TRANSITIVE CENTRALIZER AND FIBERED PARTIALLY HYPERBOLIC SYSTEMS

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ABSTRACT. We prove several rigidity results about the centralizer of a smooth diffeomorphism, concentrating on two families of examples: diffeomorphisms with transitive centralizer, and perturbations of isometric extensions of Anosov diffeomorphisms of nilmanifolds.

We classify all smooth diffeomorphisms with transitive centralizer: they are exactly the maps that preserve a principal fiber bundle structure, acting minimally on the fibers and trivially on the base.

We also show that for any smooth, accessible isometric extension $f_0 \colon M \to M$ of an Anosov diffeomorphism of a nilmanifold, subject to a spectral bunching condition, any $f \in \mathrm{Diff}^\infty(M)$ sufficiently C^1-close to f_0 has centralizer a Lie group. If the dimension of this Lie group equals the dimension of the fiber, then f is a principal fiber bundle morphism covering an Anosov diffeomorphism.

Using the results of this paper, we classify the centralizer of any partially hyperbolic diffeomorphism of a 3-dimensional, nontoral nilmanifold: either the centralizer is virtually trivial, or the diffeomorphism is an isometric extension of an Anosov diffeomorphism, and the centralizer is virtually $\mathbb{Z} \times \mathbb{T}$.

CONTENTS

1. Introduction

Let M be a closed, connected smooth manifold. The *centralizer* $\mathcal{Z}(f)$ of a diffeomorphism $f\colon M \to M$ is the set of all diffeomorphisms that commute with f under composition. The centralizer may be regarded as the set of smooth symmetries of f. The group $\mathcal{Z}(f)$ always contains the iterates of f as a normal subgroup: $\mathcal{Z}(f) \rhd \langle f \rangle$. For the C^1-generic diffeomorphism f of a closed manifold, the centralizer is no bigger than its iterates, i.e. $\mathcal{Z}(f) = \langle f \rangle$ [4]. In the latter case, we say that f has *trivial centralizer*. Whether this genericity of trivial centralizers remains true in higher regularity classes (say C^2) remains an open question, except on the circle [19].

In this paper and our earlier work [10] we address the general question:

If $\mathcal{Z}(f)$ is nontrivial, what can we say about f?

Consider the most extreme situation in which the centralizer f is $\mathrm{Diff}(M)$. In this case, if there is a point x that is not fixed by f, then there is an element $g \in \mathcal{Z}(f)$ fixing x and *not* fixing $f(x)$. This leads to a contradiction, since $g(f(x)) = f(g(x)) = f(x) = x$. Thus $\mathcal{Z}(f) = \mathrm{Diff}(M)$ implies that $f = Id_M$. Indeed, as long as $\mathcal{Z}(f)$ acts doubly transitively on M, the same conclusion holds.[1]

Even the weaker case in which $\mathcal{Z}(f)$ acts *transitively* on M puts strong constraints on f. For example if f is minimal and $\mathcal{Z}(f)$ acts transitively, then f is a minimal translation on a torus (after a smooth conjugacy). More generally, we show:[2]

Theorem 1. *Let f be homeomorphism of a connected closed manifold M, and denote by $\mathcal{Z}^r(f)$ the C^r centralizer of f for $r \in \mathbb{Z}_{\geq 1}$. Then $\mathcal{Z}^r(f)$ acts transitively on M if and only if M is a C^r principal $\mathbb{T}^k \times \mathbb{Z}/d\mathbb{Z}$ fiber bundle for some $k \geq 0, d \geq 1$, and f acts as the identity on the base and a constant minimal translation on the fibers.*

Moreover if $\mathcal{Z}^r(f)$ acts transitively on M, then f is minimal on M if and only if f is C^r conjugate to a minimal translation on a torus.

Thus *a sufficiently rich centralizer characterizes some systems of low complexity.*

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[1]All this argument requires is that for every $x, y \in M$, there exists $g \in \mathcal{Z}(f)$ such that $g(x) = x$ and $g(y) \neq y$.

[2]For $r \in (0, 1)$, it is possible there is a similar result assuming only Hölder regularity. For $r = 0$, this appears to be open at least in dimension at least 2.

At the other extreme of dynamical complexity are the Anosov diffeomorphisms. A diffeomorphism $f\colon M \to M$ is *Anosov* if there exists a Riemann structure on M, a Df-invariant splitting of the tangent bundle

$$TM = E^u \oplus E^s$$

and a constant $\lambda \in (0,1)$ such that for all $v^u \in E^u$, $v^s \in E^s$ and $n \geq 0$, we have

$$\|Df^n(v^s)\| \leq \lambda^n \|v^s\|, \quad \text{and } \|Df^{-n}(v^u)\| \leq \lambda^n \|v^u\|.$$

Anosov diffeomorphisms are well-studied and exhibit striking chaotic dynamical features, such as positive entropy, topological mixing, and exponential growth of periodic orbits. The simplest examples of Anosov diffeomorphisms are given by the automorphism of the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ induced by a hyperbolic matrix $A \in SL(n, \mathbb{Z})$, and more generally by a "linear" diffeomorphism of G/Γ, where G is a nilpotent Lie group and $\Gamma < G$ is cocompact, induced by a hyperbolic automorphism of G preserving Γ. As the Anosov condition is C^1-open, all perturbations of such linear examples are also Anosov. All known Anosov of diffeomorphisms are (up to finite covers) on nilmanifolds, and all such diffeomorphisms are topologically conjugate to such automorphisms (which can be defined by their action on π_1).

For an *irreducible* Anosov diffeomorphism of a torus[3], there are only finitely many possibilities for the centralizer (up to isomorphism and finite index subgroups), all free abelian. More generally, the centralizer of a transitive Anosov diffeomorphism is always discrete.

A much broader class of diffeomorphisms retaining some of the chaotic features of the Anosov diffeomorphisms (e.g. positive entropy) and containing the Anosov diffeomorphisms are the *partially hyperbolic diffeomorphisms*, which we define in Section 3.1. In this paper, we further develop the methods in the proof of Theorem 1 to understand the centralizer of a class of partially hyperbolic diffeomorphisms known as the *isometric extensions of Anosov diffeomorphisms*. We discuss our main result for such systems in the next section and remark here on a striking corollary.

Non-Anosov partially hyperbolic diffeomorphisms first appear in dimension 3. The possibilities for their centralizer is in general a much more delicate question than for the Anosov diffeomorphisms. On $\mathbb{T}^3$, every partially hyperbolic diffeomorphism is homotopic to a partially hyperbolic automorphism [6], and the possibilities for the centralizer depend strongly on the reducibility of this automorphism.

Reducible partially hyperbolic automorphisms on $\mathbb{T}^3$ are of the form $\mathrm{Anosov}|_{\mathbb{T}^2} \times \pm \mathrm{id}|_{\mathbb{T}^1}$ and themselves have a huge centralizer, due to the triviality of the second factor; we study perturbations of such a system in some depth in [10]. Such perturbations are *homotopically reducible*, meaning they are homotopic to a reducible automorphism of $\mathbb{T}^3$. It would be interesting

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[3]For a toral Anosov diffeomorphism f, *irreducibility* means that the linearization f_0 of f does not preserve a proper invariant subtorus.

to undertake a classification of the centralizer of such homotopically reducible systems; this would require a better topological understanding than currently available of the dynamics that can occur in this case.

For the *homotopically irreducible* partially hyperbolic diffeomorphisms, which are homotopic to (necessarily hyperbolic) irreducible automorphisms of $\mathbb{T}^3$, a type of rigidity arises for the centralizer. Gan et al [11] prove that for such "derived from Anosov" systems, the centralizer is either virtually $\mathbb{Z}$ or virtually $\mathbb{Z} \times \mathbb{Z}$; in the latter case, f is smoothly conjugate to a linear system.

A class 3-manifolds that strictly contains the tori are the nilmanifolds. Non-toral nilmanifolds in dimension 3 are (up to finite covers) diffeomorphic to a Heisenberg nilmanifold, and they admit partially hyperbolic diffeomorphisms in the homotopy class of any automorphism with a non-unital eigenvalue. Such systems have been coarsely classified (up to leaf conjugacy) by Hammerlindl and Potrie [13].

Combining work in [13] with our second main result (Theorem 3 below), we obtain a complete classification of the centralizer of such partially hyperbolic diffeomorphisms, in a similar spirit to the derived from Anosov systems, but with a subtler answer. In particular we show:

Theorem 2. *Let $f \in \mathrm{Diff}^{\infty}(M)$ be a partially hyperbolic diffeomorphism of a 3-dimensional nilmanifold M, $M \neq \mathbb{T}^3$. Then $\mathcal{Z}^{\infty}(f)$ is either virtually $\mathbb{Z}$ or virtually $\mathbb{Z} \times \mathbb{T}^1$.*

In the latter case, there exists an f-invariant, C^{∞} principal $\mathbb{T}^1$-bundle structure $\mathbb{T} \hookrightarrow M \to \mathbb{T}^2$ such that the diffeomorphism of $\mathbb{T}^2$ induced by f is Anosov.

The affine models of partially hyperbolic diffeomorphisms on Heisenberg nilmanifolds are isometric extensions of linear Anosov automorphisms of $\mathbb{T}^2$, with constant rotation on the circle fibers, and one might ask whether the conclusion of Theorem 2 can be strengthened to obtain a conclusion similar to the derived from Anosov case in [11]; that is, when the centralizer of f is not virtually $\mathbb{Z}$, then is f smoothly conjugate to an affine diffeomorphism of the Heisenberg manifold? The answer is no, as it is possible to start with such an automorphism and perturb both the Anosov diffeomorphism on the base $\mathbb{T}^2$, and the constant rotation on the fibers, to obtain new, non-affine isometric extensions. What Theorem 2 implies is that beyond such perturbations, there is no other way to obtain a centralizer "larger than" (up to finite index subgroups) $\mathbb{Z}$.

One byproduct of the proof of Theorem 1 allows us to obtain a principal fiber bundle structure on any manifold which has a transitive action by a "nice" group of diffeomorphisms (Theorem 4 in Section 4.4) by studying the centralizer of the acting group. This leads to our main result for partially hyperbolic systems which we describe in the next section.

2. A GENERAL RESULT ABOUT ISOMETRIC EXTENSIONS OF ANOSOV DIFFEOMORPHISMS

In this section we formulate our second main result. It concerns systems that combine features of low and high complexity on manifolds of any dimension. Namely, we consider *isometric extensions of Anosov diffeomorphisms* of arbitrary nilmanifolds, and their perturbations.

