both $int(f^n(C^u))$ and C^s, there there is an arc in c through x obtained by intersecting transverse disks in $f^n(C^u)$ and C^s; hence such an x cannot be in ∂c. Since n was chosen greater than k, the curves γ, γ_1 and $f^n(\gamma_1)$ are either disjoint from or contained in the intersection $C^s \cap f^n(C^u)$. In the second case, these curves are connected components of this intersection. In both cases, the curves cannot contain an extremity of another component. Hence, each component c is boundaryless, proving that it's a circle. □

Remark 2.2 *In fact we can take $k = 0$ in Lemma 2.1. The proof follows the same outline, but we must show that, for all n, the curves γ, γ_1, and $f^n(\gamma_1)$, are either disjoint or are connected components of the intersection $C^s \cap f^n(C^u)$. To see this, we note that since f expands C^u and contracts C^s, we have that $f^{n+i}(C^u) \supset f^n(C^u)$ and $f^{-i}(C^s) \supset C^s$, for $i > 0$ large enough. Hence the curves γ, γ_1, and $f^n(\gamma_1)$ are contained in the intersection $f^{n+i}(C^u) \cap f^{-i}(C^s)$, and so are disjoint circles, by Lemma 2.1.*

We next show that the circles in the intersections $C^s \cap f^n(C^u)$, $n \in \mathbb{Z}$ form a dense family in C^s.

Corollary 2.3 *The connected components of $C^s \cap f^n(C^u)$, $n \in \mathbb{Z}$ form a family $\mathcal{C}_1$ of disjoint circles dense in C^s, tangent to E^c, and therefore uniformly transverse to the strong stable foliation.*

Proof: From Lemma 2.1, we know that these circles form a disjoint family tangent to E^c, and so uniformly transverse to the strong stable foliation. The density of these circles follows from Corollary 1.2. $\square$

In the same way, we define $\mathcal{C}_2$ to be the family of circles obtained by intersecting C^u with the f-iterates of C^s. Let $W^u_+ = \bigcup_{n>0} f^n(C^u) \subset W^u(\gamma)$. It is a surface tangent to E^{cu} whose boundary is contained in the orbit of γ_2. Similarly let $W^s_+ = \bigcup_{n>0} f^{-n}(C^s) \subset W^s(\gamma)$ Let $\mathcal{C}_1^\infty$ be the union of the iterates $f^n\mathcal{C}_1$, $n > 0$, and similarly define $\mathcal{C}_2^\infty$.

Proposition 2.4 *The family $\mathcal{C}_1^\infty$ can be completed in a unique way to an f-invariant foliation $\mathcal{F}_1^c$ of W^u_+ by circles tangent to E^c.*

Analogously, the family $\mathcal{C}_2^\infty$ can be completed in a unique way to a foliation $\mathcal{F}_2^c$ of W^s_+ by circles tangent to E^c.

Proof: First we show that the circles in $\mathcal{C}_1^\infty$ are disjoint and that the family is f-invariant. This follows directly from the following lemma.

Lemma 2.5 *Let n, m be two integers such that $f^n(C^s) \subset f^m(C^s)$. Then the family $f^n(\mathcal{C}_1)$ is the restriction to $f^n(C^s)$ of the family $f^m(\mathcal{C}_1)$.*

Proof:

$$f^n(\mathcal{C}_1) = \bigcup_{-\infty}^{+\infty} \left(f^i(C^u) \cap f^n(C^s)\right) = \left(\bigcup_{-\infty}^{+\infty} f^i(C^u)\right) \cap f^n(C^s) =$$

$$= \left(\left(\bigcup_{-\infty}^{+\infty} f^i(C^u)\right) \cap f^m(C^s)\right) \cap f^n(C^s) = f^m(\mathcal{C}_1) \cap f^n(C^s)$$

$\square$

To show that the family $\mathcal{C}_1^\infty$ extends to a foliation, it suffices to show that $\mathcal{C}_1$ can be uniquely extended to a foliation of C^s. The preceding lemma and the definition of $\mathcal{C}_1^\infty$ then imply the result. The proof is reduced to the following lemma. $\square$

Lemma 2.6 *The family $\mathcal{C}_1$ can be completed in a unique way to a foliation $\mathcal{F}_1^c$ of C^s by circles tangent to E^c.*

Analogously, the family $\mathcal{C}_2$ can be completed in a unique way to a foliation $\mathcal{F}_2^c$ of C^u by circles tangent to E^c.

Proof: C^u is trivially foliated by the strong unstable leaves. By uniform transversality, the dense family $\mathcal{C}_2$ cuts each unstable segment L^u in a dense subset. Moreover, given any two strong unstable tranversals L^u_1 and L^u_2, the holonomy map along the circles of $\mathcal{C}$ induces an order preserving bijection from $L^u_1 \cap \mathcal{C}_2$ to $L^u_2 \cap \mathcal{C}_2$. Recall that an order preserving bijection between two dense subsets of segments extends in a unique way to a homeomorphism between the segments. Now the leaf through a point x in the transversal L^u is the set of its images in the other transversals by the extended holonomy.

Each such leaf is an uniform limit of leaves tangent to E^c and so is also tangent to E^c. $\square$

Lemma 2.7 *W^u_+ and W^s_+ are complete in the metric induced by M on $W^u(\gamma_2)$.*

Proof: Clearly W^u_+ is tangent to E^{cu} because C^u is tangent to E^{cu} which is f-invariant. There is a fundamental domain of W^u_+ foliated by circles tangent to E^c, so that W^u_+ is foliated by center circles. As E^c is a continuous bundle, the length of these circles is greater than some constant δ. Moreover, W^u_+ is by construction trivially foliated by half strong unstable leaves originating on the orbit of γ_2. Hence for any point $x \in W^u_+$ that is not in the local unstable manifold of the orbit of γ, W^u_+ contains the union of strong unstable segments of length δ centered at a point of the center circle through x.

As a consequence, there is a constant $c_0 > 0$, uniform over W^u_+ and independent of small δ by continuity of the partially hyperbolic splitting, such that W^u_+ contains a disk of radius $c_0\delta$ centered x, proving that W^u_+ is complete.

$\square$

At this point, we know that W^u_+ and W^s_+ are codimension 1, injectively immersed submanifolds with compact boundary, tangent to E^{cu} and E^{cs}, respectively. Since they are f-invariant, they are dense in M by Lemma 1.1. We have just shown in Lemma 2.7 that W^u_+ and W^s_+ are complete. Using Proposition 1.6, we conclude that W^u_+ and W^s_+ extend uniquely to f-invariant foliations $\mathcal{F}^{cu}$ and $\mathcal{F}^{cs}$.

In fact, by Proposition 1.11, we obtain:

Corollary 2.8 *The diffeomorphism f is dynamically coherent.*

Lemma 2.9 *Each center stable or center-unstable leaf $\mathcal{L}^{cs}$ contains a dense subset (for the topology of leaf) of points whose center leaf is a circle.*

Proof: By Corollary 1.2, we have that the intersection of $W^u_+ = \bigcup_{n>0} f^n(C^u)$ with $\mathcal{L}^{cs}$ is dense in $\mathcal{L}^{cs}$ in the leaf topology. By dynamical coherence, $\mathcal{L}^{cs}$ intersects W^u_+ in entire center leaves. Since the center leaves in W^u_+ are all compact, the dense set of leaves in the intersection $W^u_+ \cap \mathcal{L}^{cs}$ are all compact. $\square$

Arguing as in Lemma 2.6 we obtain:

Lemma 2.10 *For each compact periodic center leaf σ, its invariant manifolds $W^s(\sigma)$ and $W^u(\sigma)$ are foliated by entire center leaves, which are all compact.*

As a consequence (arguing as in Lemma 2.7), we have:

Corollary 2.11 *For every compact periodic center leaf σ, the invariant manifolds $W^s(\sigma)$ and $W^u(\sigma)$ are complete in the metric induced by the metric on the manifold, and so are an entire leaves of $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ respectively.*

Applying the above corollary to γ, we now obtain that $W^s(\gamma)$ and $W^u(\gamma)$ are complete leaves of $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$, respectively, and are foliated by compact center leaves (we previouly knew the same result for W^s_+ and W^u_+).

2.2 Compact leaves for the center foliation in a neighborhood of γ

The aim of the two next sections is to prove

Proposition 2.12 *Each leaf of the center foliation $\mathcal{F}^c$ is compact (so is diffeomorphic to a circle) and the length of the center leaves is uniformly bounded on M.*

We know that f is dynamically coherent and that the center foliation contains a dense set of compact leaves. To show all all leaves of $\mathcal{F}^c$ are compact, we take two steps. First, we show that a tubular neighborhood of γ is foliated by compact leaves. As f is transitive, this implies that on an open-dense subset of M, the leaves of $\mathcal{F}^c$ are compact. This alone does not imply Proposition 2.12: it is possible to construct a 1-dimensional foliation of a compact 3-manifold with a noncompact leaf that nonetheless has an open-dense set of compact leaves.[4]

In the second step, to conclude that all of the leaves of $\mathcal{F}^c$ are compact of bounded length[5], we use the fact that $\mathcal{F}^c$ is subordinate to two codimension 1 foliations.

We now prove the first step.

The curve γ is tangent to a leaf of $\mathcal{F}^{cs}$ and to a leaf of $\mathcal{F}^{cu}$. Consider the holonomy of these foliations along γ. Recall that the *germ of holonomy* of a foliation along a closed curve contained in a leaf is a homeomorphism of a tranverse disk to the foliation through a point p of curve. The germ at p of this homeomorphism is well-defined, depends only on the homotopy class of the curve in the leaf, and does not depend (up to conjugacy) on the choice of the transverse disk.

Denote by H^{cs}_γ and H^{cu}_γ the (germs of the) holonomies of $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ along γ: they are germs of homeomorphisms of a segment transverse to $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ through a point of γ.

Lemma 2.13 $(H^{cs}_\gamma)^2 = id$ *and* $(H^{cu}_\gamma)^2 = id$

Proof: As the germ of holonomy of $\mathcal{F}^{cs}$ along γ does not depend, up to conjugacy, on the transverse segment, one can choose a segment of strong unstable leaf through a point of γ. This segment is contained in $W^u(\gamma)$. Notice that $W^u(\gamma)$ is foliated by center leaves, which are the intersection of $W^u(\gamma)$ with the leaves of $\mathcal{F}^{cs}$. One deduces that the holonomy of $\mathcal{F}^{cs}$ along γ coincides with the holonomy along γ of the restriction $\mathcal{F}^c|_{W^u(\gamma)}$ of the center foliation to $W^u(\gamma)$.

Recall that $W^u(\gamma)$ is an annulus or a Mœbius band and that the leaves of $\mathcal{F}^c|_{W^u(\gamma)}$ are circles transverse to the strong instable leaves: as a consequence, this holonomy H^{cs}_γ is either the identity (if $W^u(\gamma)$ is a cylinder) or is a period two diffeomorphisme i.e. $(H^{cs}_\gamma)^2 = id$, if $W^u(\gamma)$ is a Mœbius band. In the same way, the holonomy H^{cu}_γ of $\mathcal{F}^{cu}$ along γ coincides with the holonomy of $\mathcal{F}^c|_{W^s(\gamma)}$ along γ, and so satisfies $(H^{cu}_\gamma)^2 = id$. □

In order to understand the behavior of the leaves of $\mathcal{F}^{cs}$, $\mathcal{F}^{cu}$, and finally of $\mathcal{F}^c$ in the neighborhood of γ, we will use the following basic idea of foliation theory:

Given a compact connected domain D of a leaf of a foliation, the restriction of the foliation to a small tubular neighborhood of D is completely determined, up to conjugacy, by the holonomy group of the leaf restricted to the closed paths contained in D.

