

Centralizers of C^1 -generic diffeomorphisms

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Abstract

On the one hand, we prove that the spaces of C^1 symplectomorphisms and of C^1 volume-preserving diffeomorphisms both contain residual subsets of diffeomorphisms whose centralizers are trivial. On the other hand, we show that the space of C^1 diffeomorphisms of the circle and a non-empty open set of C^1 diffeomorphisms of the two-sphere contain dense subsets of diffeomorphisms whose centralizer contains a subgroup isomorphic to \mathbb{R} .

Key words: Trivial centralizer, trivial symmetries, Mather invariant, C^1 generic properties.

Introduction

Let M be a connected compact manifold. The *centralizer* of a C^r diffeomorphism $f \in \text{Diff}^r(M)$ is defined as

$$C(f) := \{g \in \text{Diff}^r(M) : fg = gf\}.$$

Clearly $C(f)$ always contains the group $\langle f \rangle$ of all the powers of f . We say that f has *trivial centralizer* if $C(f) = \langle f \rangle$. A diffeomorphism f with trivial centralizer possesses no smooth symmetries, such as those that would arise if, for example, f embedded in a flow or were the lift of another diffeomorphism. Smale asked the following:

Question 0.1 ([Sm1, Sm2]) *Let $T^r(M) \subset \text{Diff}^r(M), r \geq 1$ denote the set of C^r diffeomorphisms of a compact manifold M with trivial centralizer.*

1. *Is $T^r(M)$ dense in $\text{Diff}^r(M)$?*
2. *Is $T^r(M)$ residual in $\text{Diff}^r(M)$?*
3. *Is $T^r(M)$ open in $\text{Diff}^r(M)$?*

This question has been answered in several special cases. To summarize these results in rough chronological order, we have:

- $\mathcal{T}^r(S^1)$ is open and dense in $\text{Diff}^r(S^1)$ for $r \geq 2$ [Ko];
- $\mathcal{T}^1(M)$ is residual among the Axiom A diffeomorphisms in $\text{Diff}^1(M)$; in particular, $\mathcal{T}^1(S^1)$ is residual in $\text{Diff}^1(S^1)$ [To1, To2];
- $\mathcal{T}^\infty(M)$ is open and dense among the Axiom A diffeomorphisms in $\text{Diff}^\infty(M)$ possessing at least one periodic sink or source [PY1];
- $\mathcal{T}^\infty(M)$ is open and dense among the Anosov diffeomorphisms in $\text{Diff}^\infty(\mathbb{T}^n)$, where \mathbb{T}^n is the n -torus [PY2];
- $\mathcal{T}^\infty(M)$ is locally residual among the partially hyperbolic diffeomorphisms with 1-dimensional center [Bu].

There are two main results in this paper. In the first (Theorem 0.2), we give a complete answer to the first two parts of Question 0.1 for all compact M in the case of volume-preserving and symplectic C^1 -diffeomorphisms. In the second result (Theorem 0.7), we answer the third part of Question 0.1 for the circle S^1 and the sphere S^2 , again in the case $r = 1$.

A) Trivial centralizer for C^1 -generic symplectomorphisms and volume-preserving diffeomorphisms

In order to state our first main result precisely we will need some notation. If M carries a volume μ , then we denote by $\text{Diff}_\mu^1(M)$ the space of C^1 diffeomorphisms of M that preserve μ . If M is a symplectic manifold, then $\text{Symp}^1(M)$ denotes the space of C^1 symplectomorphisms of M . The spaces $\text{Diff}^1(M)$, $\text{Diff}_\mu^1(M)$, $\text{Symp}^1(M)$ are Baire spaces in the C^1 topology. Recall that a *residual* subset of a Baire space is one that contains a countable intersection of open-dense sets.

Theorem 0.2 *Let M be a compact, connected manifold of dimension at least 2. Then:*

- (a) $\mathcal{T}^1(M) \cap \text{Diff}_\mu^1(M)$ is residual in $\text{Diff}_\mu^1(M)$.
- (b) $\mathcal{T}^1(M) \cap \text{Symp}^1(M)$ is residual in $\text{Symp}^1(M)$.

Theorem 0.2 is a corollary of parts (b) and (c) of the following result:

Theorem 0.3 (a) *There is a residual set $\mathcal{R} \subset \text{Diff}^1(M)$ such that, for any diffeomorphism $f \in \mathcal{R}$, for any $g \in C(f)$ and any periodic point $x \in \text{Per}(f)$,*

the point x is hyperbolic and, there exist $m, n \in \mathbb{Z}$ such that g coincides with f^n on $W^s(x)$ and with f^m on $W^u(x)$.

(b) There is a residual set $\mathcal{R}_{\text{symp}} \subset \text{Symp}^1(M)$ such that, for any diffeomorphism $f \in \mathcal{R}_{\text{symp}}$ for any $g \in C(f)$ and any hyperbolic periodic point $x \in \text{Per}(f)$, there exist $m, n \in \mathbb{Z}$ such that g coincides with f^n on $W^s(x)$ and with f^m on $W^u(x)$.