Such extensions are given by a smooth fiber bundle $F \hookrightarrow M \to N$, where M is closed and connected and N is a nilmanifold, and a diffeomorphism $f_0 \in \mathrm{Diff}^\infty(M)$ preserving the fiber bundle structure, acting isometrically on fibers, and projecting to an Anosov diffeomorphism $\bar{f}_0$ of N, see [3].

We assume a generic condition on f_0 called *accessibility* and that $\bar{f}_0$ satisfies a spectral condition (satisfied, for example, by any C^1-small perturbation of an affine diffeomorphism of a nilmanifold). For precise definitions and further discussion, see Section 3. Under these assumptions, we obtain a type of centralizer rigidity, whose proof uses elements of the proof of Theorem 1.

Theorem 3. *Let $f_0 \in \mathrm{Diff}^\infty(M)$ be an accessible isometric extension of an Anosov diffeomorphism $\bar{f}_0 \colon N \to N$ of a closed nilmanifold N. Assume that $D\bar{f}_0|_{E^s_{\bar{f}_0}}$ and $D\bar{f}_0^{-1}|_{E^u_{\bar{f}_0}}$ have narrow spectrum. Then there exists $s_0 = s_0(f_0) > 0$ such that for any $s \geq s_0$ and any $f \in \mathrm{Diff}^\infty(M)$ that is sufficiently C^1-close to f_0, the group $\mathcal{Z}^s(f)$ is a C^0-closed, k-dimensional Lie subgroup of $\mathrm{Diff}^s(M)$, with $k \leq \dim E^c_{f_0}$. Moreover, at least one of the following possibilities holds:*

(1) $k = 0$, and $\mathcal{Z}^s(f)$ is a countable discrete subgroup of $\mathrm{Homeo}(M)$, or

(2) $k > 0$, and there exist a k-dimensional compact Lie group G and $r \gg s$ such that M admits a C^r, f-invariant principal G-bundle structure that is subordinate to $\mathcal{W}^c_f$. Along each G-fiber f acts as left translation, and $\mathcal{Z}^s(f)$ is a countable extension by the group ($\cong G$) formed by all the right translations along G-fibers.

(3) $k = \dim E^c_{f_0}$, and f is an automorphism of a C^∞ principal fiber bundle projecting to an Anosov diffeomorphism with either narrow or (pointwise) $\frac{1}{2}$-pinched spectrum.

Remark 1. Note that in Theorem 3 (in contrast to Theorem 1), the group G might not be abelian, as the following example shows. Fix a compact group G, and an Anosov diffeomorphism $\bar{f} \colon N \to N$ of a closed manifold N. Let $\theta \colon N \to G$ be a smooth function. The manifold $M = N \times G$ is a principal G-bundle with respect to the right action $g \cdot (x, h) = (x, hg)$. The diffeomorphism $\bar{f}_\theta \colon M \to M$ defined by $\bar{f}_\theta(x, h) = (\bar{f}(x), \theta(x)h)$ is partially hyperbolic and commutes with the right action of G.

Theorem B of [7] shows that if N is nilmanifold, then for the generic such θ (in a very strong sense), the map $\bar{f}_\theta$ is also accessible. Indeed [7, Theorem B] classifies all non-accessible examples: for example, if G is semisimple, and $\bar{f}_\theta$ is ergodic, then $\bar{f}_\theta$ is accessible.

Remark 2. Since any infinite compact Lie group contains subgroups isomorphic to $\mathbb{T}^1$, Theorem 3 has the following consequence: if $\mathcal{W}_f^c$ does not admit a free $\mathbb{T}^1$ action, then $\mathcal{Z}^s(f)$ is C^0-discrete subgroup of $\mathrm{Diff}^s(M)$. For example, any accessible isometric extension of an Anosov automorphism with $\mathbb{S}^2$ fiber has discrete countable centralizer. More generally, if the fiber has non-zero Euler characteristic, then the centralizer is discrete.

Remark 3. The centralizer of a diffeomorphism in general is not C^0-closed. For example, consider a diffeomorphism f of the circle with Liouville rotation number, which is not C^1 conjugate to a rotation. Then $\mathcal{Z}^2(f) = \langle f \rangle$, which is not closed in $\mathrm{Homeo}(\mathbb{S}^1)$ but is discrete, and hence closed, in $\mathrm{Diff}(\mathbb{S}^1)$.

Theorem 3 and its variants have various applications in the study of the centralizer of partially hyperbolic systems. Theorem 2 is one application which we feature in this paper. Another application is that by using Theorem 3, it is now possible to strengthen some the results in [10] by removing the volume-preserving hypotheses. In this prior work, we classified possible centralizers for volume preserving perturbations of isometric (circle) extensions on tori and certain discretized Anosov flows. Barthelmé and Gogolev [5] extended this work to discretized Anosov flows on 3-manifolds. Both in [10] and [5] the center foliation of the partially hyperbolic systems in question is 1-dimensional. Theorem 3 is the first step in our forthcoming work which aims at understanding centralizers of partially hyperbolic systems on nilmanifolds with higher dimensional center foliations.

Recently, Wendy Zhijing Wang [24] has generalized Theorem 3 to the more general setting of accessible dynamically coherent partially hyperbolic diffeomorphisms, showing that the so-called *center fixing centralizer* (see Section 5.1 for a definition) is a Lie group. This work can also be used to remove the volume preserving assumptions for the results on discretized Anosov flows in [10, 5]. In particular, [24] obtains as a corollary, using [5], that for any 3-manifold M with nonsolvable fundamental group supporting an Anosov flow φ_t, any C^1-small, smooth perturbation of φ_1 either has virtually trivial centralizer or embeds in a smooth Anosov flow.

3. Preliminaries

3.1. **Partially hyperbolic dynamics.** We recall some definitions from smooth dynamics. Let M be a complete Riemannian manifold, and let $f \in \text{Diff}(M)$. A *dominated splitting* for f is a direct sum decomposition of the tangent bundle

$$TM = E_f^1 \oplus E_f^2 \oplus \cdots \oplus E_f^k$$

such that

- the bundles E_f^i are Df-*invariant*: for every $i \in \{1, \ldots, k\}$ and $x \in M$, we have $D_x f(E_f^i(x)) = E_f^i(f(x))$; and
- $Df|_{E_f^i}$ *dominates* $Df|_{E_f^{i+1}}$: there exists $N \geq 1$ such that for any $x \in M$ and any unit vectors $u \in E_f^{i+1}$, and $v \in E_f^i$:

$$\|D_x f^N(u)\| \leq \frac{1}{2} \|D_x f^N(v)\|.$$

A C^1 diffeomorphism $f : M \to M$ of a complete Riemannian manifold M is *partially hyperbolic* if there is a dominated splitting $TM = E_f^u \oplus E_f^c \oplus E_f^s$ and $N \geq 1$ such that for any $x \in M$, and any choice of unit vectors $v^s \in E_f^s(x)$ and $v^u \in E_f^u(x)$, we have

$$\max\{\|D_x f^N(v^s)\|, \|D_x f^{-N}(v^u)\|\} < 1/2.$$

We always assume the bundles E_f^s and E_f^u are nontrivial. If E_f^c is trivial then f is *Anosov*[4].

In many cases of interest here, we consider a partially hyperbolic diffeomorphism f equipped with a *center foliation* $\mathcal{W}_f^c$ that is tangent to E_f^c and whose leaves are compact, forming a fibration. We distinguish between several cases of such fibered systems.

Definition 1. Let f be a partially hyperbolic diffeomorphism of a closed manifold M. Assume that there exists an $f-$invariant center foliation $\mathcal{W}_f^c$ with compact leaves.

- If $\mathcal{W}_f^c$ is a topological fibration of M, i.e. the quotient space $M/\mathcal{W}_f^c$ is a topological manifold[5], then f is called a *fibered partially hyperbolic system*, and the map $\bar{f} : M/\mathcal{W}_f^c \to M/\mathcal{W}_f^c$ canonically induced by f is called the *base map*.
- A fibered partially hyperbolic system f is *smoothly fibered* (or C^r *fibered*, for $r \geq 1$) if $\mathcal{W}_f^c$ is a C^∞ (respectively C^r) foliation, and f is C^∞ (resp. C^r). In this case, the base map is an Anosov diffeomorphism.
- A smoothly fibered partially hyperbolic system f is an *isometric extension of an Anosov diffeomorphism*, if there is a C^∞, $f-$invariant metric on E_f^c.

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[4] The equivalence of this definition of Anosov with the one given in the introduction can be found in [9, Section 2.4]

[5] Or, equivalently, if $\mathcal{W}_f^c$ has trivial holonomy; see [3]

- A smoothly fibered partially hyperbolic system f is a *compact Lie group extension of an Anosov diffeomorphism* if there is a compact Lie group G and a C^∞ $f-$invariant principle G-bundle structure $\pi : M \to M/\mathcal{W}_f^c$ such that, restricted to every fiber, f acts as a left G-translation.

Remark 4. For the last two items in Definition 1, the partial hyperbolicity assumptions are not necessary, since any isometric extension of an Anosov diffeomorphism is partially hyperbolic; see [12, Proposition 5.3].

If f is partially hyperbolic, then there are foliations $\mathcal{W}_f^u$ and W_f^s tangent to the bundles E_f^u and E_f^c, respectively.

Definition 2. We say that $x, y \in M$ are in the same *accessibility class* if they can be joined by an *su-path*, that is, a piecewise C^1 path such that every arc (or *leg*) is contained in a single leaf of $\mathcal{W}_f^s$ or a single leaf of $\mathcal{W}_f^u$. We say that f is *accessible* if M consists of a single accessibility class.

Definition 3. For $r \geq 1$, we say that a partially hyperbolic diffeomorphism f of a Riemannian manifold M is $r-$*bunched* if there exists $k \geq 1$ such that:

$$\sup_p \left\{ \|D_p f^k|_{E^s}\| \cdot \|(D_p f^k|_{E^c})^{-1}\|^r, \|(D_p f^k|_{E^u})^{-1}\| \cdot \|D_p f^k|_{E^c}\|^r \right\} < 1,$$

$$\sup_p \|D_p f^k|_{E^s}\| \cdot \|(D_p f^k|_{E^c})^{-1}\| \cdot \|D_p f^k|_{E^c}\|^r < 1, \text{ and}$$

$$\sup_p \|(D_p f^k|_{E^u})^{-1}\| \cdot \|D_p f^k|_{E^c}\| \cdot \|(D_p f^k|_{E^c})^{-1}\|^r < 1.$$

If f is an isometric extension of an Anosov diffeomorphism, then f is ∞-bunched (i.e. r-bunched, for all r). For any such f, and any finite $r > 1$, if $\hat{f}$ is C^r and sufficiently C^1-close to f, then $\hat{f}$ is r-bunched.