Let Γ^s be a compact tubular neighborhood of γ in $W^s(\gamma)$ bounded by 1 or 2 center leaves. As the intersection $W^s(\gamma) \cap W^u(\gamma)$ is dense in $W^s(\gamma)$ in the leaf topology, one can assume that these center leaves are contained in $W^s(\gamma) \cap W^u(\gamma)$. Notice that Γ^s is an annulus or a Mœbius band, and is the union of segments of strong stable leaves centered at points of γ.

[4]Let $\varphi : S^1 \to \mathbf{R}$ be a nonconstant continuous function such that $\varphi^{-1}(\mathbf{Q})$ contains an open-dense subset of S^1. This defines a homeomorphism of $S^1 \times S^1$ by $(x, y) \mapsto (x, y + \varphi(x))$. If we suspend the example, the orbit foliation has an open-dense set of compact leaves but also noncompact leaves. Note that this foliation is tangent to a codimension 1 foliation, but not two of them.

[5]A theorem by Epstein in [Ep] asserts that any foliation of a compact 3-manifold by circles is topologically conjugate to a Seifert bundle; this is not true for open 3-manifolds like $\mathbb{R}^3$, nor on compact manifolds of dimension ≥ 4. In our situation a very elementary argument gives at the same time that the leaves are compact and of bounded length.

Lemma 2.14 *Γ^s admits a compact tubular neighborhood $\pi\colon U \to \Gamma^s$ such that:*

- *the fibers of π are segments of leaves of the strong unstable foliation,*
- *for any leaf L^{cs} of the restriction $\mathcal{F}^{cs}|_U$ of $\mathcal{F}^{cs}$ to U* [6]*, the projection $\pi\colon L^{cs} \to \Gamma^s$ is either a homeomorphism or a 2-fold covering map, according to if H^{cs}_γ is the identity or a period 2 homeomorphism.*
- *the boundary of U is the union of $\pi^{-1}(\partial\Gamma^s)$ and of 1 or 2 leaves of $\mathcal{F}^{cs}|_U$ contained in $W^s(\gamma)$ (with the number of leaves equal to the period of H^{cs}_γ).*

Proof: Since the strong unstable foliation is transverse to Γ^s, there is a tubular neighborhood $\pi_0\colon \Delta \to \Gamma^s$ whose fibers are segments of strong unstable leaves. Fix a point $x_0 \in \gamma$ and consider the strong unstable segment $\pi_0^{-1}(x_0)$.

The holonomy H^{cs}_γ is either the identity or a period 2 homeomorphism. It follows that there is a neighborhood σ_0 of x_0 in σ such that, for any $x \in \sigma_0$, the leaf of $\mathcal{F}^{cs}|_\Delta$ through x projects by π_0 onto Γ^s as a homeomorphism or a 2-fold covering map. (Here is where we use that the holonomy of a compact part of a leaf determines the foliation of its tubular neighborhood.)

As $W^s(\gamma)$ is dense in M, there is a dense set of points x in σ_0 such that the leaf of $\mathcal{F}^{cs}|_\Delta$ through x is contained in $W^s(\gamma)$. So up to shrinking σ_0 we may assume that the endpoints of σ_0 belong to $W^s(\gamma)$.

Now we define U to be the union of the leaves of $\mathcal{F}^{cs}|_\Delta$ through points of σ_0, and π to be the restriction of π_0 to U. □

Lemma 2.15 *With the notations above, the center leaves through the points of U are all compact with uniformly bounded length.*

Proof: Let L^{cs} be a leaf of $\mathcal{F}^{cs}|_U$, and x a point of L^{cs}. Consider $y = \pi(x) \in \Gamma^s$ and let γ_y be the center leaf through y. It is a circle contained in Γ^s, by construction of Γ^s. Then $\pi^{-1}(\gamma_y) \cap L^{cs}$ is a 1 or 2 fold cover of γ_y; in particular the connected component containing x is a circle γ_x. However $\pi^{-1}(\gamma_y)$ is an annulus or Mœbius band contained in a center unstable leaf (because it is the union of strong unstable segments through a center leaf, and f is dynamically coherent). Then γ_x is a closed curve in the intersection of a center-stable and a center-unstable leaf, so that γ_x is a closed center leaf.

Finally, each curve γ_x is tangent to a continuous bundle, and is a 1 or 2 fold cover of a curve γ_y which is itself a 1 or 2 fold cover of γ. We deduce that the length of γ_x is uniformly bounded in Γ^s. □

Notice that the boundary of U consists of 1 or 2 annuli or Mœbius bands contained in $W^u(\gamma)$, and 1 or 2 annuli or Mœbius bands contained in $W^s(\gamma)$.

Remark 2.16 *In fact we have shown that the holonomy H^c_γ of the foliation $\mathcal{F}^c$ along γ is the cartesian product of the holonomies H^{cs}_γ and H^{cu}_γ, so that H^c_γ is an homeomorphism (of a 2-disk transverse to γ) which is periodic of period at most 2 (the cartesian product of two homeomorphisms of period 2 also has period 2).*

2.3 The center leaves are compact

Lemma 2.17 *Each connected component of $W^s(\gamma) \cap (M \setminus \overset{o}{U})$, where $\overset{o}{U}$ denotes the interior of U, is a compact cylinder.*

[6] that is, L^{cs} is a connected component of the intersection of a leaf of $\mathcal{F}^{cs}$ with U .

Proof: Let C be such a component. By construction, the boundary of C is 1 or 2 center leaves. If C is not a compact cylinder, then it is diffeomorphic to $S^1 \times \mathbb{R}$ and is complete. Note that C is obtained from W^s_+ by removing a compact part (either cylinder or Mœbius band) from one of the separatrices of γ. Since C is disjoint from $\overset{o}{U}$, this separatrix can cut $\overset{o}{U}$ in only finitely many components; in particular it does not accumulate on γ. If f fixes this separatrix, it must be dense by Lemma 1.1, and we immediately obtain a contradiction.

Otherwise, there are two separatrices, and f must permute them. Since γ is invariant, and one of these separatrices does not accumulate on γ, the other does not accumulate on γ either. But this contradicts the density of $W^s(\gamma)$. $\square$

Let $\tilde{M} = (M \setminus \overset{o}{U})$ and denote by $\tilde{\mathcal{F}}^{cs}$ and $\tilde{\mathcal{F}}^{cu}$ the restrictions to $\tilde{M}$ of $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$, respectively.

Proposition 2.18 *Each leaf of $\tilde{\mathcal{F}}^{cs}$ and of $\tilde{\mathcal{F}}^{cu}$ is compact.*

Proof: This is the classical result of A. Haefliger [Ha]:

For any codimension 1 foliation on a compact manifold, the set of points contained in a compact leaf is compact.

Here $\tilde{M}$ is a compact manifold (with boundary and corners) on which $\tilde{\mathcal{F}}^{cs}$ admits a dense subset of compact leaves (Lemma 2.17). So that the reader won't be disturbed by the boundary and corners, we'll reproduce the argument of Haefliger in this case.

Recall that U is bounded by 1 or 2 center-unstable annuli or Mœbius bands and 1 or 2 center-stable annuli or Moebius bands. Each center-unstable component of ∂U is foliated by strong unstable segments. Choose one such segment in each component, and let T^u be the union of these segments. Recall that the center-stable boundary of U is also foliated by center circles, and note that each circle meets T^u in at most 2 points.

Lemma 2.19 *Each leaf $\tilde{\mathcal{F}}^{cs}$ meets the interior of T^u.*

Proof: We first show that each leaf of $\mathcal{F}^{cs}$ meets $\overset{o}{U}$. Since $W^u(\gamma)$ cuts every leaf of $\mathcal{F}^{cs}$, and M is compact, there is a compact part of $W^u(\gamma)$ that cuts every center-stable leaf. Iterating by f^{-1}, and using the fact that γ is fixed by f, we obtain that any local unstable manifold of γ meets every leaf of $\mathcal{F}^{cs}$. Hence there is a local untable manifold of γ contained in $\overset{o}{U}$ that meets every leaf of $\mathcal{F}^{cs}$.

Now let L^{cs} be a leaf of $\tilde{\mathcal{F}}^{cs}$. Since the corresponding leaf of $\mathcal{F}^{cs}$ (in M) meets $\overset{o}{U}$, the boundary of L^{cs} must be nonempty. Its boundary consists of circles in the unstable boundary of U, and therefore meets the interior of T^u. $\square$

Lemma 2.20 *Each component of $W^s(\gamma)$ in $\tilde{M}$ meets the interior of T^u in at most 4 points.*

Proof: Let C be a component of $W^s(\gamma) \cap \tilde{M}$. Then C is a cylinder by Lemma 2.17, and so its boundary consists of 2 circles contained in the unstable boundary of U. Since each circle meets T^u in at most 2 points, the boundary of C meets T^u in at most 4 points. $\square$

Now suppose that L^{cs} is a noncompact leaf of $\tilde{\mathcal{F}}^{cs}$.

Lemma 2.21 *L^{cs} meets the interior of T^u in infinitely many points.*

Proof: Since L^{cs} is not compact, it must accumulate on another leaf $\hat{L}^{cs}$ of $\tilde{\mathcal{F}}^{cs}$. The leaf $\hat{L}^{cs}$ must cut the interior of T^u at least once. Hence if L^{cs} accumulates on the entire leaf $\hat{L}^{cs}$, it must intersect the interior of T^u infinitely many times. The other possibility is that L^{cs} does not accumulate on the entire $\hat{L}^{cs}$ leaf. This means that there is a path in $\hat{L}^{cs}$ that cannot be approximated by a path in L^{cs}. In the foliation $\mathcal{F}^{cs}$, every path in the leaf containing $\hat{L}^{cs}$ can be approximated by a path in the leaf containing L^{cs}. Hence every approximating path in the leaf containing L^{cs} must cross $\overset{o}{U}$. But the only way this can happen is if L^{cs} itself intersects the interior of T^u infinitely many times. $\square$

It is clear that L^{cs} does not meet the stable boundary of U. Consequently, given any path $\sigma : [0,1] \to L^{cs}$, if y is sufficiently close to $\sigma(0)$, then the leaf of $\tilde{\mathcal{F}}^{cs}$ through y contains a path close to σ. Fix a point $x_0 \in L^{cs}$. Since L^{cs} meets the interior of T^u infinitely many times, it now follows that for any $k \in \mathbb{N}$, any leaf of $\tilde{\mathcal{F}}^{cs}$ containing a point close to x_0 meets T^u at least k times. But there is dense set of leaves in $\tilde{\mathcal{F}}^{cs}$ meeting T^u at most 4 points, namely, the components of $W^s(\gamma) \cap \tilde{M}$. This gives a contradiction. $\square$

We are now ready to prove Proposition 2.12.

Proof of the proposition: By Proposition 2.18, each center leaf in $\tilde{M}$ is the intersection of two compact leaves and so is compact. By Lemma 2.15, each leaf of the center foliation in U is also compact. Moreover the holonomy of any center leaf is the cartesian product of the holonomies of the center-stable and center-unstable holonomies along this curves, so that it is either the identity or a periodic map of period ≤ 2 (depending on whether the foliations are orientable or not). As a consequence, the length of the center leaves in a neighborhood of a given center leaf is bounded. Compactness of M implies that these lengths are uniformly bounded. $\square$

2.4 f is a skew product

Let f be a transitive partially hyperbolic diffeomorphism on a 3-manifold satisfying the hypotheses of Theorem 1. We have seen that f is dynamically coherent and that the center foliation is by compact leaves having an at most period 2 holonomy. Moreover each leaf has trivial holonomy if the center stable and center unstable foliations are transversaly orientable. Let $\tilde{f}$ be a lift of f on the covering $\tilde{M}$ corresponding to the transverse orientations of the foliations $\mathcal{F}^{cs}$, $\mathcal{F}^{cu}$. Then $\tilde{f}$ is a partially hyperbolic diffeomorphism, possibly non-transitive.

Notice that the center stable and center unstable foliations for $\tilde{f}$ are now transversely orientable, and that the lifted center leaves remain compact. By taking this lift, we have now assured that the lifted center-stable and center-unstable holonomies along any center leaf have period 1; that is, they are both the identity map. Hence, the holonomy of the center foliation along any center leaf is also the identity map. This implies that the center foliation $\tilde{\mathcal{F}}^c$ of $\tilde{f}$ is a locally trivial fibration over the space of leaves, which is a compact surface S. Moreover, the (tranversally oriented) center stable and center unstable foliations $\tilde{\mathcal{F}}^{cs}$ and $\tilde{\mathcal{F}}^{cu}$ are subfoliated by the center leaves, so that they induce regular, transversally oriented, topologically transverse foliations on S. Since S admits a foliation, it has Euler characteristic 0. Since both foliations are transversally orientable, they together give an orientation of S, and so S is orientable. It follows that S is the torus T^2. Finally, $\tilde{f}$ preserves the center foliation so that it passes to the quotient in an homeomorphism h of S.

It remains to show that $h : S \to S$ is topologically conjugate to an Anosov diffeomorphism. Here are two possible approaches to this problem. In the first approach, one shows that the action of h on $H^1(S) = H^1(T^2)$ is given by a hyperbolic matrix $A \in SL(2,\mathbb{R})$. Then one constructs by hand a conjugacy between h and the linear Anosov diffeomorphism induced by A

on T^2. In the second approach, one shows that h is an expansive homeomorphism. Appealing to a result of [Le, Hi], it then follows that h is Anosov. In the remainder of this section, we complete the details of this second proof.

For points x_1 and x_2 in the quotient space S, let $\gamma(x_1)$ and $\gamma(x_2)$ be the corresponding fibers in $\tilde{M}$, and denote by $\delta(x_1, x_2)$ the Hausdorff distance between the fibers $\gamma(x_1)$ and $\gamma(x_2)$ with respect to some fixed Riemannian metric on $\tilde{M}$.

Proposition 2.22 *The homeomorphism $h: S \to S$ is expansive; that is, there exists $\alpha_0 > 0$ such that, if x_1 and x_2 satisfy $\delta\left(h^k(x_1), h^k(x_2)\right) \le \alpha_0$ for all $k \in \mathbb{Z}$, then $x_1 = x_2$.*

Proof:
First, just using that $\tilde{\mathcal{F}}^{cs}$ and $\tilde{\mathcal{F}}^{u}$ are transverse foliations and that $\tilde{\mathcal{F}}^{cs}$ is subfoliated by $\tilde{\mathcal{F}}^{c}$ and $\tilde{\mathcal{F}}^{s}$ which are transverse, one deduces:

Lemma 2.23 *For any $\eta > 0$ there exists $\delta > 0$ such that, for any x_1, x_2 with $\delta(x_1, x_2) \le \delta$, there is a strong unstable segment $\sigma_1 \subset \tilde{M}$ and a strong stable segment $\sigma_2 \subset \tilde{M}$ both of length bounded by η, such that $\sigma_1 \cup \sigma_2$ is a segment joining a point of $\gamma(x_1)$ to a point of $\gamma(x_2)$.*

Let $C = sup\{\|D\tilde{f}(y)\|, \|(D\tilde{f}(y))^{-1}\|, y \in \tilde{M}\}$.

Lemma 2.24 *There exist $\alpha > 0$ and $\eta > 0$ such that, if σ_1 and σ_2 are strong stable and strong unstable segments, respectively, joining points y_1 to y_2 and y_2 to y_3, respectively, and if $\sup\{\ell(\sigma_1), \ell(\sigma_2) \in [\eta, C\cdot\eta]$, then $\delta(\gamma(x_1), \gamma(x_3)) \ge \alpha$, where x_1 and x_3 are the projections of y_1 and y_3, respectively.*

Proof: For any $\eta > 0$, the set of pairs (σ_1, σ_2) joining points y_1 to y_2 and y_2 to y_3 satisfying $\sup\{\ell(\sigma_1), \ell(\sigma_2)\} \in [\eta, C\cdot\eta]$ is compact, and the Hausdorff distance between the corresponding fibers is a continuous function. So it is enough to find η such that the path obtained as union of σ_1 and σ_2 cannot have both endpoints on the same center leaves.

For that, first notice that, as the center foliation is a locally trivial fibration tangent to a continuous bundle, there is constant β such that, if points in the same center leaf have a distance in $\tilde{M}$ less that β, then they are joined by a center path of length less than 2β. Now, just by transversality of the strong stable, strong unstable and center bundles, one gets, for β small enough, that there is no "triangle" whose sides are strong stable, strong unstable and center segments of length less than 2β.

Now it is enough to choose η such that $2C\eta < \beta$. $\square$

Let η and α be given by Lemma 2.24 above and let δ be the constant associated to η by Lemma 2.23. Let $\alpha_0 = \inf\{\alpha, \delta\}$. Then α_0 is a constant of expansivity for h:

Consider $x_1 \ne x_2$ such that $\delta(x_1, x_2) \le \alpha_0$. Let σ_1 and σ_2 be the strong stable and strong unstable segments of length less than η, given by Lemma 2.23, whose union joins $\gamma(x_1)$ to $\gamma(x_2)$. One of these two segments is nontrivial, so that there is some iterate (positive or negative) for which one of these segments has length greater than η. Let $k \ge 0$ be the smallest integer such that the supremum $\sup\{\ell(\tilde{f}^k(\sigma_1)), \ell(\tilde{f}^{-k}(\sigma_1)), \ell(\tilde{f}^k(\sigma_2)), \ell(\tilde{f}^{-k}(\sigma_2))\}$ is greater than η. Notice that, by the choice of C, this supremum belongs to $[\eta, C\cdot\eta]$. Then at least one of the two pairs $(\tilde{f}^k(\sigma_1)), (\tilde{f}^k(\sigma_2))$ and $(\tilde{f}^{-k}(\sigma_1)), (\tilde{f}^{-k}(\sigma_2))$ satisfies the hypothesis of Lemma 2.24 so that $\sup\{\delta\left(h^k(x_1), h^k(x_2)\right), \delta\left(h^{-k}(x_1), h^{-k}(x_2)\right)\} \ge \alpha \ge \alpha_0$.

$\square$

Any expansive homeomorphism on a compact surface is conjugate to a pseudo Anosov homeomorphism; this result has been proved independently by [Le] and [Hi]. On the torus T^2, any pseudo-Anosov homeomorphism is conjugate to a linear Anosov map. So we get:

Corollary 2.25 *The homeomorphism h is conjugate to an Anosov diffeomorphism of the torus T^2.*

3 Behavior seen in Anosov flows

The aim of this section is to prove Theorem 2. In this section, f denotes a partially hyperbolic, dynamically coherent diffeomorphism on a compact 3-manifold M, and we fix a Riemannian metric on M. Let $\mathcal{F}^{ss}, \mathcal{F}^{cs}, \mathcal{F}^{cu}, \mathcal{F}^{uu}$ and $\mathcal{F}^{c}$ be the invariant foliations of f. We assume that there is a closed periodic center leaf γ such that each center leaf in $W^s_{loc}(\gamma)$ is periodic for f.

In contrast with Section 2, we admit here, by hypothesis, the existence of foliations, allowing us to use tools like holonomies. The main difficulty consists in understanding the relation between invariant manifolds of a center leaf c and the center stable and center unstable leaves through c. The hypothesis of dynamical coherence implies that the union $W^s(c)$ of the strong stable leaves through the points of c is contained in the center stable leaf through c; however, dynamical coherence does not say that $W^s(c)$ is complete. We must rule out the possibility that some center leaf intersects $W^s(c)$ along an open bounded interval and then exits $W^s(c)$ through a strong stable leaf in the accessible boundary of $W^s(c)$. (As far as we know, this incompleteness of $W^s(c)$ is only possible in theory; we have no example of a transitive partially hyperbolic diffeomorphism in which this situation arises.)

Given these considerations, an important step of the proof is to show that the invariant manifolds are saturated by the center foliation. Since these manifolds are already saturated by the strong stable or unstable foliation, it then follows that they are complete and coincide with the whole corresponding center stable or center unstable leaf. We begin by solving this problem for the invariant manifold of the periodic compact center leaf γ.

3.1 Properties of the invariant leaves through γ

In this subsection we establish some basic properties of the stable and unstable manifolds $W^u(\gamma)$ and $W^s(\gamma)$ of the periodic leaf γ and of the center leaves contained in these manifolds. The general theory of normally hyperbolic foliations implies that $W^u(\gamma)$ and $W^s(\gamma)$ must be contained in the center-unstable and center-stable leaves through γ, respectively. A priori, it is possible that this containment is strict; see the discussion in Section 2. We show here in Lemma 3.5 that periodicity of the center leaves in $W^s(\gamma)$ prevents this from happenining: both $W^u(\gamma)$ and $W^s(\gamma)$ must be complete and equal to leaves of $\mathcal{F}^{cu}$ and $\mathcal{F}^{cs}$, respectively.

Lemma 3.1 *Fix an orientation of γ. This orientation induces an orientation of the restriction of the center foliation to $W^s(\gamma)$. Then for any strong stable leaf $\mathcal{L}^{ss}$ in $W^s(\gamma)$, the first return map on $\mathcal{L}^{ss}(\gamma)$ of the center leaves in $W^s(\gamma)$ is a homeomorphism of $\mathcal{L}^{ss}$, having a unique periodic point: $\mathcal{L}^{ss} \cap \gamma$.*

The analogous statement also holds for $W^u(\gamma)$.

Proof: We first remark that there is a well-defined first return map of the center foliation to any strong stable manifold intersecting γ. This fact uses only the periodicity and compactness of γ. To see this, note that the first return map of the center foliation on the strong stable leaves, and its inverse, are well-defined on a sufficiently small local stable manifold $W^s_{loc}(\gamma)$ of γ, and this first return map coincides with the holonomy of γ. Let k_γ be the period of γ under f. As f^{k_γ} leaves invariant both center and strong stable foliations of $W^s(\gamma)$, one deduces that the first return map of the center foliation on the strong stable leaves and its inverse are also well-defined on every negative iterate of $W^s_{loc}(\gamma)$. Hence, they are both well-defined on the whole stable manifold $W^s(\gamma)$.

Then for any strong stable leaf $\mathcal{L}^{ss}$, the first return map of $\mathcal{F}^c|_{W^s(\gamma)}$ on $\mathcal{L}^{ss}$ is a homeomorphism. Notice that $\mathcal{L}^{ss} \cap \gamma$ is a fixed point of this return map. Moreover, as $\mathcal{L}^{ss}$ is diffeomorphic to $\mathbb{R}$, any periodic point of the first return map has period less than or equal to 2.

Now assume that this first return map has a periodic point x. This means that the corresponding center leaf is compact. Notice that by definition of the stable manifold of γ, any point of this compact leaf converges to γ under positive iterates of f^{k_γ}. In particular, none of its iterates can be periodic for f, contradicting the hypothesis on the center leaves in the local stable manifold of γ.