If f is an r-bunched, C^r fibered partially hyperbolic diffeomorphism, then the leaves of $\mathcal{W}_f^c, \mathcal{W}_f^{cs}, \mathcal{W}_f^{cu}$ are C^r, moreover, if f is C^{r+1}, then the stable and unstable *holonomies* and the bundles E_f^s, E_f^u are C^r along $\mathcal{W}_f^c$; see [20, 25].

Definition 4. Denote by $\mathcal{CZ}^s(f)$ the group of all $g \in \mathcal{Z}^s(f), s = 0, 1, \ldots$ fixing the center leaves of f; that is, $g \in \mathcal{CZ}^s(f)$ if and only if $g\mathcal{W}^c(x) = \mathcal{W}^c(x)$, for all $x \in M$.

3.2. *su*-**holonomy.** Every fibered partially system $f \in \mathrm{Diff}(M)$ admits *global su-holonomies*, meaning that for any *su-path* γ in M from x to y, there exists a unique homoemorphism $H_\gamma \colon \mathcal{W}_f^c(x) \to \mathcal{W}_f^c(y)$ with the properties:

- $H_\gamma(x) = y$;
- $H_{\gamma_1 \cdot \gamma_2} = H_{\gamma_2} \circ H_{\gamma_1}$, where $\gamma_1 \cdot \gamma_2$ is the concatenation of two *su*-paths; and
- if γ is tangent to $\mathcal{W}_f^s$ (resp $\mathcal{W}_f^u$), then H_γ is the $\mathcal{W}_f^s$ (resp $\mathcal{W}_f^u$) holonomy, restricted to $\mathcal{W}_f^c$ leaves.

The set of su-holonomies from a fixed center leaf $\mathcal{W}_f^c(x_0)$ to itself form a group, which we denote by $\mathcal{H}_f(x_0)$. If f is r-bunched, then $\mathcal{H}_f(x_0) < \mathrm{Diff}^r(\mathcal{W}_f^c(x_0))$.

3.3. Leafwise structural stability.
In the setting of fibered partially hyperbolic systems, there is a variety of results we will use, starting with the Hirsch-Pugh-Shub perturbation theory [15, Theorems 7.5 and 7.6] (see Remark 4 on p. 117), [15, Theorem 7.1], and [20, Theorems A and B]. See also the discussion in [20, Section 3].

3.4. Stable accessibility.
In this section we establish the stable accessibility of any isometric extension of an Anosov diffeomorphism.

Proposition 5. *Let $f \in \mathrm{Diff}^\infty(M)$ be an isometric extension of an Anosov diffeomorphism. If f is accessible, then f is stably accessible, i.e. any g sufficiently C^1-close to f is accessible.*

Proof. We follow the strategy of [7]. Since most of the proofs are the same or similar, we only sketch the proof here and recommend reading [7] for background. Fix a center leaf $\mathcal{W}_f^c(x_0)$.

Step 1: accessibility implies local accessibility. Any f-invariant G-structure on the leaves of $\mathcal{W}_f^c$ is also invariant under the stable and unstable holonomies between $\mathcal{W}_f^c$ leaves; since f is an isometric extension, all the su-holonomies from $\mathcal{W}_f^c(x_0)$ to itself are isometries of $\mathcal{W}_f^c(x_0)$. Let $\mathcal{H} = \mathcal{H}_f(x_0)$ be the group all such holonomies, and denote by $\mathcal{H}^0$ the subgroup of $\mathcal{H}$ consisting of all the holonomies H_γ with $\pi \circ \gamma$ a closed, contractible path in the base. Note that such a $\pi \circ \gamma$ is also an su-path, for the base Anosov diffeomorphism $\bar{f}$ and that H_γ depends only on the projection $\pi \circ \gamma$.

Then $\mathcal{H}$ is a subgroup of the Lie group $\mathrm{Iso}(\mathcal{W}_f^c(x_0))$, and $\mathcal{H}^0$ is the identity component of $\mathcal{H}$. Any element H_γ of $\mathcal{H}^0$ can be isotoped to the identity through maps H_{γ_t} with $\pi \circ \gamma_t$ an isotopy to the trivial path through closed su-paths for $\bar{f}$ ("Brin's argument"). Thus $\mathcal{H}^0$ is path connected. A result of Kuranashi-Yamabe [26] then implies that $\mathcal{H}^0$ is a Lie subgroup of $\mathrm{Iso}(\mathcal{W}_f^c(x_0))$.

The quotient $\mathcal{H}/\mathcal{H}^0$ has at most countably many elements (since each coset corresponds to an element of $\pi_1(M/\mathcal{W}_f^c, \bar{x}_0)$). The Baire category theorem then implies that $\mathrm{Image}(\mathcal{H}^0 \cdot x_0)$ has nonempty interior. Since $\mathcal{H}^0$ is a Lie subgroup of the isometry group of $\mathcal{W}_f^c(x_0)$, this implies that $\mathrm{Image}(\mathcal{H}^0 \cdot x_0) = \mathcal{W}_f^c(x_0)$. Hence f is accessible through those su-paths that project to null-homotopic, closed paths in $M/\mathcal{W}^c$.

Step 2: a homogeneous space structure on the leaves of the center foliation. From Step 1, we have that $\mathcal{H}^0$ is a connected Lie group acting transitively on $\mathcal{W}_f^c(x_0)$. This extends to a transitive action of $\mathcal{H}^0$

on every center leaf via conjugation (for each $x \in M$, one fixes an su-holonomy between $\mathcal{W}_f^c(x)$ and $\mathcal{W}_f^c(x_0)$ and conjugates the $\mathcal{H}^0$-action by this map). Thus $\mathcal{W}_f^c$ is an $\mathcal{H}^0$-bundle; i.e., there is an f-invariant homogeneous space structure on the foliation $\mathcal{W}_f^c$ obtained by identifying each $\mathcal{W}_f^c(x)$ with $\mathcal{H}^0(x)/\text{Stab}(x)$, where $\text{Stab}(x)$ is the stabilizer of x.

Step 3: a useful criterion for stable accessibility. The paper [7] contains a useful criterion for stable accessibility of dynamically coherent partially hyperbolic diffeomorphisms, which will be used in our proof. We say that a partially hyperbolic diffeomorphism f is *dynamically coherent* if there exist f-invariant center stable and center unstable foliations $\mathcal{W}_f^{cu}$ and $\mathcal{W}_f^{cs}$ tangent to the bundles $E_f^c \oplus E_f^u$ and $E_f^c \oplus E_f^s$, respectively. The fibered partially hyperbolic maps we consider here are dynamically coherent (see Theorem 8 in [10]).

Let f be a dynamically coherent partially hyperbolic diffeomorphism, and let d_c be the dimension of E_f^c. A point $q_0 \in M$ is *centrally engulfed* from a point $p_0 \in M$ if there is a continuous map $\Gamma : Z \times [0,1] \to M$ such that:

(1) Z is a compact, connected, orientable, d_c-dimensional manifold with boundary;
(2) for each $z \in Z$, the curve $\gamma_z(\cdot) = \Gamma(z, \cdot)$ is an su-path with $\gamma_z(0) = p_0$ and $\gamma_z(1) \in \mathcal{W}_f^c(q_0)$;
(3) there is a constant C such that every path γ_z has at most C legs;
(4) $\gamma_z(1) \neq q_0, \forall z \in \partial Z$;
(5) the map $(Z, \partial Z) \to (\mathcal{W}_f^c(q_0), \mathcal{W}_f^c(q_0) - \{q_0\})$ defined by $z \to \Gamma(z, 1)$ has positive degree.

We have the following criterion for stable accessibility.

Theorem 6 (Corollary 5.3 in [7]). *Let f be a dynamically coherent partially hyperbolic diffeomorphism. Suppose that f is accessible and that exist $p_0 \in M$ such that p_0 can be centrally engulfed from p_0. Then f is stably accessible.*

Step 4: achievable and approximable paths. By Theorem 6, it suffices to show x_0 is centrally engulfed from x_0. We follow the strategy of Section 9 of [7]: the key is to use the homogeneous space structure of $\mathcal{W}_f^c(x_0)$ to construct the map Γ. We say that $\psi \colon Z \to \mathcal{H}^0$ is *achievable* if it is the "endpoint map" of a continuous family of su-holonomies that begin and end in $\mathcal{W}_f^c(x_0)$ and project to null-homotopic loops in $M/\mathcal{W}_f^c$ (i.e. through elements of $\mathcal{H}_{x_0}^0(f)$).

More precisely, ψ is achievable if there exist a continuous function

$$\Xi = (\Xi_1, \Xi_2) \colon Z \times [0,1] \to \{(\bar{x}, g_x) : \bar{x} \in M/\mathcal{W}_f^c, g_x \in \text{Iso}(\mathcal{W}_f^c(x_0), \mathcal{W}_f^c(x))\}$$

and an integer $C \geq 1$ such that

(1) for each $z \in Z$, $\Xi_1(z,t)$ is an su-path on the base $M/\mathcal{W}_f^c$ beginning and ending in $\bar{x}_0$, with at most C legs;

(2) for each $z \in Z$ and $t \in [0,1]$, $\Xi_2(z,t)$ is the (isometric) su-holonomy induced by lifting the su-path $\Xi_1(z,t)$; and

(3) the path $\xi_z : [0,1] \to M$ defined by $\xi_z(t) = \Xi_2(z,t)$ has the property that $\xi_z(1)$ maps any $y \in \mathcal{W}_f^c(x_0)$ to $\psi(z) \cdot y$.

Following [7], we say a map $\psi \colon Z \to \mathcal{H}^0$ is *approximable* if for each $\epsilon > 0$ there is an achievable map $\psi^\epsilon : Z \to \mathcal{H}^0$ such that $\mathrm{dist}_{C^0}(\psi, \psi^\epsilon) < \epsilon$.

Lemma 7 (Lemma 9.2 of [7]). *Let $\psi_i \colon Z_i \to G, 1 \le i \le k$ be approximable maps. Then the product map*

$$(z_1, \ldots, z_k) \mapsto \psi_k(z_k)\psi_{k-1}(z_{k-1}) \cdots \psi_2(z_2)\psi_1(z_1)$$

is approximable.