Now consider the center foliation in $W^u(\gamma)$. Arguing as above, we see that the first return map of the center leaves on a strong unstable leaf $\mathcal{L}^{uu}$ is a homeomorphism of $\mathcal{L}^{uu}$. We next show that the unique periodic point of this homeomorphism is $\mathcal{L}^{uu} \cap \gamma$: assume that x is another periodic point, so that the corresponding center leaf γ_1 is compact. Let $m > 0$ be the period of the unstable separatrix containing γ_1. Then γ_1 and $f^m(\gamma_1)$ bound an open annulus which is saturated by the center foliation, and whose iterates are pairwise disjoint: in particular no center leaf through a point of this annulus may be periodic. However, as f is transitive, Corollary 1.2 implies that the iterates of W^s_γ cut W^u_γ in a dense subset (in its leaf topology), and each center leaf through these intersection points is periodic for f. These leads us to a contradiction. $\square$

In a 1-dimensional foliation with a noncompact leaf, we can speak of a *ray* of the foliation, that is, a complete injective immersion of $[0, +\infty[$ into a noncompact leaf of the foliation. In this way, we define the notion of *strong stable, strong unstable* and *center rays* in M. An *end* of a non-compact 1-dimensional leaf is an equivalence relation of rays for the relation "$\subset$ or $\supset$"; that is, two rays are equivalent if one contains the other. Notice that each non-compact leaf has two ends.

The following very simple remarks play a non-trivial role in what follows:

Remark 3.2
1. *If an end of a non-compact leaf of an invariant foliation is periodic under f, then both ends are periodic and have the same period. If the leaf itself has period n, then the ends of a leaf have period n or $2n$ depending on whether f^n exchanges the ends or not.*
2. *By Lemma 3.1, each leaf of the restriction of $\mathcal{F}^c$ to $W^s(\gamma)$ contains at least one end of the corresponding entire center leaf: namely, the end spiraling in to γ (the same statement holds for the unstable manifold of γ).*
3. *As a consequence of Lemma 3.1, each leaf of the restriction of $\mathcal{F}^c$ to $W^s_{loc}(\gamma)$ corresponds to an end of an entire leaf of $\mathcal{F}^c$. So the hypothesis of the Theorem 2 may be restated as: each end of center leaf contained in $W^s_{loc}(\gamma)$ is periodic.*
4. *If a point x belongs to $W^s(\gamma) \cap W^u(\gamma)$, then the center leaf through x contains one end in $W^s_{loc}(\gamma)$ and one end in $W^u_{loc}(\gamma)$. These two ends cannot be the same end because $W^s_{loc}(\gamma) \cap W^u_{loc}(\gamma) \setminus \gamma = \emptyset$. However, the end contained in $W^s_{loc}(\gamma)$ is periodic by hypothesis and so the corresponding end in $W^u_{loc}(\gamma)$ is periodic of the same period, by the first item of this remark. Moreover, they cannot be exchanged by any iterate of f so that the corresponding center leaf has the same period too.*

Lemma 3.3 *There exists $n \in \mathbb{N}$ such that any end of a center leaf in $(W^s_{loc}(\gamma) \cup W^u_{loc}(\gamma)) \setminus \gamma$ is periodic of (least) period n.*

Proof: Let W^s_0 be one of the (1 or 2) stable separatrices of γ. Fixing a strong stable ray σ in W^s_0 with initial point $\sigma(0)$ on γ, Lemma 3.1 implies that the holonomy of the center foliation (restricted to W^s_0) on this ray has a unique fixed point (its base point on γ). Hence the space of ends of center leaves of $W^s_{0,loc} \setminus \gamma$ is a circle. Notice that the orientation of σ "from $\sigma(0)$ to the end point at infinity", gives a natural orientation of the circle of ends of center leaves.

Let k be the period of W^s_0: this period is equal to either the period of γ or twice the period of γ. Then f^k acts continuously on the space of ends of center leaves as a homeomorphism of the circle. Since the family of strong stable rays originating in γ is preserved by f^k, the orientation of the circle given by the orientation of σ "from $\sigma(0)$ to the end point at infinity" is preserved by the induced homeomorphism. Moreover, by assumption, every end of a center leaf of $W^s_{0,loc}$ is periodic, so that this action is conjugate to a rational rotation, and all the center leaves in $W^s_{0,loc}$ have the same period. Let n be this period.

Now let W^u_0 be an unstable separatrix of γ, and let m be its period. From Corollary 1.2 the orbit of W^s_0 intersects $W^u_{0,loc}$ in a dense subset of points. These points are on periodic center leaves of period n (in particular n is a multiple of m). Once more, Lemma 3.1 implies that the set of ends of center leaves in W^u_0 is also a circle and the action of f^m on this circle has a dense subset of period n orbits so that it is a rotation of period n. We have so far proved that any center leaf (except possibly γ) of any local unstable separatrix of γ has period n.

Having shown this, we can now prove that any center leaf (except possibly γ) of any local stable separatix (and not just W^s_0) of γ has period n: the center leaves of period n in the unstable separatrices now intersect all local stable separatrices in a dense subset. $\square$

Now we prove that the translation distance of f^n along these center leaves is continuous. Given 2 points x and y in the same center leaf, we denote by $d_c(x,y)$ the length of the shortest center segment joining x to y.

Lemma 3.4 *Let $n > 0$ be the period of the center leaves in $W^s(\gamma)$, given by Lemma 3.3. Then $d_c(x, f^n(x))$ is uniformly bounded on $W^s_{loc}(\gamma) \cup W^u_{loc}(\gamma)$ and continuous on $W^s(\gamma) \setminus \gamma$ (in the leaf topology) and on $W^s(\gamma) \setminus \gamma$.*

Proof: Fix an orientation of γ. This orientation induces an orientation of the restriction of the center foliation to $W^s(\gamma)$. For any $x \in W^s(\gamma) \setminus \gamma$ let $\mathcal{L}^{ss}_+(x)$ be the strong stable ray through x with initial point on γ. Let h^c_x be the first return map of the center foliation (restricted to $W^s(\gamma)$) on $\mathcal{L}^{ss}(x)$. By Lemma 3.1, h^c_x is a homeomorphism of $\mathcal{L}^{ss}_+(x)$ with a unique periodic point (the initial point of $\mathcal{L}^{ss}_+(x)$). Let $I \subset \mathcal{L}^{ss}_+(x)$ be an open interval containing x, and contained in a fundamental domain of h^c_x, that is, disjoint from all of its iterates by the holonomy.

Each center leaf (of $\mathcal{F}^c|_{W^s(\gamma)}$) [7] cuts I in at most one point, and the same property holds for $f^n(I)$.

However, by Lemma 3.3, for each point $y \in I$ the point $f^n(y)$ belongs to the same center leaf as y, and so is the unique intersection point of the center leaf through y and the segment $f^n(I)$. This implies that the holonomy map of the center foliation is an homeomorphism from I to $f^n(I)$ which coincides with f^n. Now, this shows that the unique center segment σ_y joining y to $f^n(y)$ varies continuoulsy for $y \in I$. This implies the continuity of $d_c(x, f^n(x))$ when x varies on a strong stable leaf, and one easily deduces the continuity of this function on $W^s(\gamma) \setminus \gamma$.

Let's show that this function remains bounded when x tends to γ. Notice that the argument above shows that, for each point y in $\mathcal{L}^{ss}_+(x)$, the projection of the segment σ_y on γ along the strong stable leaves coincides with the projection of σ_x. Then the length of this segment converges to the length of the projection when y converges to γ, showing that this length remains bounded. $\square$

Lemma 3.5 *The stable manifold $W^s(\gamma)$ is saturated by the center foliation. As a consequence, $W^s(\gamma)$ is complete (in the leaf topology) and so coincides with the center-stable leaf through γ. In particular this leaf is a cylinder or a Mœbius band. The same holds for $W^u(\gamma)$.*

[7] if a center leaf has both ends in $W^s_{loc}(\gamma)$ the entire leaf might cut I twice, but each connected component of its intersection with $W^s(\gamma)$ cuts at most once.

Proof: As $W^s(\gamma)$ is saturated by the strong stable foliation, to prove that $W^s(\gamma)$ is complete, Proposition 1.13 implies that it is enough to show that it is saturated by the center foliation.

Assume for the sake of contradiction that $W^s(\gamma)$ is not saturated by $\mathcal{F}^c$. Then there is a point $x \in W^s(\gamma)$ whose center leaf $\mathcal{L}^c_x$ is not contained in $W^s(\gamma)$. However, by Lemma 3.1 we know that one center ray R^c_x through x spirals in to γ inside $W^s(\gamma)$, so that it is completely contained in $W^s(\gamma)$. Then the first point of this leaf outside of $W^s(\gamma)$ is well-defined; denote it by z_x. As the end corresponding to R^c_x is periodic of period n, we obtain that $f^n(z_x) = z_x$.

We have just proved that any point of the accessible boundary of $W^u(\gamma)$ is a periodic point of period n. However, as $W^s(\gamma)$ is saturated by $\mathcal{F}^{uu}$, its accessible boundary is composed of strong stable leaves, according to Proposition1.13. Each of these leaves contains at most one periodic point, contradicting the periodicity of every point of the accessible boundary. □

3.2 Lifting

Many properties of the foliations can be easier to see on the universal cover of M, where recurrences of the leaves due to the fundamental group of the manifold disappear, and where the topology of the lifted leaves is simpler.

Let $\pi\colon \tilde{M} \to M$ be the universal cover of M. We denote by $\tilde{\mathcal{F}}^c, \tilde{\mathcal{F}}^{cs}, \tilde{\mathcal{F}}^{cu}, \tilde{\mathcal{F}}^{uu}$ and $\tilde{\mathcal{F}}^{ss}$ the lifts to $\tilde{M}$ of the invariant foliations of f. These foliations are all orientable and tranversely orientable.

Let us first show a simple properties of the lifted foliations:

Proposition 3.6 *The lift $\tilde{\mathcal{L}}$ of any center-stable or center-unstable leaf to $\tilde{M}$ is a plane which is properly embedded in $\tilde{M}$. For any $x, y \in \tilde{\mathcal{L}}$ there is at most one point of intersection between $\tilde{\mathcal{L}}^c(x)$ and $\tilde{\mathcal{L}}^s(y)$ or $\tilde{\mathcal{L}}^u(y)$, respectively.*

Proof of the proposition: Novikov's Theorem states that, if a codimension one foliation $\mathcal{F}$ of a compact 3 manifold M admits a non-null homotopic closed path in a leaf which is homotopic to 0 in M, then $\mathcal{F}$ has a Reeb component[8]. The same holds if the foliation admits a closed transversal which is homotopic to 0 in M.

Lemma 3.7 *The center stable and the center unstable foliations of a transitive partially hyperbolic diffeomorphism on a compact 3-manifold cannot have Reeb components.*

Proof: The center-unstable foliation has at most finitely many Reeb components. The union of their boundaries is an invariant compact manifold that is normally hyperbolic (contracting), so that it is an attractor, contradicting the transitivity of f. □

If a leaf $\tilde{\mathcal{L}}$ of $\tilde{\mathcal{F}}^{cu}$ is not simply connected, there is a path $\tilde{\sigma}$ in $\tilde{\mathcal{L}}$ which fails to be null-homotopic in $\tilde{\mathcal{L}}$. This path is null-homotopic on $\tilde{M}$. Now the projection σ of $\tilde{\sigma}$ on M satisfies the hypotheses of Novikov's Theorem, which implies that $\mathcal{F}^{cu}$ has a Reeb component. This contradicts Lemma 3.7.

In the same way, if the leaf $\tilde{\mathcal{L}}$ is not properly embedded in $\tilde{M}$, then it accumulates on some leaf. Then there exists a transverse segment cutting $\tilde{\mathcal{L}}$ twice, and a classical argument of Haefliger shows that $\tilde{\mathcal{F}}^{cu}$ a dmits a closed transversal. But then $\mathcal{F}^{cu}$ admits a null-homotopic closed transversal: once more Novikov's Theorem implies the existence of a Reeb component, contradicting Lemma 3.7

[8] A Reeb component is a 2-dimensional foliation of the solid torus $D^2 \times S^1$ for which the boundary is a compact leaf, and any other leaf is diffeomorphic to a plane $\mathbb{R}^2$. Novikov's theorem is certainly the most famous result on codimension 1 foliation on 3-manifolds. However a complete and correct proof of this theorem is not so easy to find. A proof of Novikov's theorem may be find in [CaCo, Chapter 9]

The second statement holds for any two transverse foliations on the plane. $\square$

Let n be the period of the center (end of) leaves in $W^s(\gamma)$, given by Lemma 3.3. Let H^u and H^s be connected components of $W^u(\gamma) \setminus \gamma$ and $W^s(\gamma) \setminus \gamma$, respectively, and denote $C = \bigcup_{m\in\mathbb{Z}} f^m(H^u \cup H^s)$.

Proposition 3.8 *1. The lift $\tilde{C}$ of C to $\tilde{M}$ is path-connected, and $\tilde{\mathcal{F}}^c$-saturated*

2. *For every leaf $\mathcal{L}$ of $\mathcal{F}^{uu}, \mathcal{F}^{ss}, \mathcal{F}^{cu}$ or $\mathcal{F}^{cs}$, each lift $\tilde{\mathcal{L}}$ to $\tilde{M}$ meets $\tilde{C}$ in a dense subset (in the leaf topology).*
3. *$\tilde{C}$ is invariant under any lift of any iterate of f*
4. *There is a lift $\tilde{g} : \tilde{M} \to \tilde{M}$ of $g = f^n$ to the universal cover of M such that $\tilde{f^n}(x)$ and x are on the same center leaf, for any $x \in \tilde{C}$.*

Proof of the proposition: Item 3 follows directly from the f-invariance of C. Item 2 follows from the density and completeness of the orbit of any separatrix of γ.

As $W^s(\gamma)$ and $W^u(\gamma)$ are saturated by $\mathcal{F}^c$ (see Lemma 3.5), and as γ is a leaf of $\mathcal{F}^c$, the components H^s and H^u are saturated by $\mathcal{F}^c$. Then the set C is saturated by $\mathcal{F}^c$, and finally $\tilde{C}$ is saturated by $\tilde{\mathcal{F}}^c$. Then item 1 is a direct consequence of the following lemma:

Lemma 3.9 *Suppose that $\sigma : [0,1] \to M$ is a path in M whose endpoints lie in C. Then σ is homotopic to a path σ' in C via an endpoint-fixing homotopy. Moreover, the image of σ' can be chosen to consist of finitely many segments, each lying in H^u or H^s. Consequently, the lift $\tilde{C}$ of C to $\tilde{M}$ is path-connected.*

Proof: Let σ be given, and choose $0 = t_1 < t_2 < \cdots < t_k = 1$ so that $\sigma[t_i, t_{i+1}]$ is contained in a neighborhood B_i contained in foliation charts for all of the invariant foliations. By Lemma 3.5 $W^s(\gamma)$ and $W^u(\gamma)$ are entire leaves and Lemma 1.1 implies that the orbit of each separatrix is dense. As a consequence, H^u and H^s meet each B_i in a dense set of plaques, so we may approximate $\sigma(t_i)$ for $2 < i < k$ by a point x_i lying in one of the plaques. Hence, by an initial homotopy, we may assume that $\sigma(t_i)$ lies in an H^u or H^s plaque. Inside of each B_i, we can find a path in C joining $\sigma(t_i)$ to $\sigma(t_{i+1})$. $\square$

Now choose any center leaf $\mathcal{L}^c_0$ contained in C and a lift $\tilde{\mathcal{L}}^c_0$ of $\mathcal{L}^c_0$ to $\tilde{M}$. Since $\mathcal{L}^c_0$ is simply connected (as is any center leaf in C) and f^n-invariant, there is a unique lift $\tilde{g}$ of $g = f^n$ such that $\tilde{g}(\tilde{\mathcal{L}}^c_0) = \tilde{\mathcal{L}}^c_0$.

Lemma 3.10 *Every center leaf $\tilde{\mathcal{L}}^c$ in $\tilde{C}$ is fixed by $\tilde{g}$.*

Proof: Let $y \in \tilde{\mathcal{L}}^c$ be a point in a center leaf $\tilde{\mathcal{L}}^c$, let $x \in \tilde{\mathcal{L}}^c_0$, where $\tilde{\mathcal{L}}^c_0$ is the center leaf we chose to be fixed by $\tilde{g}$. By Lemma 3.9, there is a path $\tilde{\sigma}$ from x to y, such that $\tilde{\sigma} = \tilde{\sigma}_1 \cdot \tilde{\sigma}_2 \cdots \tilde{\sigma}_k$, each $\tilde{\sigma}_i$ lying in a lift of H^u or H^s. We proceed by induction. Suppose that the center leaf through $\tilde{\sigma}_i(0)$ is fixed by $\tilde{g}$; we show that the leaf through $\tilde{\sigma}_i(t)$ is also fixed, for $t \in [0,1]$.

Let σ_i be the projection of $\tilde{\sigma}_i$ to M. By Lemma 3.4, there is a continuous family $\lambda_{i,t}$ of center paths connecting $\sigma_i(t)$ to $g(\sigma_i(t))$ for each $t \in [0,1]$. Choose a lift of this family passing through $\tilde{\sigma}_i(0)$; this gives a family $\tilde{\lambda}_{i,t}$ of paths originating in $\tilde{\sigma}_i(t)$. Since $\tilde{g}$ fixes the center leaf through $\tilde{\sigma}_i(0)$, it must send $\tilde{\sigma}_i(0)$ to the lift of $g(\sigma_i(0))$ in this family of paths. Consider the family of endpoints of this family of lifted segments. It is a path over $g(\sigma_i)$ originating at $\tilde{g}(\tilde{\sigma}_i(0)$. So it coincides with $\tilde{g}(\tilde{\sigma}_i)$, ending the proof. $\square$

This ends the proof of the proposition. $\square$

As a consequence of Propositions 3.6 and 3.8 we get:

Corollary 3.11 *$\tilde{g}$ has no periodic points.*

Proof: Suppose x is a periodic point of $\tilde{g}$ with period k. Then each point y in $\tilde{\mathcal{F}}^{uu}(x) \cap \tilde{C}$ is a fixed point of $\tilde{g}^k$: y is the unique intersection of its $\tilde{g}$-invariant) center leaf with the k-periodic leaf $\tilde{\mathcal{F}}^{uu}(x)$. Since these points form a dense subset of $\tilde{\mathcal{F}}^{uu}(x)$, we obtain a contradiction: any strong unstable leaf contains at most 1 periodic point. $\square$

3.3 Invariance of all center leaves

In this section we prove items 1. and 2. of Theorem 2, which are direct consequences of the following proposition.

Proposition 3.12 *Any central leaf $\tilde{\mathcal{L}}$ of $\tilde{\mathcal{F}}^c$ is fixed by $\tilde{g}$, and the central distance $d_c(x, \tilde{g}(x))$ is continuous.*

Proof of the proposition:

Fix a lifted center-unstable leaf $\tilde{\mathcal{L}}^{cu}$, and consider any point z in that leaf. There are points $x, y \in \tilde{C}$, one on each connected component of $\tilde{\mathcal{L}}^{uu}(z) \setminus \{z\}$. By Proposition 3.8 we know that the center leaves through x and y are fixed by $\tilde{g}$.

Consider the closed path σ obtained by putting end to end $[x, \tilde{g}(x)]^c$, $\tilde{g}([x, y]^{uu}) = [\tilde{g}(x), \tilde{g}(y)]^{uu}$, $[\tilde{g}(y), y]^c$ and $[y, x]^{uu}$ (where the notations $[a, b]^{uu}$, $[a, b]^{ss}$ or $[a, b]^c$ are used for the strong unstable strong stable or center segments joining two points a and b on the same strong unstable strong stable or center leaf).

Lemma 3.13 *The closed path σ defined above bounds on $\tilde{\mathcal{L}}^{cu}$ a disk trivially bi-foliated by $\tilde{\mathcal{F}}^{uu}$ and $\tilde{\mathcal{F}}^c$.*

Proof: We begin by showing that σ is a simple closed curve; i.e. it has no self-intersections. Since σ is the union of 4 intervals, it suffices to show that any two such intervals have only the trivial intersection. Suppose first that the two center intervals, $[x, \tilde{g}(x)]^c$ and $[\tilde{g}(y), y]^c$, intersect nontrivially; it follows that $y \in [x, \tilde{g}(x)]^c$. But then $y \in \tilde{\mathcal{L}}^{uu}(x) \cap \tilde{\mathcal{L}}^c(x)$, and so $\tilde{\mathcal{L}}^{uu}(x)$ and $\tilde{\mathcal{L}}^c(x)$ intersect in more than one point, contradicting Proposition 3.6. If the two unstable intervals, $[y, x]^{uu}$ and $\tilde{g}([x, y]^{uu}) = [\tilde{g}(x), \tilde{g}(y)]^{uu}$, intersect nontrivially, then $\tilde{\mathcal{L}}^{uu}(x) = \tilde{\mathcal{L}}^{uu}(\tilde{g}(x))$. Since the restriction of $\tilde{g}$ to $\tilde{\mathcal{L}}^{uu}(x)$ is expanding, it must have a fixed point, contradicting Corollary 3.11. Finally, by Proposition 3.6, an unstable interval can intersect a center interval in at most one point, and so any such pair in σ intersects in exactly one endpoint.

The following lemma is a direct consequence of the Poincaré-Bendixon Theorem and completes the proof of the lemma.

Lemma 3.14 *Let $\mathcal{F}$ be a foliation of the plane, and let σ be a simple closed curve consisting of two segments tangent to $\mathcal{F}$ and two segments transverse to $\mathcal{F}$, then the disk bounded by σ is trivially foliated by $\mathcal{F}$.*

$\square$

Consider now the intersection of $\tilde{C}$ with the curve σ. It meets the unstable segment $[x, y]^u$ in a dense set of points. For each such point $z \in \tilde{C} \cap [x, y]^u$, the center leaf $\tilde{\mathcal{L}}^c(z)$ is $\tilde{g}$-invariant, so that the point $\tilde{g}(z)$ is the unique (by Proposition 3.6) intersection of $\tilde{\mathcal{L}}^c(z)$ with $\tilde{\mathcal{L}}^{uu}(\tilde{g}(z)) = \tilde{\mathcal{L}}^{uu}(\tilde{g}(x))$. However, Lemma 3.13 implies that $\tilde{\mathcal{L}}^c(z) \cap \tilde{\mathcal{L}}^{uu}(\tilde{g}(x))$ has a point in $[\tilde{g}(x), \tilde{g}(y)]^{uu}$, so that we proved that $\tilde{g}(z) = [\tilde{g}(x), \tilde{g}(y)]^{uu} \cap \tilde{\mathcal{L}}^c(z)$; since σ bounds a disk trivially foliated by $\tilde{\mathcal{F}}^c$ (once more Lemma 3.13), the action of $\tilde{g}$ on $[x, y]^u$ therefore coincides with the $\tilde{\mathcal{F}}^c$-holonomy on a dense subset, and hence everywhere. $\square$

Corollary 3.15 *Each fixed or periodic point of f belongs to a compact center leaf.*

Proof: Let $x \in M$ be a periodic point of f, and let $\mathcal{L}^c(x)$ be its center leaf. Consider a lift $\tilde{x}$ of the point x so that the center leaf $\tilde{\mathcal{L}}^c(\tilde{x})$ through $\tilde{x}$ is a lift of $\mathcal{L}^c(x)$. The point x is periodic for $g = f^n$; let k be its period. Then there is an automorphism φ of the universal cover $\tilde{M}$ of M such that $\tilde{g}^k(\tilde{x}) = \varphi(\tilde{x})$. Moreover, by Corollary 3.11 the point $\tilde{x}$ is not periodic for $\tilde{g}$ so that the automorphism φ is not trivial. However, Proposition 3.12 implies that $\tilde{g}^k(\tilde{x})$ belongs to the center leaf $\tilde{\mathcal{L}}^c(\tilde{x})$. This implies that this center leaf is invariant by the non-trivial automorphism φ, so that its quotient on M is compact. □

3.4 Topology of the center-stable leaves

We now prove item 3. in Theorem 2; that is, each leaf of $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ is a cylinder or Mœbius band, if it is the invariant manifold of a compact center leaf, and a plane otherwise.