Step 5: geodesics in $\mathcal{H}^0$ are approximable. By the same proof as Proposition 9.3 of [7], we have that for any v in the Lie algebra of $\mathcal{H}^0$, the map $\sigma_v \colon [-1,1] \to \mathcal{H}^0, z \mapsto \exp(zv)$ is approximable. Hence by Lemma 7, for any $v_1, \ldots, v_{d_c}$ in the Lie algebra of $\mathcal{H}^0$, the map $\psi_{v_1, \ldots, v_{d_c}} \colon [-1,1]^{d_c} \to \mathcal{H}^0$ defined by $(z_1, \ldots, z_{d_c}) \mapsto \prod_{i=d_c}^{1} \exp(z_i v_i)$ is approximable. Let $\langle \cdot, \cdot \rangle$ be the invariant inner product on the Lie algebra of $\mathrm{Iso}(\mathcal{W}_f^c(x_0))$. With respect to $\langle \cdot, \cdot \rangle$, we choose the v_i such that

(1) $\langle v_i, v_j \rangle = \eta \delta_{ij}$, for some small η, so that exp is "almost an isometry" on the η-neighborhood of 0; and

(2) the angle between the subspace spanned by the v_i and the Lie algebra of $\mathrm{Stab}(x_0)$ is greater than $1/10$.

Then $\psi_{v_1, \ldots, v_{d_c}}$ is a parametrization of a d^c-dimensional submanifold of $\mathcal{H}^0$, whose interior contains the identity element id and is uniformly transverse to $\mathrm{Stab}(x_0)$. Since $\psi_{v_1, \ldots, v_{d_c}}$ is approximable, it has an achievable approximation $\psi \colon [-1,1]^{d_c} \to \mathcal{H}^0$. Let $Z = [-1,1]^{d_c}$, and let

$$\Xi = (\Xi_1, \Xi_2) \colon Z \times [0,1] \to \{(\bar{x}, g_x) \colon \bar{x} \in M/\mathcal{W}_f^c, g_x \in \mathrm{Iso}(\mathcal{W}_f^c(x_0), \mathcal{W}_f^c(x))\}$$

be the map satisfying properties (1)-(3) in the previous step for the achievable map ψ, for some $C \ge 1$.

Now consider the evaluation map $\Gamma \colon [-1,1]^{d_c} \times [0,1] \to M$ defined by $\Gamma(z_1, \ldots, z_{d_c}, t) = \Xi_2(z_1, \ldots, z_{d_c}, t)(x_0)$. By the $1/10$-transversality of the subspaces $\mathrm{Span}(v_1, \ldots, v_{d_c})$ and the Lie algebra of $\mathrm{Stab}(x_0)$, if η is sufficiently small and the achievable approximation ψ is chosen sufficiently close, the map Γ will satisfy conditions (1)-(5) in Step 3 (with p_0, q_0 equal to x_0), and so x_0 is centrally engulfed from x_0. Theorem 6 then implies that f is stably accessible, completing the proof of Proposition 5. $\square$

3.5. $\mathcal{Z}^1(f)$-invariance of f-invariant foliations.

If a diffeomorphism g commutes with a partially hyperbolic diffeomorphism f, the derivative Dg preserves the Df-invariant partially hyperbolic splitting for f and the foliations $\mathcal{W}_f^{s/u}$. Perhaps surprisingly, it is unknown in general whether if, under the additional assumption that f is dynamically coherent, such a g

must also preserve even one of the foliations $\mathcal{W}_f^{cu}, \mathcal{W}_f^{cs}$, or $\mathcal{W}_f^c$. Even in the case of a fibered partially hyperbolic diffeomorphism, this appears to be an open question. We do have the following result in the case that the base is a nilmanifold (and this alone accounts for our assumption that the base be a nilmanifold in Theorem 3).

Proposition 8. *Let $f \in \mathrm{Diff}(M)$ be a fibered partially hyperbolic diffeomorphism such that $M/\mathcal{W}_f^c$ is homeomorphic to a nilmanifold N. Then for any $g \in \mathcal{Z}^1(f)$, we have $g\mathcal{W}_f^c = \mathcal{W}_f^c$.*

Proof. For any fibered partially hyperbolic diffeomorphism f, [3] shows that the quotient dynamics $\bar{f} : M/\mathcal{W}_f^c \to M/\mathcal{W}_f^c$ is expansive and has the pseudo orbit tracing property (unique shadowing of pseudo-orbits). Theorem D of [12] then implies that $\bar{f}$ is topologically conjugate to an Anosov automorphism of N. From now on we identify $M/\mathcal{W}_f^c$ with N through the conjugacy map so that $f : M \to M$ can be viewed as a bundle automorphism of a continuous fiber bundle $M \to N$ over an Anosov automorphism $\bar{f}$.

In particular, $\bar{f}$ has a fixed point $\bar{x}_0$, and so f has an invariant fiber $\mathcal{W}_f^c(x_0) = \pi^{-1}(\bar{x}_0)$. The universal cover of N is a nilpotent Lie group G_N. We may assume that $\bar{x}_0$ lifts to the identity $0 \in G_N$. Consider the canonical covering map $p : (G_N, 0) \to (M/\mathcal{W}_f^c, \mathcal{W}_f^c(x_0))$. The pullback fiber bundle $p^*(\mathcal{W}_f^c)$ is a covering space over the contractible space G_N and hence is a trivial bundle. In particular, for every $g \in \mathcal{Z}_{\mathrm{Diff}(M)}(f)$ there exist lifts $\hat{f}, \hat{g}$ of f, g to $p^*(\mathcal{W}_f^c)$ such that $\hat{f}$ commutes with $\hat{g}$.

To establish Proposition 8 it suffices to show that $\hat{g}$ preserves the fibration $\widehat{\mathcal{W}_f^c}$ of the bundle $p^*(\mathcal{W}_f^c)$. By uniform compactness of the leaves of $\mathcal{W}_f^c$ and f-invariance of $\mathcal{W}_f^c$, it follows that for any two points $\hat{x}, \hat{y} \in p^*(\mathcal{W}_f^c)$, we have $\hat{y} \in \hat{\mathcal{W}}_f^c(\hat{x})$ if and only if for any $n \in \mathbb{Z}$, there is a C^1-path γ_n connecting $\hat{f}^n(\hat{x})$ to $\hat{f}^n(\hat{y})$ whose length is bounded by some $C > 0$ independent of n; in fact we can take $C = \max_{x \in M} \mathrm{diam}(\mathcal{W}_f^c(x))$. Here we use the fact the induced action of $\bar{f}$ is an Anosov automorphism of N.

Now consider the points $\hat{g}(x), \hat{g}(y)$. For any $n \in \mathbb{Z}$ the points $\hat{f}^n(\hat{g}(x))$ and $\hat{f}^n(\hat{g}(y))$ can be linked by the path $\hat{g}(\gamma_n)$, where γ_n is defined in the previous paragraph. Moreover we have

$$\text{the length of } \hat{g}(\gamma_n) \leq \|g\|_{C^1} \cdot \text{the length of } \gamma_n \leq \|g\|_{C^1} \cdot C,$$

where C is defined in the previous paragraph and is independent of the choice of n. This implies that $\hat{g}(x), \hat{g}(y)$ are contained in the same $\widehat{\mathcal{W}_f^c}$ leaf; hence $\hat{g}$ preserves $\widehat{\mathcal{W}_f^c}$. $\square$

3.6. **Normal form theory.** In this section we introduce some useful aspects of the normal form theory for contracting foliations from [17], omitting some technical details, for which we refer the reader to the source.

Definition 5. Let f be a C^∞ diffeomorphism of a compact manifold X and let $\mathcal{W}$ be an f-invariant continuous foliation of X with uniformly C^∞ leaves. Suppose that $\|Df|_{T\mathcal{W}}\| < 1$. We say that $Df|_{T\mathcal{W}}$

(1) satisfies *the (pointwise) $\frac{1}{2}$-pinching condition* if there exist $C > 0$ and $\gamma < 1$ such that

$$\|(Df^n|_{T_x\mathcal{W}})^{-1}\| \cdot \|Df^n|_{T_x\mathcal{W}}\|^2 \le C \cdot \gamma^n, \text{ for all } x \in X, n \in \mathbb{N}.$$

(2) *has (χ, ϵ)-spectrum* if there exists an f-invariant dominated splitting of $T\mathcal{W} = \bigoplus E_i$ and a continuous Riemannian metric on $\|\cdot\|$ on $T\mathcal{W}$ such that

$$e^{\chi_i - \epsilon}\|v\| \le \|Df(v)\| \le e^{\chi_i + \epsilon}\|v\| \text{ for all } v \in E_i.$$

Here $\chi = (\chi_1, \chi_2, \ldots, \chi_l)$ satisfies $\chi_1 < \cdots < \chi_l < 0$, and we assume $\epsilon > 0$.

(3) *has narrow spectrum* if $Df|_{T\mathcal{W}}$ has (χ, ϵ)-spectrum for some $\chi_1 < \cdots < \chi_l < 0$ and $\epsilon \in (0, \epsilon_0(\chi))$, where $\epsilon_0(\chi)$ is defined in [17, Section 3.4] to guarantee that the "bands" $(\chi_i - \epsilon, \chi_i + \epsilon)$ are narrow (see [17] for more details).

The following lemma summarizes results from Section 3.4 and Theorem 4.6 of [17].

Lemma 9. *Let $f, X, \mathcal{W}$ be as in Definition 5, and suppose that $Df|_{T\mathcal{W}}$ satisfies the $\frac{1}{2}$-pinching condition or has narrow spectrum. Then there exist $d, s_0 = s_0(f) \ge 1$ and a family $\{\Psi_x\}_{x \in M}$ of C^∞ diffeomorphisms $\Psi_x : \mathcal{W}(x) \to E_x := T_x\mathcal{W}(x)$ such that*

(1) $P_x = \Psi_{f(x)} \circ f \circ \Psi_x^{-1} : E_x \to E_{f(x)}$ *is a polynomial map with degree at most d for each $x \in M$;*

(2) $\Psi_x(x) = 0$ *and $D_x\Psi_x$ is the identity map for each $x \in M$,*

(3) Ψ_x *depends continuously on $x \in M$ in the C^∞ topology and depends smoothly on x along the leaves of $\mathcal{W}$.*

(4) *For any $g \in \mathcal{Z}^0(f)$, if g is uniformly C^{s_0} along $\mathcal{W}$, then $\Psi_{g(x)} \circ g \circ \Psi_x^{-1} : E_x \to E_{g(x)}$ is also a polynomial map with degree at most d, for each $x \in M$. In particular, g is C^∞ when restricted to the leaves of $\mathcal{W}_f^{s/u}$.*

Remark 10. If f has narrow spectrum, i.e. f has specified (χ, ϵ)-spectrum on $\mathcal{W}$, the parameter $s_0(f)$ can be taken to depend only on χ, ϵ. In particular, for (χ', ϵ') sufficiently close to (χ, ϵ), and for any f' that has (χ', ϵ') spectrum on a contracting foliation $\mathcal{W}'$, we have that $s_0(f')$ is close to $s_0(f)$. The same stability under C^1-small perturbations holds for the 1/2-pinching condition.