Lemma 3.16 *In $\tilde{M}$ each leaf $\tilde{\mathcal{L}}^{cu}$ is trivially bi-foliated by $\tilde{\mathcal{F}}^c$ and $\tilde{\mathcal{F}}^{uu}$.*

Proof: Fix $x \in \tilde{M}$ and consider the strip S that is the union of the $\tilde{\mathcal{F}}^{uu}$-leaves meeting $[x, \tilde{g}(x)]^c$. By Lemma 3.13, the strip S is trivially bi-foliated by $\tilde{\mathcal{F}}^c$ and $\tilde{\mathcal{F}}^{uu}$; since $\tilde{g}$ has no fixed points and leaves invariant each center leaf, the union $\bigcup_k \tilde{g}^k(S)$ consists of entire $\tilde{\mathcal{F}}^c$-leaves and coincides with $W^u(\tilde{\mathcal{L}}^c(x))$, which is therefore complete and coincides with $\tilde{\mathcal{L}}^{cu}(x)$. □

Lemma 3.17 *If a leaf $\mathcal{L}^{cu}$ of $\mathcal{F}^{cu}$ contains a compact center leaf, then $\mathcal{L}^{cu}$ is cylinder or a Mœbius band.*

Proof: Let γ_0 be a compact center leaf contained in $\mathcal{L}^{cu}$. Then γ_0 is periodic, since every center leaf is periodic, as is every center leaf in $W^u(\gamma_0)$. We are now in the setting of Lemma 3.5, and applying this lemma to γ_0, we conclude that the leaf $\mathcal{L}^{cu}$ coincides with the unstable manifold of γ_0. But $W^u(\gamma_0)$ is the union of the strong unstable leaves through γ_0, and so is a cylinder or a Mœbius band. □

The aim of the rest of this section is to prove:

Proposition 3.18 *Each leaf of $\mathcal{F}^{cu}$ which does not contain a compact center leaf is a plane.*

The proof is divided in two very different arguments. In the first part, examining the action of the fundamental group on the universal cover, we will show that the fundamental group of the leaf is commutative. Hence, after eliminating the possibility of the torus or the Mœbius band, we obtain that the leaf is a cylinder or a plane.

In the second part, we argue by contradiction: we assume that there is a center unstable leaf, containing no compact center leaf, that is a cylinder. Using the invariance of this leaf and the uniform contraction of the strong unstable leaves by f^{-n}, we then show that the cylinder cannot be saturated by the strong unstable foliation: there must be a segment of strong unstable leaf with finite length that exits the cylinder, contradicting the fact that it is a center unstable leaf.

Lemma 3.19 *Each leaf of $\mathcal{F}^{cu}$ that does not contain a compact center leaf is a plane or a cylinder.*

Proof: Consider the natural action of the fundamental group $\pi_1(M)$ on $\tilde{M}$ by automorphisms of the universal cover. This action leaves invariant all of the lifted invariant foliations of f, and in particular it induces an action on the set of the leaves of $\tilde{\mathcal{F}}^{cu}$.

Fix a center unstable leaf $\mathcal{L}^{cu}$ that does not contain any compact center leaf, and let $\tilde{\mathcal{L}}^{cu}$ be a lift of $\mathcal{L}^{cu}$. Let $\Gamma \subset \pi_1(M)$ be the stabilizer of $\tilde{\mathcal{L}}^{cu}$ for this action. Any $\alpha \in \Gamma$, considered as an automorphism of $\tilde{M}$, induces an homeomorphism of $\tilde{\mathcal{L}}^{cu}$, so that one has a natural action of Γ on $\tilde{\mathcal{L}}^{cu}$. The corresponding leaf $\mathcal{L}^{cu}$ in M is the quotient of $\tilde{\mathcal{L}}^{cu}$ by the action of Γ.

By Lemma 3.16, the leaf $\tilde{\mathcal{L}}^{cu}$ is a plane trivially foliated by $\mathcal{F}^c$ and $\mathcal{F}^u$, which is Γ-invariant, so that the action of Γ is a product action. More precisely, Γ acts on the space of $\tilde{\mathcal{F}}^c$-leaves and on the space of $\tilde{\mathcal{F}}^{uu}$-leaves, and the full action of Γ is the Cartesian product of these 2 actions.

Let $\alpha \in \Gamma$ be a non-trivial element. Notice that α (considered as an automorphism of $\tilde{M}$) is fixed point free. Hence, if α leaves invariant some strong unstable leaf of $\tilde{\mathcal{F}}^{uu}$, then the quotient of this leaf by α is a circle and the corresponding leaf of $\mathcal{F}^{uu}$ is compact. Since there are no closed $\mathcal{F}^{uu}$-leaves, the action on the space of $\tilde{\mathcal{F}}^{uu}$-leaves is therefore free.

Once more, as $\tilde{\mathcal{L}}^{cu}$ is trivially foliated by $\tilde{\mathcal{F}}^{uu}$, the space of strong unstable leaves in $\tilde{\mathcal{L}}^{cu}$ is homeomorphic to $\mathbb{R}$. Hölder's Theorem states that any group acting freely on $\mathbb{R}$ is commutative (and also orientation preserving), so that the action of Γ on this space of strong unstable leaves is abelian and orientation preserving.

Futhermore, by hypothesis $\mathcal{L}^{cu}$ does not contain any compact center leaf, and so the action of Γ on $\tilde{\mathcal{F}}^c$-leaves is also free, and hence abelian and orientation preserving. Then the full action of Γ on $\tilde{\mathcal{L}}^{cu}$ is abelian and orientation preserving.

We have now proved that the fundamental group of the surface $\mathcal{L}^{cu}$ is commutative and that $\mathcal{L}^{cu}$ is orientable, so that $\mathcal{L}^{cu}$ is a torus T^2, a cylinder or a plane. Recall that any compact leaf of $\mathcal{F}^{cu}$ is an attractor, violating transitivity; from this it follows that $\mathcal{L}^{cu}$ a plane or a cylinder. $\square$

Proof of the proposition: We now argue by contradiction. We assume that $\mathcal{L}^{cu}$ is a center unstable leaf that is a cylinder and that does not contain any compact center leaf; as before, we denote by $\tilde{\mathcal{L}}^{cu}$ a lift of $\mathcal{L}^{cu}$ to $\tilde{M}$.

Lemma 3.20 *The leaf $\mathcal{L}^{cu}$ contains a simple closed path α, topologically transverse to the center and strong unstable foliations and cutting each leaf of these foliations in exactly* 1 *point.*

Proof: As the leaf is a cylinder, its fundamental group is $\mathbb{Z}$. Consider a generator σ of π_1 and look at its action φ_σ on $\tilde{\mathcal{L}}^{cu}$. This action induces a free action on the space of strong unstable leaves and also on the space of strong stable leaves. So fix a point $z_0 \in \tilde{\mathcal{L}}^{cu}$, and consider $\varphi_\sigma(z_0)$. By Lemma 3.16, $\tilde{\mathcal{L}}^{cu}$ is trivially bifoliated by the center and strong unstable foliations. As z_0 and $\varphi_\sigma(z_0)$ are not on the same center leaf nor on the same strong unstable leaf, there is a rectangle R in $\tilde{\mathcal{L}}^{cu}$ bounded by two segments of center leaves and two segments of strong unstable leaves, having z_0 and $\varphi_\sigma(z_0)$ as opposite corners. Consider a bifoliated chart of this rectangle, that is, a homeomorphism $h: R \to [0,1]^2$, sending the center foliation to the horizontal one and the strong unstable foliation to the vertical one. Let $\tilde{\alpha}$ be the diagonal of the rectangle in this chart, joining z_0 to $\varphi_\sigma(z_0)$.

Then the projection of $\tilde{\alpha}$ to $\mathcal{L}^{cu}$ is a closed path, topologically transverse to both center and strong stable foliations.

Notice that the projection of $\tilde{\alpha}$ to the space of center leaves in $\tilde{\mathcal{L}}^{cu}$ is a fundamental domain of the action of φ_σ, and so of the action of $\pi_1(\mathcal{L}^{cu})$ on this space. As a consequence α meets every center leaf of $\mathcal{L}^{cu}$ in exactly one point, which implies that α is a simple closed path.

The same argument shows that α meets every strong unstable leaf of $\mathcal{L}^{cu}$ in exactly one point. $\square$

Lemma 3.21 *The preimage $f^{-n}(\alpha)$ is a closed curve in $\mathcal{L}^{cu}$ disjoint from α. The set $\alpha \cup f^{-n}(\alpha)$ bounds in $\mathcal{L}^{cu}$ a compact cylinder $\mathcal{C}$ foliated by strong unstable segments joining α to $f^{-n}(\alpha)$.*

Proof: Recall that f^{-n} leaves invariant each center leaf of $\mathcal{L}^{cu}$. Moreover, the center leaves in $\mathcal{L}^{cu}$ are non-compact by hypothesis, and so Corollary 3.15 implies that there is no periodic point in $\mathcal{L}^{cu}$. As a consequence $f^{-n}(x)$ is a point different from x in the same center leaf, for any $x \in \alpha$. Since α meets each center leaf of $\mathcal{L}^{cu}$ in exactly one point, this implies that $f^{-n}(\alpha) \cap \alpha = \emptyset$. Moreover, both α and $f^{-n}(\alpha)$ meet every strong unstable leaf in exactly one point each, so that $\alpha \cup f^{-n}(\alpha)$ bounds in $\mathcal{L}^{cu}$ a compact cylinder $\mathcal{C}$ foliated by strong unstable segments joining α and $f^{-n}(\alpha)$. $\square$

Lemma 3.22 *The set $\mathcal{C}^\infty = \bigcup_{i=0}^{+\infty} f^{-n\cdot i}(\mathcal{C})$ is the union of center rays through α. Hence it is a complete surface with boundary equal to α. As a consequence, $\mathcal{C}^\infty$ contains entire strong unstable rays originating on α.*

Proof: Fix a center leaf $\mathcal{L}^c(x)$ with $x \in \alpha$. By construction, $\mathcal{C}$ intersects $\mathcal{L}^c(x)$ in a fundamental domain for the action of f^n on this leaf. Then $\mathcal{C}^\infty$ contains the union of the negative iterates f^{-ni} of this fundamental domain, that is, the entire center ray through the initial point x. We obtain that $\mathcal{C}^\infty$ is the union of α and of one of the connected components of $\mathcal{L}^{cu} \setminus \alpha$. It follows that $\mathcal{C}^\infty$ is a complete surface bounded by α.

Now consider a point $y \in \alpha$ and the strong unstable ray that enters $\mathcal{C}^\infty$ at y. This ray cannot cross α in another point because α cuts any strong unstable leaf in $\mathcal{L}^{cu}$ exactly in 1 point. Since $\mathcal{C}^\infty$ is complete, this ray must be contained in $\mathcal{C}^\infty$. $\square$

Lemma 3.23 *There is $\ell > 0$ such that any strong unstable ray contained in $\mathcal{C}^\infty$ has length less than ℓ.*

Proof: Fix an integer $k_0 > 0$ such that, for any strong unstable vector $v \in E^{uu}(x)$, with $x \in M$, one has $\|Df_x^{n\cdot k_0}(v)\| \geq 2\|v\|$. Let $n_0 = n \cdot k_0$.

Let ℓ_0 denote the length of the largest strong unstable segment in $\mathcal{C}^{k_0} = \bigcup_{i=0}^{k_0-1} f^{-i\cdot n}\mathcal{C}$. Then the total length of any strong unstable ray in $\mathcal{C}^\infty = \bigcup_{i\geq 0} f^{-i\cdot n_0}(\mathcal{C}^{k_0})$ is bounded by $\sum_{i=0}^{\infty} \frac{1}{2^i}\ell_0 \leq 2\ell_0$. We conclude the proof of lemma by choosing $\ell = 2\ell_0$.

$\square$

As any strong unstable ray has infinite length, by definition of a ray, Lemmas 3.23 and 3.22 lead to a contradiction, concluding the proof of Proposition 3.18.

$\square$

3.5 The center foliation is expansive

To conclude the proof of Theorem 2, in this section we prove that (under the hypotheses of Theorem 2) the center foliation carries an expansive flow.

A dynamical system is expansive if there is an $\alpha > 0$ so that any two distinct orbits are α-separated. For flows, the definition of α-separated must take into account possible time reparametrizations of the orbit. Here are the precise definitions.

Definition 3.24 Let $X = \{X_t\}$ be a topological flow on a compact Riemannian manifold M.

1. Given $\alpha > 0$, we say that two points x, y are *α-separated for X* if there exists $t \in \mathbb{R}$ such that, for any continuous function $\varphi\colon [0,t) \to \mathbb{R}$ with $\varphi(0) = 0$, there is an $s \in [0,t]$ such that $\sup\{d\left(X_s(x), X_{\varphi(s)}(y)\right), d\left(X_s(y), X_{\varphi(s)}(x)\right)\} \geq \alpha$.
2. The flow X is *expansive* if there exists $\alpha > 0$ such that, if x and y are not α-separated by X, then there is a $t \in [-1,1]$ such that $y = X_t(x)$.

To prove that the foliation $\mathcal{F}^c$ carries an expansive flow, we will use the criterion given by the following easy lemma:

Lemma 3.25 *Let $\mathcal{F}$ be an oriented 1-dimensional foliation on a manifold M whose leaves are tangent to a continuous, nonsingular vector field, and let $\tilde{\mathcal{F}}$ be the lift of $\mathcal{F}$ to the universal cover $\tilde{M}$ of M. Endow $\tilde{M}$ with the lifted Riemannian metric. Suppose that there exists $\alpha > 0$ with the following property:*

For any two distinct leaves $\tilde{\mathcal{L}}_1$ and $\tilde{\mathcal{L}}_2$ of $\tilde{\mathcal{F}}$, there is a point $x \in \tilde{\mathcal{L}}_1$ whose distance to $\tilde{\mathcal{L}}_2$ is greater than $\alpha > 0$ (that is, $\inf\{d(x,y), y \in \tilde{\mathcal{L}}_2\} \geq \alpha$).

Then $\mathcal{F}$ carries an expansive flow.

In order to apply this criterion, we first need to show that the center foliation $\mathcal{F}^c$ is orientable:

Lemma 3.26 *Under the hypotheses of Theorem 2, the center foliation $\mathcal{F}^c$ is orientable.*

Proof: By Proposition 3.12, there is a lift of $g = f^n$ to $\tilde{M}$ that fixes the leaves of the lifted center foliation $\tilde{\mathcal{F}}^c$. Let $\tilde{g}$ be such a lift. Corollary 3.11 implies that $\tilde{g}$ has no periodic points, and so there is a natural continuous orientation "*from x to $\tilde{g}(x)$*" of each center leaf $\tilde{\mathcal{L}}^c$ of $\tilde{\mathcal{F}}^c$. We show that this orientation is invariant under all covering automorphisms of $\tilde{M}$, which will imply that it projects to an orientation of the leaves of $\mathcal{F}^c$ in M.

Let $C \subset M$ be the set defined in Section 3.2. Then C contains no periodic points or compact leaves of $\mathcal{F}^c$, so that the orientation "*from x to $g(x)$*" is well-defined on C. This implies that any covering automorphism of $\tilde{M}$ respects the orientation "*from x to $\tilde{g}(x)$*" on the subset $\tilde{C} \subset \tilde{M}$. Since $\tilde{C}$ is dense in $\tilde{M}$, we obtain that every covering automorphism must preserve this orientation on all of $\tilde{M}$. Hence this orientation projects to an orientation of the foliation $\mathcal{F}^c$, concluding the proof. □

Now Theorem 2 is a direct consequence of:

Proposition 3.27 *There exists $\varepsilon > 0$ with the following property: Given any two distinct center leaves $\tilde{\mathcal{L}}^c_1$ and $\tilde{\mathcal{L}}^c_2$ of $\tilde{\mathcal{F}}^c$, there is a point $z \in \tilde{\mathcal{L}}^c_1$ whose distance $d(z, \tilde{\mathcal{L}}^c_2)$ to the other leaf is greater than ε.*

Remark 3.28

1. Any curve transverse to $\tilde{\mathcal{F}}^{cs}$ or $\tilde{\mathcal{F}}^{cu}$ cuts each leaf of the corresponding foliation in at most one point (if such a transversal cut a leaf twice, then it would be possible to build a closed transverse curve, implying the existence of a Reeb component for the corresponding foliation on M).
2. Let $\tilde{\mathcal{L}}^{cs}$ and $\tilde{\mathcal{L}}^{cu}$ be leaves of $\tilde{\mathcal{F}}^{cs}$ and $\tilde{\mathcal{F}}^{cu}$, respectively. Then $\tilde{\mathcal{L}}^{cs} \cap \tilde{\mathcal{L}}^{cu}$ is either empty or consists in exactly 1 center leaf:

   Proof: By Lemma 3.16, any strong unstable leaf $\tilde{\mathcal{L}}^{uu}$ in $\tilde{\mathcal{L}}^{cu}$ cuts any center leaf in $\tilde{\mathcal{L}}^{cu}$ in exactly one point; hence $\tilde{\mathcal{L}}^{uu}$ intersects each center leaf in $\tilde{\mathcal{L}}^{cu} \cap \tilde{\mathcal{L}}^{cs}$. However, item (1) asserts that $\tilde{\mathcal{L}}^{uu} \cap \tilde{\mathcal{L}}^{cs}$ consists in at most 1 point. □

3. There exist $\varepsilon, \delta > 0$ such that, if if x and y are points in $\tilde{M}$ with $d(x,y) \leq \varepsilon$, then $\tilde{\mathcal{L}}^{uu}(x,\delta) \cap \tilde{\mathcal{L}}^{cs}(y,\delta) \neq \emptyset$ and $\tilde{\mathcal{L}}^{ss}(x,\delta) \cap \tilde{\mathcal{L}}^{cu}(y,\delta) \neq \emptyset$, where "$\mathcal{L}(x,\delta)$" denotes the ball of radius δ centered at x in the leaf of the foliation $\mathcal{F}$ containing x.

Fix ε and δ satisfying the conditions in item 3 of the previous remark.

Lemma 3.29 *Let $x \in \tilde{M}$ and let $y \in \tilde{\mathcal{L}}^{uu}(x)$ be a point in the strong unstable leaf through x such that the length of the strong unstable segment $[x,y]^{uu}$ is greater then 2δ. Then the distance between x and the center stable leaf $\tilde{\mathcal{L}}^{cs}(y)$ through y is greater than ε.*

Analoguously, the distance between x and the center unstable manifolds through the points $z \in \tilde{\mathcal{L}}^{ss}(x)$ is greater than ε if the length of $[x,z]^{ss}$ is greater than 2δ.

Proof: Arguing by contradiction, assume that there is a point $z \in \tilde{\mathcal{L}}^{cs}(y)$ such that $d(x,z) < \varepsilon$. Then the choice of ε in Remark 3.28(3) implies that $\tilde{\mathcal{L}}^{cs}(y) \cap \tilde{\mathcal{L}}^{uu}(x)$ contains a point y_1 with the length of $[x, y_1]^{uu}$ less than δ. In particular, y and y_1 are two different points in $\tilde{\mathcal{L}}^{cs}(y) \cap \tilde{\mathcal{L}}^{uu}(x)$ violating Remark 3.28(1). $\square$

We are now ready to prove Proposition 3.27, which in turn implies that the center foliation carries an expansive flow.

Proof of the proposition: Let $\tilde{\mathcal{L}}^c_1$ and $\tilde{\mathcal{L}}^c_2$ be two different leaves of $\tilde{\mathcal{F}}^c$. Denote by $\tilde{\mathcal{L}}^{cs}$ and $\tilde{\mathcal{L}}^{cu}$ the center stable and center unstable leaves, respectively, containing $\mathcal{L}^c_2$. According to Remark 3.28(2) these two leaves cannot be simultanuously on the same center stable leaf and on the same center unstable leaf. Assume without loss of generality that $\tilde{\mathcal{L}}^c_1$ is not contained in $\tilde{\mathcal{L}}^{cs}$.

Let x be a point in $\tilde{\mathcal{L}}^c_1$. If $d(x, \tilde{\mathcal{L}}^{cs}) \geq \varepsilon$, then $d(x, \tilde{\mathcal{L}}^c_2) \geq \varepsilon$, and we are done. If $d(x, \tilde{\mathcal{L}}^{cs}) < \varepsilon$, then Remark 3.28(3) implies that the strong unstable leaf $\tilde{\mathcal{L}}^{uu}(x)$ through x cuts the leaf $\tilde{\mathcal{L}}^{cs}$ in a point $y \neq x$. Denote by $[x, y]^{uu}$ the segment of strong unstable leaf joining x and y.

Recall that the diffeomorphism $\tilde{g} : \tilde{M} \to \tilde{M}$ uniformly expands the vectors in the strong unstable direction. Thus there is a $k > 0$ such that the length of $\tilde{g}^k\left([x,y]^{uu}\right) = \left[\tilde{g}^k(x), \tilde{g}^k(y)\right]^{uu}$ is strictly greater than 2δ. By Proposition 3.12, $\tilde{g}$ fixes the leaves of $\tilde{\mathcal{F}}^c$, and so $\tilde{\mathcal{L}}^c_1$ and $\tilde{\mathcal{L}}^{cs}$ are $\tilde{g}$-invariant. Hence the point $z = \tilde{g}^k(x)$ belongs to $\tilde{\mathcal{L}}^c_1$ and $\tilde{g}^k(y) \in \tilde{\mathcal{L}}^{cs}$. Furthermore, Lemma 3.29 ensures that $d(z, \tilde{\mathcal{L}}^{cs}) \geq \varepsilon$, and so $d(z, \tilde{\mathcal{L}}^c_2) \geq \varepsilon$, concluding the proof of the proposition. $\square$

4 Examples

The aim of this section is to build examples showing that the informal conjecture of Pujals, presented in the introduction, needs to be slightly adapted:

- there are examples of diffeomorphisms satisfying the hypotheses of Theorem 1 in which the center foliation is a Seifert bundle with singular leaves. In order to get a "skew product" over an Anosov map of T^2 by diffeomorphisms of the circle, it is necessary to lift these examples to a finite cover. We also describe examples in which the center foliation is a nontrivial circle bundle.
- in the hypotheses of Theorem 2, the center leaves are all periodic but their periods may be distinct: we give an example where the circle γ is fixed but the center leaves in its stable manifold are periodic.