4. Proof of Theorem 1

We return to the topic of diffeomorphisms with transitive centralizer. Let $f : M \to M$ be a homeomorphism of a closed manifold M, and suppose that $\mathcal{Z}^r(f)$ acts transitively on M, for some $r \ge 1$.

4.1. **A useful regularity result.** We have the following useful proposition about the regularity of mappings.

Proposition 11. *Let $r \geq 1$ be an integer, and let $g : M \to M$ be a continuous map on a compact manifold M. If for any $x, x' \in M$ there exists $\varphi \in \mathcal{Z}^r(g)$ such that $\varphi(x) = x'$, then g is a C^r map.*

Proof. By continuity of g and compactness of M, the graph $\Gamma_g := \{(x, g(x)), x \in M\}$ of g is a compact subset of $M \times M$. For any two points $(x, g(x)), (x', g(x'))$ in Γ_g, there exists $\varphi \in \mathcal{Z}^r(g)$ such that $\varphi(x) = x'$. Then

$$\begin{aligned} \varphi \times \varphi(x, g(x)) &= (\varphi(x), \varphi(g(x))) \\ &= (x', g(\varphi(x))) = (x', g(x')). \end{aligned}$$

Since $\varphi \times \varphi$ is a C^r diffeomorphism, this means that the compact set Γ_g is C^r *homogeneous* (in the sense of [25, p.8]). Every locally compact, C^r homogeneous subset of a manifold is a C^r submanifold [25, Theorem B] (see also [21]); it follows that Γ_g is a C^r submanifold in $M \times M$, and hence the projection maps $\mathrm{Pr}_{1,2}|_{\Gamma_g}$ to each factor of $M \times M$ are C^r when restricted to Γ_g. Moreover, $\mathrm{Pr}_1|_{\Gamma_g}$ is a C^r homeomorphism. By Sard's theorem, (see [14]), there exists a point $(x, g(x)) \in \Gamma_g$ such that $(x, g(x))$ is not a critical point of $\mathrm{Pr}_1|_{\Gamma_g}$.

The inverse mapping theorem implies that $\mathrm{Pr}_1|_{\Gamma_g}$ has a C^r inverse in a neighborhood of x, which implies that g is C^r in a neighborhood of x. Now for an arbitrary point $x' \in M$ and a sufficiently small neighborhood $B(x') \subset M$ of x', we take a C^r diffeomorphism $\psi \in \mathcal{Z}^r(g)$ such that $\psi(x) = x'$. Then in a suitably small neighborhood U of x, we have

$$g|_{B(x')} = \psi|_{g(x)} \circ g|_U \circ \psi^{-1}|_{x'},$$

which implies g is also C^r in a possibly smaller neighborhood of x'. As $x' \in M$ was arbitrary, g is C^r. $\qquad\square$

4.2. **Nice subgroups and nice filtrations.**

Definition 6. Let H be a topological group. A subgroup $\mathcal{K} \subset H$ is called *nice* if there exists a countable increasing family of compact sets $K_i, i = 1, 2, \ldots$ such that $\mathcal{K} = \cup_{i=1}^{\infty} K_i$, and for any $i, j \in \mathbb{Z}^+$, there exists $l = l(i, j) \in \mathbb{Z}^+$ such that $K_i \cdot K_j \subset K_l$ and $K_i^{-1} \subset K_l$. The family $\{K_i\}$ is called a *nice filtration* of $\mathcal{K}$. It is easy to see a nice subgroup $\mathcal{K}$ may contain many nice filtrations.

Examples:

- Let M be a compact manifold and $H = \mathrm{Homeo}(M)$ be the group of homeomorphisms of M. Then the subgroup of H formed by the bi-Lipchitz homeomorphisms $\mathrm{Lip}(M)$ is nice since $\mathrm{Lip}(M) = \cup_{k=1}^{\infty} \mathrm{Lip}^k(M)$, where $\mathrm{Lip}^k(M)$ is the set of all bi-Lipchitz homeomorphism of M with Lipchitz constant bounded by k.

- Similarly, the subgroup of H formed by all bi-Hölder homeomorphisms of M, $\mathrm{Hol}(M) = \cup_{k=1}\mathrm{Hol}^k(M)$ is nice, here $\mathrm{Hol}^k(M)$ denote the set of bi-Hölder homeomorphisms with exponents bounded by $1/k$ and constants bounded by k.
- Lemma 18 gives examples arising in partially hyperbolic dynamics.

Combining the nice property with Proposition 11, we have the following crucial proposition.

Proposition 12. *Let $r \geq 0$ be an integer, and let $g\colon M \to M$ be a homeomorphism of a connected closed manifold M. Suppose that $\mathcal{K}$ is a nice subgroup of $\mathcal{Z}^r(g)$ such that for every $x, x' \in X$ there exists $\varphi \in \mathcal{K}$ such that $\varphi(x) = x'$. Then the following holds:*

(1) g is a C^r diffeomorphism (a homeomorphism if $r = 0$).

(2) Let $\{K_i\}_{i \in \mathbb{Z}^+}$ be a nice filtration of $\mathcal{K}$. There exists $N \geq 1$ such that for every $\varepsilon > 0$, there is $\delta > 0$ satisfying: for all $x, x' \in M$, $d(x, x') < \delta$ implies existence of $\varphi \in K_N$ with $d_{C^r}(\varphi, \mathrm{id}) < \varepsilon$ and $\varphi(x) = x'$.

(3) The family $\{g^n \colon n \in \mathbb{Z}\}$ is precompact in the C^0 topology. The action by $\mathcal{G} := \overline{\{g^n : n \in \mathbb{Z}\}}^{C^0}$ is a free compact abelian group action on M.

(4) Moreover, if g is minimal, then there exist a C^r diffeomorphism $h : \mathbb{T}^d \to M$ and $\rho \in \mathbb{T}^d$ such that $h^{-1} \circ g \circ h(x) = x + \rho$, for all $x \in \mathbb{T}^d$.

Question: *Does the same result in (3) hold if we remove the σ-compact assumption and only assume X is a compact metric space? That is, suppose that $g\colon X \to X$ is a minimal homeomorphism and that for every $x, x' \in X$, there exists a homeomorphism $h\colon X \to X$ such that $hg = gh$ and $h(x) = x'$. Is X a compact abelian group?*

Proof of Proposition 12. Since $\mathcal{Z}^r(g) = \mathcal{Z}^r(g^{-1})$, Proposition 11 and the transitivity of $\mathcal{Z}^r(g)$ imply that both g and g^{-1} are C^r, and so g is a C^r diffeomorphism.

Fix a nice filtration $\{K_i\}$ of $\mathcal{K}$. Fix $z \in M$ and consider the evaluation map $\Phi\colon \mathcal{Z}^r(g) \to M$ defined by

$$\Phi(\varphi) = \varphi(z).$$

The compactness of K_i in $\mathcal{Z}^r(g)$ implies that for any $z \in M$, $K_i \cdot z$ is compact in M (since the evaluation map is continuous) A straightforward application of Baire category theorem gives the following.

Lemma 13. *There exists $N_0 > 0$ such that for any $y, z \in M$, there exists $\varphi \in K_{N_0}$ such that $\varphi(z) = y$.*

Proof. Since $\mathcal{K}$ admits a nice filtration $\{K_i\}$, it suffices to show a slightly weaker version of this lemma: for any $z \in M$, there exists $N_1 \in \mathbb{Z}^+$ such that for every $y \in M$, there exists $\varphi \in K_{N_1}$ such that $\varphi(z) = y$.

Fix $z \in M$. The transitivity of the $\mathcal{K}$ action implies that

$$M = \bigcup_{i \in \mathbb{N}} K_i \cdot z.$$

is a countable union of closed subsets of a complete metric space M. The Baire category theorem implies that there exists $i \in \mathbb{N}$ such that $K_i \cdot z$ has nonempty interior in M. Fix $z' \in M$ and $\delta > 0$ such that $B(z', \delta) \subset K_i \cdot z$. For $y \in M$, there exists $\varphi = \varphi_{z',y} \in \mathcal{K}$ such that $\varphi(z') = y$. Then there exists N_y such that for every $y' \in \varphi(B(z', \delta))$, there exists $\varphi' \in K_{N_y}$ such that $\varphi'(z) = y'$: one just composes an element of K_i to go from z to an element of $B(z', \delta)$ with $\varphi_{z',y}$. One thus obtains an open cover of M; extracting a finite subcover gives the result. □

Following the treatment in Avila-Santamaria-Viana [2], we consider a continuous map $\Phi \colon \mathcal{A} \to \mathcal{B}$ between topological spaces $\mathcal{A}$ and $\mathcal{B}$. We say a point $x \in \mathcal{A}$ is *regular* if for every neighborhood $\mathcal{V}$ of x we have $\Phi(x) \in \operatorname{int}(\Phi(\mathcal{V}))$. A point $y \in \mathcal{B}$ is a *regular value* of Φ if every point of $\Phi^{-1}(y)$ is regular. We will use the following result from [2], which is a type of Sard's theorem for continuous maps.

Proposition 14. [2, Proposition 7.6] *Let $\mathcal{A}$ be a compact metrizable space, and let $\mathcal{B}$ be a locally compact Hausdorff space. If $\Phi \colon \mathcal{A} \to \mathcal{B}$ is continuous, then the set of regular values of Φ is residual.*

Note that Proposition 14 implies that for such a continuous map Φ, either the image of Φ is meager or Φ has regular points.

Lemma 15. *Fix $z \in M$, and suppose that $K_i \cdot z$ has nonempty interior in M, for some $i \in \mathbb{Z}^+$. Then there exists $\varphi_0 \in K_i$ such that for every C^r neighborhood $\mathcal{V}$ of φ_0 in K_i, we have*

$$\varphi_0(z) \in \operatorname{int}(\mathcal{V} \cdot z).$$

Proof. We consider the evaluation map $\Phi \colon K_i \to M$ defined by

$$\Phi(\varphi) = \varphi(z).$$

It is a continuous map from a compact metric space to a compact Hausdorff space. Proposition 14 implies that the set of regular values of Φ is residual. Let $y_0 \in K_i$ be a regular value, and let $\varphi_0 \in \Phi^{-1}(y_0)$ be a regular point. Then by the definition of regular point, φ_0 satisfies the desired property. □

Now we show (2) in Proposition 12. Fix $z \in M$, and let $N_0 \in \mathbb{Z}^+$ is given by Lemma 13, then by Lemma 15, we can pick $\varphi_0 \in K_{N_0}$ and $y_0 := \varphi_0(z)$ satisfying Lemma 15. Given $\varepsilon > 0$, let $\mathcal{V}$ be a neighborhood of φ_0 in K_{N_0} such that

$$d_{C^r}(\mathcal{V}^{-1}\mathcal{V}, \operatorname{id}) < \varepsilon.$$

Note that since φ_0 satisfies the conclusions of Lemma 15, we have that $\mathcal{V} \cdot z$ contains a neighborhood W_{y_0} of y_0.