The examples we present in this section are all volume-preserving; by [BMVW] they can be C^1-approximated by stably ergodic volume-preserving diffeomorphisms (In fact, combining the results of [BW, BPW], we obtain that the examples we present are generically stably ergodic, in any C^r topology). We believe that these examples can also be smoothly perturbed to produce robustly transitive diffeomorphisms. Here is a sketch of why this should be possible. In [BD], the trivial skew product (A, id), where A is an Anosov map, is C^1-perturbed to produce a robustly transitive diffeomorphism. The perturbation there is localized around a periodic orbit and a homoclinic circle. We think that the same proof should allow us to perturb the skew product examples presented in this section to produce robustly transitive diffeomorphisms. In the same way, in [BD], the time-1 of a transitive Anosov flow is perturbed to become robustly transitive. We also think that this proof can be adapted to the example presented in Section 4.2 below, again producing a robustly transitive example. However the proofs in [BD] are somewhat technical, and adapting them to this context is beyond the scope of this paper.

4.1 Skew-product like examples

The term "partially hyperbolic skew product" is often reserved for the following construction (see, e.g. [BW]). One begins with a an Anosov diffeomorphism $A : N \to N$, a compact Lie group G, and a smooth map $\theta : N \to G$. The skew product over A induced by θ is the diffeomorphism $A_\theta : N \times G \to N \times G$ defined by the formula:

$$A_\theta(p, g) = (A(p), \theta(p)g).$$

In this definition, the bundle $N \times G$ can be replaced without too much technical difficulty by a nontrivial principal G-bundle over N. For a skew product in dimension 3, the manifold N is necessarily the 2-torus T^2, and G is necessarily the circle S^1. Such a skew product takes the form: $A_\theta((x, y), z) = (A(x, y), z + \theta(x, y))$.

In this paper, we adopt a more general definition of a skew product over an Anosov map. Let $\pi: M \to T^2$ be any circle bundle over the 2 torus, and let $A : T^2 \to T^2$ be an Anosov map. We say that a homeomorphism $F : M \to M$ is a *topological skew product over* A if F preserves the fibration and projects to A. In particular, we do not require that M be a trivial bundle, or even a principal bundle, over T^2, and we do not require that F act by translations on the fiber.

4.1.1 Non-orientable bundles

Here we construct an example where the bundle $\pi : M \to \mathbf{T}^2$, and hence the center foliation, is non-orientable.

Let $A \in GL(2, \mathbb{Z})$ be a hyperbolic matrix such that $A(0, \frac{1}{2}) \equiv (0, \frac{1}{2})(\bmod \mathbb{Z})$. Examples of such a matrix are $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ -1 & -3 \end{pmatrix}$. Let $\theta: T^2 \to \mathbb{R}$ be a map satisfying $\theta(x, y+\frac{1}{2}) = -\theta(x, y)$.

Let A_θ be the diffeomorphism of the torus T^3 defined by $A_\theta((x, y), z) = (A(x, y), z+\theta(x, y))$. It is a (usual) skew product of the Anosov map A by rotations on the circle.

Consider now the diffeomorphism φ of the torus T^3 defined by $\varphi((x, y), z) = ((x, y+\frac{1}{2}), -z)$. Notice that φ induces a free action of $\mathbb{Z}/2\mathbb{Z}$ on T^3 so that the quotient of T^3 by this action is a smooth manifold M. Furthermore the natural projection $(x, y, z) \mapsto (x, y)$ induces a projection of M on the torus $T = \mathbb{R}^2/(\mathbb{Z} \oplus \frac{1}{2}\mathbb{Z})$, and this projection is a non-orientable circle bundle.

Proposition 4.1 *The map A_θ projects to a partially hyperbolic diffeomorphism $F_{A,\theta}: M \to M$. Furthermore, if $\theta(0)$ is irrational, then $F_{A,\theta}$ is transitive.*

Proof: For any $((x, y), z)$ one has

$$A_\theta \circ \varphi\,(((x, y), z)) = \left(A(x, y + \frac{1}{2}), -z + \theta(x, y + \frac{1}{2})\right) =$$

$$= \left(A(x, y) + A(0, \frac{1}{2}), -z - \theta(x, y)\right) = \varphi \circ A_\theta\,((x, y), z) + \left(A(0, \frac{1}{2}) - (0, \frac{1}{2}), 0\right).$$

This means that A_θ and φ commute as diffeomorphisms of T^3, so that A_θ induces a diffeomorphism on the quotient space M. Since A_θ preserves the trivial vertical circle bundle on T^3, its quotient preserves the corresponding non-orientable circle bundle of M over the torus.

Since the diffeomorphisms A_θ and $F_{A,\theta}$ act on the fibers by isometries and project to an Anosov diffeomorphism, they are partially hyperbolic.

To see that $F_{A,\theta}$ is transitive if $\theta(0)$ is irrational, first note that the center leaf over 0 is fixed by $F_{A,\theta}$, and its stable and unstable manifolds are dense in M. Further, these stable and unstable manifolds intersect all strong unstable and strong stable segments, respectively. If $\theta(0)$

is irrational then the positive orbit of any strong unstable segment accumulates on the whole local unstable manifold of the center leaf $\pi^{-1}(0)$; hence, the positive orbit of any strong unstable segment is dense in M. The same holds for the negative orbits of strong stable segments. As a consequence, $F_{A,\theta}$ is transitive. $\square$

4.1.2 Orientable bundles with non-trivial Euler class

The oriented circle bundles over the torus T^2 have a unique invariant (up to fibered diffeomorphisms): the Euler class of the bundle, which can be regarded as an integer $k \in \mathbb{Z}$. Let $\pi_k\colon M_k \to T^2$ be the circle bundle whose Euler class is k.

Proposition 4.2 *For any diffeomorphism $f \in \mathit{Diff}_+(T^2)$ and any $k \in \mathbb{Z}$ there is a fibered diffeomorphism $F : M_k \to M_k$ projecting to f under π_k, and acting on the fibers by rotations.*

Proof: Any orientation preserving diffeomorphism $f : T^2 \to T^2$ may be written as a composition of 2 diffeomorphisms, $f = g_2 \circ g_1$ where g_i is the identity on a neighborhood U_i of a compact disk $D_i \subset T^2$.

For each i, the circle bundle $\pi_k\colon \pi_k^{-1}(T^2 \setminus \mathrm{int}(D_i)) \to T^2 \setminus \mathrm{int}(D_i)$ is trivializable by a fibered chart ψ_i that induces rotations in the fibers. Now g_i induces a diffeomorphism of $T^2 \setminus \mathrm{int}(D_i)$, and we can define a diffeomorphism G_i of M_k as follows: G_i coincides with the identity map on $\pi_k^{-1}(U_i)$ and, in the trivialization given by the chart ψ_i, coincides with (g_i, id_{S^1}).

Setting $F = G_2 \circ G_1$, we obtain a diffeomorphism with the desired properties. $\square$

As a direct consequence we get that any oriented fiber bundle M_k carries a transitive, partially hyperbolic diffeomorphism that is a skew product over an Anosov map:

Corollary 4.3 *Let $A : T^2 \to T^2$ be a linear Anosov diffeomorphism. For any $k \in \mathbb{Z}$ there is a transitive, partially hyperbolic diffeomorphism F preserving the natural projection $\pi_k\colon M_k \to T^2$, inducing a rotation in each fiber, and projecting to A under π_k.*

Proof: Just choose F in Proposition 4.2 so that the restriction of F in the fiber over a fixed point of A is an irrational rotation. $\square$

4.1.3 Seifert bundles

We now show how to construct a transitive, partially hyperbolic diffeomorphism whose center foliation is given by the fibers of a Seifert bundle.

Let $\phi\colon T^2 \to \mathbb{R}$ be a symmetric function, meaning that $\phi = \phi \circ (-Id)$, where $-Id$ is the map on T^2 induced by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Let $A : T^2 \to T^2$ be a linear Anosov diffeomorphism.

Let $A_\phi : T^2 \times S^1 \to T^2 \times S^1$ be the skew product defined by: $A_\phi(p,t) = (A(p), t + \phi(x))$. Notice that A_ϕ commutes with the period 2 diffeomorphism $\mathcal{S}$ of $T^2 \times S^1$ defined by: $\mathcal{S}((x,t)) = (-x, t + \frac{1}{2})$.

The quotient space M of $T^2 \times S^1$ by $\mathcal{S}$ is a smooth compact manifold. It is a Seifert bundle over the sphere S^2 having 4 singular leaves (corresponding to the fixed points of $-Id$ on T^2). Denote by $F_{A,\phi}$ the diffeomorphism of M induced by A_ϕ. The proof of the following proposition is identical to the proof Proposition 4.1.

Proposition 4.4 *For any A and ϕ, $F_{A,\phi}$ is partially hyperbolic, with center foliation defined by the Seifert fibration. If $\phi(0)$ is irrational, then $F_{A,\phi}$ is transitive.*

4.2 A diffeomorphism whose center leaves are all periodic, but with different periods.

Proposition 4.5 *There is a diffeomorphism f satisfying the hypotheses of Theorem 2 and with the following properties. There is an f-invariant closed curve γ, such that the center leaves in $W^u(\gamma)$ are periodic of period k strictly greater than 1. Furthermore, f also has periodic compact center leaves of period k.*

Proof: Let φ_t be a smooth Anosov flow on a compact 3-manifold M that has two periodic orbits $\hat{\gamma}_1, \hat{\gamma}$ whose homology classes in $H_1(M, \mathbb{Z})$ are not colinear. Then there is morphism $\rho_0 \colon \pi_1(M) \to \mathbb{Z}$ such that $\rho_0(\hat{\gamma}_1) = 0$ and $\rho_0(\hat{\gamma}) = k > 0$. Let l be an integer relatively prime to k; in other words, $l \wedge k = 1$. Then ρ_0 induces a morphism $\rho \colon \pi_1(M) \to \mathbb{Z}/l\mathbb{Z}$, such that $\rho(\hat{\gamma}_1) = 0$, and such that $\rho_0 = k$ is a generator of $\mathbb{Z}/l\mathbb{Z}$.

Let $\pi_\rho M_\rho \to M$ be the finite cyclic covering associated to ρ; that is, the regular covering whose automorphism group is $\mathbb{Z}/l\mathbb{Z}$ so that ρ agrees with the natural morphism $\pi_1(M) \to Aut(\pi_\rho)$. Let g be a generator of $Aut(\pi_\rho)$.

Let $f = g \circ \tilde{\varphi}_1$ be the partially hyperbolic diffeomorphism obtained by composing g with the time-1 map $\tilde{\varphi}_1$ of the lift $\tilde{\varphi}$ of φ to M_ρ.

Then f is a partially hyperbolic diffeomorphism. The lift γ of $\hat{\gamma}$ is fixed by f but the lift of $\hat{\gamma}_1$ consists of k closed curves permuted by f. Let γ_1 be a connected component of the lift of $\hat{\gamma}_1$. Since γ_1 has period k, the heteroclinic curves of $W^s(\gamma_1) \cap W^u(\gamma)$ have period k. It follows that all the center leaves in $W^u(\gamma)$ have period k, and in particular are not fixed by f. $\square$

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