Given $y \in M$, choose $\varphi \in K_{N_0}$ such that $\varphi(y_0) = y$. Then $\varphi(\mathcal{V} \cdot z)$ contains a neighborhood $W_y = \varphi(W_{y_0})$ of y. We thus obtain an open cover $\{W_y : y \in M\}$ of M; let $\delta > 0$ be the Lebesgue number of this cover.

Now given $x, x' \in M$, if $d(x, x') < \delta$, there exists a W_y such that $x, x' \in W_y$. This means that there exist $\varphi, \varphi' \in \mathcal{V}$ such that $\varphi\varphi_0(z) = x$ and $\varphi'\varphi_0(z) = x'$. In other words, $x' = (\varphi')^{-1}\varphi(x)$. Since $(\varphi')^{-1}\varphi \in \mathcal{V}^{-1}\mathcal{V}$, we have that $d_{C^r}((\varphi')^{-1}\varphi, \mathrm{id}) < \varepsilon$. This completes the proof of (2).

Now we show (3) in Proposition 12. Given $\epsilon > 0$, let $\delta > 0$ be given by (2). If $d(x, x') < \delta$, then there exists $\varphi \in K_{N_0}$ with $d_{C^r}(\varphi^{\pm 1}, \mathrm{id}) < \varepsilon$ such that $\varphi(x) = x'$.

Then for any $n \in \mathbb{Z}$,

$$
\begin{aligned}
d(g^n(x), g^n(x')) &= d(g^n(x), \varphi \circ g^n \circ \varphi^{-1}(x')) \\
&< \varepsilon,
\end{aligned}
$$

since both φ^{-1} and φ are C^r-close to the identity in a neighborhood of x' and x, respectively. The Arzelà-Ascoli theorem then implies that $\{g^n : n \in \mathbb{Z}\}$ is precompact in $C^0(M)$. Let $\mathcal{G}$ be the C^0 closure of the iterates of g:

$$
\mathcal{G} := \overline{\{g^n : n \in \mathbb{Z}\}}^{C^0} \subset \mathrm{Homeo}(M).
$$

By definition of $\mathcal{G}$ it clearly commutes with any element of $\mathcal{Z}^r(g)$, therefore $\mathcal{G}$ commutes with $\mathcal{K}$. Since $\mathcal{K}$ acts transitively on M and commutes with $\mathcal{G}$, if $g' \in \mathcal{G}$ fixes a point $x \in M$, then g' fixes all points and is the identity. Therefore the $\mathcal{G}$ action is free. This complets the proof of (3).

To show part (4) of Proposition 12, first we consider its C^0 version. Assume that g is minimal. Using the C^0 precompactness of $\{g^n : n \in \mathbb{Z}\}$ from (3), we construct an abelian topological group structure on M. As before, let $\mathcal{G} := \overline{\{g^n : n \in \mathbb{Z}\}}^{C^0}$.

Minimality of g implies that $\mathcal{G}$ acts transitively on M: given $x, y \in M$, since g is minimal, there exists $n_j \to \infty$ such that $g^{n_j}(x) \to y$; using compactness of $\mathcal{G}$ and passing to a subsequence we obtain in the limit a map $\bar{g} = \lim_k g^{n_{j_k}} \in \mathcal{G}$ satisfying $\bar{g}(x) = y$.

Fix $z \in M$. Since the $\mathcal{G}$-action is transitive and free, the evaluation map $\Phi : \mathcal{G} \to M$ defined by $\Phi(\varphi) = \varphi(z)$ is a homeomorphism. It defines a continuous abelian group structure on M via the operation $\varphi_1(z) + \varphi_2(z) := \varphi_1(\varphi_2(z)) = \varphi_2(\varphi_1(z))$. Thus M is topologically a torus, and $\varphi \in \mathcal{G}$ is topologically conjugate via Φ to a minimal translation $x \mapsto x + \rho$ for some $\rho \in \mathbb{T}^d$.

Proposition 11 implies that the elements of $\mathcal{G}$ are C^r. Then [8, Theorem 5] implies that the action by $\mathcal{G}$ is in fact a C^r Lie group action, and M is C^r conjugate to the standard torus.

$\square$

4.3. **Proof of Theorem 1.**

Proof of Theorem 1. (1) First we assume $\mathcal{Z}^r(f)$ acts transitively on M. Proposition 11 implies that f is C^r. Let $\mathcal{K}$ be the group of all bi-Lipschitz homeomorphisms of M that commute with f, and note that $\mathcal{K}$ is a nice subgroup of $\mathrm{Homeo}(M)$ (since any C^0 limit point of a sequence of bi-Lipchitz homeomorphisms with Lipchitz constant bounded by L and commuting with f, is bi-Lipchitz and with the same Lipchitz constant and commutes with f). Then by Proposition 12, $\{f^n, n \in \mathbb{Z}\}$ is precompact in $\mathrm{Homeo}(M)$, and $P = \overline{\{f^n, n \in \mathbb{Z}\}}$ is a compact topological group acting on M. Since each $p \in P$ commutes with $\mathcal{Z}^r(f)$ which acts transitively on M, Proposition 11 implies that p is C^r. Then P is a compact topological group acting continuously on M by C^r diffeomorphisms, and so [8, Theorem 5] implies that P is a Lie group and its action is a C^r compact Lie group action on M.

Since P is the C^0-closure of $\{f^n, n \in \mathbb{Z}\}$, it follows that P is a compact abelian subgroup of $\mathrm{Homeo}(M)$, containing a dense subset $\{f^n, n \in \mathbb{Z}\}$. Therefore P is a direct product of some $\mathbb{Z}/d\mathbb{Z}$ with $\mathbb{T}^k$.

Since P is compact Lie group, its action is proper. By part (3) of Proposition 12, the P action is free. Then we apply the classical fact that a proper and free Lie group action induces a C^r principal bundle structure on M. This implies the "only if" part of Theorem 1.

(2) Now we assume that M is a C^r principal $\mathbb{T}^k \times \mathbb{Z}/d\mathbb{Z}$ bundle and f is a constant minimal translation on the bundle and identity on the base. Let $G \to M \xrightarrow{\pi} \overline{M}$ be the f-invariant principal fiber bundle structure, with $G = \mathbb{T}^k \times \mathbb{Z}/d\mathbb{Z}$. Since M is connected, to show that $\mathcal{Z}^r(f)$ acts transitively on M, it suffices to show for any $x \in M$ and any y close to x, there exists $g \in \mathcal{Z}^r(f)$ such that $g(x) = y$. To show this we take a small neighborhood $\bar{U}$ of $\pi(x)$ such that $\pi^{-1}(\bar{U})$ is C^r diffeomorphic to $\bar{U} \times E$. Then for any $y \in \pi^{-1}(\bar{U})$, it is not hard to construct a C^r diffeomorphism g of M such that

(1) g is supported on $\pi^{-1}(\bar{U})$.
(2) g preserves the principal bundle structure in $\pi^{-1}(\bar{U})$, i.e. g is translation of G along each fiber.
(3) $g(x) = y$ (such a g is constructed as a translation skew product of a map $\bar{g} : \overline{M} \to \overline{M}$ supported on $\bar{U}'$ such that $\overline{\bar{U}'} \subset \bar{U}$).

Since f is fiber fixing and a constant translation along every fiber, it follows that $fg = gf$. □

4.4. A byproduct of the proof of Theorem 1.

For later use we consider the following corollary of Propositions 11 and 12. For a set $\mathcal{K} \subset \mathrm{Homeo}(M)$, and $k \in \mathbb{N}$, we define

$$\mathcal{Z}^k(\mathcal{K}) := \{g \in \mathrm{Diff}^k(M), gg' = g'g, \ \forall g' \in \mathcal{K}\}.$$

Theorem 4. *Let M be a connected closed manifold and let $\mathcal{K}$ be a nice subgroup of $\mathrm{Homeo}(M)$ such that $\mathcal{K} \subset \mathrm{Diff}^r(M)$, for some $r \geq 1$. If $\mathcal{K}$ acts transitively on M, then M is a C^r principal fiber bundle (possibly a trivial*

one). Moreover $\mathcal{Z}^0(\mathcal{K}) = \mathcal{Z}^r(\mathcal{K})$ consists of the set of constant translations along the fibers (that is, the right action of the structure group).

Proof. Using the same method as in the proof of (3) of Proposition 12, we use the Arzelà-Ascoli theorem to conclude that $\mathcal{Z}^0(\mathcal{K})$ is a compact group (in the C^0 topology). Transitivity of the action of $\mathcal{K}$ implies that $\mathcal{Z}^0(\mathcal{K}) = \mathcal{Z}^r(\mathcal{K})$. This implies that $\mathcal{Z}^0(\mathcal{K})$ is a Lie group[22]. Then [8, Theorem 5] implies that the $\mathcal{Z}(\mathcal{K})$ action is a C^r action of a Lie group on M.

For any $g \in \mathcal{Z}^0(\mathcal{K})$, if g has a fixed point x, by transitivity of $\mathcal{K}$ action we can easily show that any point x' of M is also a fixed point of g, which means that g is the identity. Thus the action of $\mathcal{Z}^0(\mathcal{K})$ is a free and proper hence induces a C^r principal $\mathcal{Z}^0(\mathcal{K})$-bundle structure on M. $\qquad\square$

5. Applications to fibered partially hyperbolic systems: the center fixing centralizer and the proof of Theorem 3

5.1. The center fixing centralizer.
In the following theorem, the integer parameter $s_0 \geq 1$ controls the spectral behavior of $Df|_{E^{s,u}}$, and r measures in part the nonconformality of $Df|_{E^c}$ (the larger the r, the more conformal). So in general r can be arbitrarily large, i.e. close to or equal ∞, for example, when f is a perturbation of an isometric extension. In particular, r could be much larger than s_0, and in that case the following theorem demonstrates a certain bootstrapping effect on regularity.

Theorem 5. *Let $f \in \mathrm{Diff}^\infty(M)$ be a fibered partially hyperbolic diffeomorphism that is accessible and r-bunched, for some $r \in \mathbb{Z}^+$. We further assume that the cocycles $Df|_{E^s}, Df^{-1}|_{E^u}$ either satisfy the the pointwise $1/2$-pinching condition or have narrow spectrum. Then there exists $s_0 \geq 1$ such that for any $s \in \{s_0, \ldots, r\}$, $\mathcal{CZ}^s(f)$ is a compact k-dimensional Lie subgroup of $\mathrm{Diff}^s(M)$, for some $k \leq \dim E^c_{f_0}$, such that*

(1) *The action by $\mathcal{CZ}^s(f)$ is a C^r Lie group action that is uniformly C^∞ along the stable and unstable foliations of f.*
(2) *M admits an f-invariant, holonomy-invariant C^r-principal bundle structure that is subordinate to $\mathcal{W}^c_f$.*
(3) *Moreover, if $\dim \mathcal{CZ}^s(f) = \dim \mathcal{W}^c_f$, then $\mathcal{CZ}^s(f) = \mathcal{CZ}^\infty(f)$ and f is a C^∞ compact Lie group extension of an Anosov diffeomorphism.*

Remark 16. From the proof we will see our result also holds for $f \in \mathrm{Diff}^k(M)$ for some $k < \infty$ sufficiently large; for simplicity we only state the C^∞ case. We have the following corollary of the proof of Theorem 5.

Corollary 17. *Let $f \in \mathrm{Diff}^\infty(M)$ be an accessible fibered partially hyperbolic diffeomorphism with 1-dimensional center. We further assume that the cocycles $Df|_{E^s}, Df^{-1}|_{E^u}$ either satisfy the the pointwise $1/2$-pinching condition or have narrow spectrum. Then $\mathcal{CZ}^\infty(f)$ is either finite or $\mathbb{T}^1$. In the latter case M admits an f-invariant, holonomy-invariant C^∞ principal bundle structure along $\mathcal{W}^c$.*

We will prove Theorem 5 and Corollary 17 in the following subsection; we first establish some preliminaries.

Lemma 18. *Let $r \in \mathbb{Z}^+$ and f be C^r a fibered partially hyperbolic diffeomorphism that is accessible and r-bunched. Then for any fixed center leaf $\mathcal{W}^c_f(x_0)$, the set of all su-holonomies from $\mathcal{W}^c(x_0)$ to $\mathcal{W}^c(x_0)$ is a nice subgroup of $\mathrm{Diff}^k(\mathcal{W}^c(x_0))$ for $k = 0, \ldots, r$.*

Proof of Lemma 18. If f is C^r and r-bunched, then the stable and unstable holonomies between center leaves are C^r and vary continuously in C^r the topology as a function of the paths inducing them. Therefore the set of su-holonomies from a fixed center leaf $\mathcal{W}^c(x_0)$ to itself admit a nice filtration by an increasing sequence of compact sets $K_n \subset \mathrm{Diff}^k(\mathcal{W}^c(x_0))$, where K_n is the subset of $\mathrm{Diff}^r(\mathcal{W}^c(x_0))$ induced by su-paths with $\leq n$ legs each of length $\leq n$. $\square$

It is easy to see that the holonomy maps of a fibered partially hyperbolic diffeomorphism $f \in \mathrm{Diff}^1(M)$ commute with the elements of $\mathcal{CZ}^1(f)$. This is not necessarily the case for elements of $\mathcal{CZ}^0(f)$, as homeomorphisms commuting with f might not preserve the leaves of $\mathcal{W}^{s/u}_f$. This motivates the following definition.

Definition 7. For a fibered partially hyperbolic diffeomorphism $f \in \mathrm{Diff}^1(M)$, we denote by

  (1) $\mathcal{CZ}_*(f)$ the group of homeomorphisms of M that commute with f, fixing each $\mathcal{W}^c_f$ leaf and preserving the $\mathcal{W}^{s/u}_f$-foliations; and
  (2) $\mathcal{C}^r\mathcal{Z}_*(f) \subset \mathcal{CZ}_*$ the subgroup of $\mathcal{CZ}_*(f)$ consisting of the elements that are uniformly C^r on $\mathcal{W}^c_f$ leaves.

Clearly we have $\mathcal{CZ}^r(f) \subset \mathcal{C}^r\mathcal{Z}_*(f) \subset \mathcal{CZ}_*(f)$.

We have the following proposition, which is an application of Theorem 4.

Proposition 19. *Let $f \in \mathrm{Diff}^\infty(M)$ be a fibered partially hyperbolic diffeomorphism. If f is accessible and r-bunched, for some $r \in \mathbb{Z}^+$, then*

  (1) $\mathcal{C}^r\mathcal{Z}_*(f) = \mathcal{CZ}_*(f)$, i.e., every element in $\mathcal{CZ}_*(f)$ is uniformly C^r along $\mathcal{W}^c_f$.
  (2) The action of $\mathcal{CZ}_*(f)$ on M is a free compact Lie group action.

Proof of Proposition 19. For any fixed center leaf $\mathcal{W}^c_f(x_0)$, consider the group $\mathcal{K}(x_0)$ formed by su-holonomies from $\mathcal{W}^c_f(x_0)$ to $\mathcal{W}^c_f(x_0)$. Lemma 18 implies that $\mathcal{K}(x_0) \subset \mathrm{Diff}^r(\mathcal{W}^c_f(x_0))$ is a nice subgroup of $\mathrm{Homeo}(\mathcal{W}^c_f(x_0))$. Theorem 4 then implies that any element in $\mathcal{CZ}_*(f)$ is C^r along $\mathcal{W}^c_f(x_0)$.

Claim (1) of Proposition 19 is implied by the following *uniformity lemma.*

Lemma 20. *Let f be C^r fibered partially hyperbolic diffeomorphism that is r-bunched, for some $r \in \mathbb{Z}_+$. Suppose that $g \in \mathcal{CZ}_*(f)$ is C^r along a center leaf $\mathcal{W}^c_f(x_0)$. Then g is uniformly C^r along all leaves of $\mathcal{W}^c_f$. Moreover,*

$y \mapsto g|_{\mathcal{W}_f^c(y)}$ *induces a continuous map from M into the space of r-jets along leaves of $\mathcal{W}_f^c$.*

Proof of Lemma 20. For $x, y \in M$, let γ be an su-path from x to y. The induced holonomy $H_\gamma : \mathcal{W}_f^c(x) \to \mathcal{W}_f^c(y)$ is C^r (since f is assumed to be r-bunched), and it varies continuously in C^r topology with the path γ. For $g \in \mathcal{CZ}_*(f)$, we have that $H_\gamma \circ g|_{\mathcal{W}_f^c(x)} \circ H_\gamma^{-1} = g|_{\mathcal{W}_f^c(y)}$.

Let $d = \dim M - \dim E^c$. Due to the fibered structure of $\mathcal{W}_f^c$, for any $x \in M$, there is a continuous family of 2-legged su-paths $\{\gamma_\xi \mid \xi \in [-\epsilon, \epsilon]^d\}$ starting at x such that the map $\mathcal{W}_f^c(x) \times [-\epsilon, \epsilon]^d \to M$ defined by $(\xi, z) \mapsto H_{\gamma_\xi}(z)$ is a homeomorphism onto a neighborhood of $\mathcal{W}^c(x)$. This defines a continuous map from $[-\epsilon, \epsilon]^d$ into the space of C^r embeddings of $\mathcal{W}_f^c(x)$ into M via $\xi \mapsto H_{\gamma_\xi}$.

If $g|_{\mathcal{W}_f^c(x)}$ is C^r, then $y \mapsto g|_{\mathcal{W}_f^c(y)}$ is continuous in the C^r topology in this neighborhood of $g|_{\mathcal{W}_f^c(x)}$, via the parametrization $\xi \mapsto H_{\gamma_\xi} \circ g|_{\mathcal{W}_f^c(x)} \circ H_{\gamma_\xi}^{-1}$. Since M is connected, if $g|_{\mathcal{W}_f^c(x_0)}$ is C^r for some x_0, then $y \mapsto g|_{\mathcal{W}_f^c(y)}$ is continuous in the C^r topology on all of M. $\qquad\square$

Now we prove part (2) of Proposition 19. We first show that the action of $\mathcal{CZ}_*$ on M is free. To this end, suppose that $g \in \mathcal{CZ}_*$ fixes a point x. Then for every point $y \in M$, there is an su path γ from x to y, and $g(y) = H_\gamma \circ g \circ H_\gamma^{-1}(y) = y$. Thus g is the identity, and the action is free.

The proof of Lemma 20, shows that for any $x_0 \in M$, the group representation $\iota : \mathcal{CZ}_* \to \mathrm{Homeo}(\mathcal{W}_f^c(x_0))$ defined by restriction to $\mathcal{W}_f^c(x_0)$ is a topological group embedding. Lemma 18 implies that $\mathcal{K}(x_0)$ is a nice subgroup of $\mathrm{Homeo}(\mathcal{W}_f^c(x_0))$, which centralizes $\iota(\mathcal{CZ}_*)$. Accessibility of f implies that $\mathcal{K}(x_0)$ acts transitively on $\mathcal{W}_f^c(x_0)$. Theorem 4 then implies that $\iota(\mathcal{CZ}_*)$ is contained in the compact Lie group $\mathcal{Z}(\mathcal{K}(x_0)) := \{g \in \mathrm{Homeo}(\mathcal{W}_f^c(x_0)) : gh = hg, \ \forall h \in \mathcal{K}(x_0)\}$.

Since ι is faithful, to show the action by $\mathcal{CZ}_*$ is a compact Lie group action, we need only show that $\iota(\mathcal{CZ}_*)$ is closed in $\mathrm{Homeo}(\mathcal{W}_f^c(x_0))$. If not, there exists a sequence $g_n \in \mathcal{CZ}_*$ such that g_n has no limit point in $\mathrm{Homeo}(M)$ but the restriction $g_n|_{\mathcal{W}_f^c(x_0)}$ converges uniformly. But the proof of Lemma 20 shows that uniform convergence of $g_n|_{\mathcal{W}_f^c(x_0)}$ implies uniform convergence of g_n on M, a contradiction. This establishes part (2) of of Proposition 19, completing its proof. $\qquad\square$

5.2. **Proof of Theorem 5.**

Assume that f satisfies the hypotheses of Theorem 5. We denote $\mathcal{CZ}^s(f)$ by $\mathcal{CZ}^s$, and let $\mathcal{CZ}_* = \mathcal{CZ}_*(f)$ and $\mathcal{CZ}_*^r = \mathcal{CZ}_*^r(f)$ be defined as in the previous subsection. Since $Df|_{E^s}$ and $Df^{-1}|_{E^u}$ satisfy either the $\frac{1}{2}$-pinching or narrow spectrum condition, part (4) of Lemma 9 implies that there exists $s_0 > 0$, such that if $s \geq s_0$, then any $g \in \mathcal{CZ}^s$ is C^∞ along the leaves of $\mathcal{W}^{s/u}$ (and is polynomial after a uniformly C^∞ change of coordinates along leaves).

Moreover, Remark 10 (see [17] for more details), implies that by increasing $s_0(f_0)$ if necessary, we can take $s_0(f_0) = s_0(f)$ to work for all f such that $d_{C^1}(f_0, f) < \varepsilon$, for some small ε. We fix such an s_0. Proposition 19 gives that $\mathcal{CZ}^s \subset \mathcal{CZ}_* = \mathcal{C}^r \mathcal{Z}_*$ for $s_0 \le s \le r$, and in particular, every element of $\mathcal{CZ}^s$ is C^r along the leaves of $\mathcal{W}_f^c$. Since, in addition, the elements of $\mathcal{CZ}^s$ are C^∞ along the $\mathcal{W}_f^{s/u}$ leaves, Journé's lemma [16] implies that $\mathcal{CZ}^s = \mathcal{CZ}^r$. Thus any element of $\mathcal{CZ}^s$ is C^∞ along the leaves of $\mathcal{W}_f^{s/u}$.

To show the action by $\mathcal{CZ}^s$ is a C^r compact Lie group action, we recall from Proposition 19 that the action by $\mathcal{CZ}_*$ is a free compact Lie group action. It then suffices to show the following.

Claim: The group $\mathcal{CZ}^s$ is a C^0 closed subgroup of $\mathcal{CZ}_*$.

The claim implies that $\mathcal{CZ}^s$ is a compact Lie subgroup of $\mathcal{CZ}_* \subset \mathrm{Homeo}(M)$, and each element acts on M by C^r diffeomorphisms; it follows from [8, Theorem 5] that $\mathcal{CZ}^s$ acts by a C^r Lie group action.

Proof of claim. Consider an arbitrary sequence $g_n \in \mathcal{CZ}^s$ converging uniformly to $g_0 \in \mathcal{CZ}_*$. Proposition 19 implies that $g_0 \in \mathcal{CZ}_*^r$, and so g_0 is uniformly C^r along $\mathcal{W}_f^c$-leaves.

Part (4) of Lemma 9 implies that, up to a uniformly C^∞ coordinate change along $\mathcal{W}^{s/u}$, each g_n is a polynomial map along $\mathcal{W}^{s/u}$ with bounded degree and bounded coefficients; a C^0 limit of polynomial maps with bounded degree and bounded coefficients is polynomial, and so therefore, under a uniform C^∞ coordinate change, the map g_0 is also a polynomial map along $\mathcal{W}^{s/u}$. Thus g_0 is uniformly C^∞ along $\mathcal{W}^{s/u}$. Then by Journé's lemma [16], g_0 is a C^r diffeomorphism, and so $g_0 \in \mathcal{CZ}^s$. □

As in the proof of Theorem 4, we obtain from the action of $\mathcal{CZ}^s$ a principal bundle structure on M. It is f-invariant, holonomy invariant and subfoliates $\mathcal{W}_f^c$; this gives (2) of Theorem 5. Clearly $\dim \mathcal{CZ}^s$ is well-defined and not greater than $\dim W_f^c$ (a Lie group cannot act freely on a manifold with lower dimension).

To complete the proof of Theorem 5, we consider the case that $\dim \mathcal{CZ}^s = \dim \mathcal{W}_f^c$. Denote by $cc_{\mathrm{id}}(\cdot)$ the connected component of the identity in a given topological group. Since the action by $\mathcal{CZ}_*$ is free, so is the $\mathcal{CZ}^s$ action. Therefore the action of $\mathcal{CZ}^s$ is a free C^r compact Lie group action. By compactness and connectedness of $\mathcal{W}_f^c$ and $cc_{\mathrm{id}}(\mathcal{CZ}^s)$, the group $cc_{\mathrm{id}}(\mathcal{CZ}^s)$ acts freely and transitively on each $\mathcal{W}_f^c$ leaf, and by freeness of the action of $\mathcal{CZ}^s$, we have $\mathcal{CZ}^s = cc_{\mathrm{id}}(\mathcal{CZ}^s)$. In this case $\mathcal{W}_f^c$ is subfoliated by a principal $\mathcal{CZ}^s$-bundle, where $\mathcal{CZ}^s$ has the same dimension as $\mathcal{W}_f^c$.

Consider the bi-invariant metric on $cc_{\mathrm{id}}(\mathcal{CZ}^s)$: it induces an f-invariant metric on $\mathcal{W}_f^c$. Then f is ∞-bunched. But this implies that $\mathcal{CZ}^s = \mathcal{CZ}^\infty$, and $\mathcal{W}_f^c$ leaves are the orbits of the C^∞, free compact Lie group action by $\mathcal{CZ}^s$.

To summarize, in the case that $\dim \mathcal{CZ}^s = \dim \mathcal{W}_f^c$, the connected compact Lie group $\mathcal{CZ}^s$ acts on M smoothly, freely and properly with orbit foliation $\mathcal{W}_f^c$. Thus the projection $\pi : M \to M/\mathcal{W}_f^c$ gives a C^∞ principal bundle with structure group $\mathcal{CZ}^s$. In any group the centralizer of all left translations is the group of right translations. Since f commutes with $\mathcal{CZ}^s$, it follows that f is a C^∞ compact Lie group extension of an Anosov diffeomorphism. This completes the proof of Theorem 5. $\qquad\square$

Using almost the same method, we can prove Corollary 17. For completeness we give a sketch of the proof. Proposition 19 and Corollary 17 imply that $\mathcal{C}^1 \mathcal{Z}_*(f)$ is either finite or $\mathbb{T}^1$. If it finite, then $\mathcal{CZ}^\infty(f)$ is obviously finite. If $\mathcal{C}^1 \mathcal{Z}_*(f)$ is $\mathbb{T}^1$, then f is ∞-bunched. Then by almost the same proof as of Theorem 5, using Journé's lemma and normal form theory we obtain that $\mathcal{C}^1 \mathcal{Z}_*(f) = \mathcal{C}^\infty \mathcal{Z}_*(f) = \mathcal{CZ}^\infty(f)$ is $\mathbb{T}^1$. The rest of Corollary 17 is a consequence of part (3) of Theorem 5.

5.3. **Proof of Theorem 3.** Let f_0 be $\infty-$bunched, and assume that $Df_0|_{E^s}$, $Df_0^{-1}|_{E^u}$ have $1/2$-pinched or narrow spectrum. Lemma 9 gives an $s_0(f_0) \geq 1$, such that for any $s \geq s_0$, the maps f_0 and any $g \in \mathcal{Z}^s(f_0)$ preserve C^∞ normal forms along $\mathcal{W}_{f_0}^{s/u}$, and the same holds for any f sufficiently C^1 close to f_0. Fix $s \geq s_0$. If f is sufficiently C^1-close to f_0, then f is $r-$bunched for some $r > s$, is fibered (by [15]) and accessible (by Lemma 5). Therefore such an f satisfies all the assumptions of Theorem 5.

Theorem 5 implies that $\mathcal{CZ}^s(f)$ is a C^0-closed Lie subgroup of $\mathrm{Diff}^s(M)$ with dimension $k \leq \dim E_{f_0}^c$. Thus to show that $\mathcal{Z}^s(f)$ is also a C^0-closed Lie subgroup of $\mathrm{Diff}^s(M)$, it suffices to show that $\mathcal{Z}^s(f)$ is just countably many discrete copies of $\mathcal{CZ}^s(f)$ in $\mathrm{Homeo}(M)$.

Proposition 8 implies that any element in $\mathcal{Z}^s(f)$ preserves the fiber bundle structure, and so $\mathcal{CZ}^s(f)$ is a normal subgroup of $\mathcal{Z}^s(f)$, and the quotient group $\mathcal{Z}^s(f)/\mathcal{CZ}^s(f)$ has a continuous (with respect to the quotient metric) identification with a subgroup of $\mathcal{Z}^0(\bar{f})$, through the map $\iota : g \cdot \mathcal{CZ}^s(f) \mapsto \bar{g}$.

The first paragraph of the proof of Proposition 8 gives that $\bar{f}$ is Hölder conjugate to an Anosov automorphism on a nilmanifold M. Any transitive Anosov diffeomorphism (in particular an affine Anosov map on a compact nilmanifold) has countable discrete C^0 centralizer. Hence the coset partition $\mathcal{Z}^s(f) = \bigsqcup g \, \mathcal{CZ}^s(f)$ is actually formed by countably many discrete copies of $\mathcal{CZ}^s(f)$. Then $\mathcal{Z}^s(f)$ is a closed Lie group in $\mathrm{Homeo}(M)$ and we have $\dim \mathcal{Z}^s(f) = \dim \mathcal{CZ}^s(f)$. The rest of results in Theorem 3 are direct consequences of the corresponding results in Theorem 5 (for the spectral property of the base dynamics in (2), we use the fact that if $Df|_{E^s}$ has (χ, ϵ)-spectrum, and $\mathcal{W}_f^c$ is smooth then $D\bar{f}|_{E_{\bar{f}}^s}$ has (χ, ϵ)-spectrum as well, and similarly for the $1/2$-pinched case). This completes the proof of Theorem 3. $\qquad\square$

6. Proof of Theorem 2

In this section we prove Theorem 2. Let f be a partially hyperbolic diffeomorphism on a 3-dimensional non-toral nilmanifold M. Then f satisfies the following properties:

(1) By [13, Theorem 1.6], f is a fibered partially hyperbolic system over an Anosov homeomorphism $\bar{f} : N/\mathcal{W}_f^c \to N/\mathcal{W}_f^c$, which is topologically conjugate to a hyperbolic automorphism of $\mathbb{T}^2$.

(2) By [13, Proposition 6.4], f is accessible.

(3) Since $E^{s/u}$ are 1-dimensional, the restrictions $Df|_{E^{s/u}}$ satisfy the pointwise 1/2-pinching condition.

Corollary 17 then implies that $\mathcal{CZ}^\infty(f)$ is either finite or virtually $\mathbb{T}^1$. In the latter case, f preserves a C^∞ principal $\mathbb{T}^1$ bundle structure.

To complete the proof of Theorem 2, it suffices to show that $\mathcal{Z}^\infty(f)$ is virtually $\{f^n \circ \mathcal{CZ}^\infty(f), n \in \mathbb{Z}\}$. Notice that $\mathcal{Z}^\infty(f)/\mathcal{CZ}^\infty(f)$ can be naturally embedded into $\mathcal{Z}^0(\bar{f})$. Since $\bar{f}$ is topologically conjugate to a toral hyperbolic automorphism, [1] and [18, Proposition 3.7] imply that $\mathcal{Z}^0(\bar{f})$ is virtually trivial, which forces $\mathcal{Z}^\infty(f)$ to be virtually $\{f^n \circ \mathcal{CZ}^\infty(f), n \in \mathbb{Z}\} \cong \mathbb{Z} \times \mathbb{T}$.

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