(c) There is a residual set $\mathcal{R}_\mu \subset \text{Diff}_\mu^1(M)$ such that, for any diffeomorphism $f \in \mathcal{R}_\mu$ for any $g \in C(f)$ and any hyperbolic periodic point $x \in \text{Per}(f)$, there exists $n \in \mathbb{Z}$ such that g coincides with f^n on either $W^s(x)$ or $W^u(x)$.

Theorem 0.3 (a) was previously proved by Togawa [To1, To2], using different methods. Togawa's methods, combined with the results in Appendix A, can also be used to prove parts (b) and (c) of Theorem 0.3. While using Togawa's results would shorten considerably the proof of Theorem 0.3, we believe our approach, in particular Propositions 1.4 and 1.9, has independent interest. It would be interesting to see if these results have further application. We discuss the motivation and background to this approach in Section 1.

Proof of Theorem 0.2. Theorem 0.2 follows immediately from Theorem 0.3 and:

Theorem 0.4 ([BC, ABC]) For any compact connected manifold M , there are residual sets $\widetilde{\mathcal{R}}_\mu \subset \text{Diff}_\mu^1(M)$ and $\widetilde{\mathcal{R}}_{\text{symp}} \subset \text{Symp}^1(M)$ such that, every $f \in \widetilde{\mathcal{R}}_\mu \cup \widetilde{\mathcal{R}}_{\text{symp}}$ has a hyperbolic periodic point p with

$$\overline{W^s(p)} = \overline{W^u(p)} = M.$$

□

More generally, Theorem 0.3 naturally applies to the class of C^1 diffeomorphisms satisfying a property we call periodic accessibility. A diffeomorphism f on a compact manifold satisfies the *periodic accessibility property* if there is a dense subset $\mathcal{E} \subset M$ of non-periodic points such that any pair of points $x, y \in \mathcal{E}$ may be joined by a finite sequence $x_0 = x, x_1, \dots, x_n = y$, $x_i \in \mathcal{E}$ and a sequence p_i of hyperbolic periodic orbits such that for any $i \in \{0, \dots, n-1\}$ one has:

$$\{x_i, x_{i+1}\} \subset \overline{W_{\text{orb}}^s(p_i)} \quad \text{or} \quad \{x_i, x_{i+1}\} \subset \overline{W_{\text{orb}}^u(p_i)}.$$

Question 0.5 *Is periodic accessibility generic in $\text{Diff}^1(M)$?*

As a weaker problem, one can also ask if, for generic diffeomorphisms, the union of the stable manifolds of the periodic points are dense in M .

Theorem 0.3 has the immediate corollary:

Corollary 0.6 *Furthermore, if $f \in \mathcal{R}$ satisfies the periodic accessibility property then $C(f)$ is trivial.*

The periodic accessibility property is satisfied by Axiom A diffeomorphisms, by C^1 -generic tame diffeomorphisms (i.e. by C^1 -generic diffeomorphisms having finitely many homoclinic classes), and by C^1 -generic conservative (volume preserving or symplectic) diffeomorphisms. In this way, one can recover Togawa’s result that the C^1 -generic Axiom A diffeomorphism has trivial centralizer.

B) Large centralizer for a locally C^1 dense set of diffeomorphisms

Our next main result addresses the third part of Question 0.1: is $\mathcal{T}^r(M)$ open in $\text{Diff}^r(M)$? In the case of the circle, recall that Kopell proved that $\mathcal{T}^r(S^1)$ is open-dense in $\text{Diff}^r(S^1)$ for $r \geq 2$, and Togawa proved that $\mathcal{T}^1(S^1)$ is residual in $\text{Diff}^1(S^1)$. It is natural to ask whether Togawa’s result can be strengthened to show that $\mathcal{T}^1(S^1)$ is open-dense.

Our next main result shows that the answer is “no”: the answers to the third part of Question 0.1 are genuinely different in the C^1 and C^2 topologies, at least for the circle. We are also able to answer the third part of Question 0.1 for the 2-sphere. Specifically, we have:

Theorem 0.7 *$\mathcal{T}^1(S^1)$ and $\mathcal{T}^1(S^2)$ are not open. Moreover:*

(a) *There is a dense subset $\mathcal{D}^1 \subset \text{Diff}^1(S^1)$ such that every $f \in \mathcal{D}^1$ leaves invariant a C^∞ Morse-Smale vector field. In particular, $C(f)$ contains a subgroup isomorphic to \mathbb{R} .*

(b) *Let $\mathcal{O} \subset \text{Diff}^1(S^2)$ denote the (open) subset of Morse-Smale diffeomorphisms g such that the nonwandering set $\Omega(g)$ consists of two fixed points, one source N_g and one sink S_g , such that the derivatives $D_{N_g}g$ and $D_{S_g}g$ have each a complex (non real) eigenvalue.*

There is a dense subset $\mathcal{D}^2 \subset \mathcal{O}$ such that every $f \in \mathcal{D}^2$ is the time 1 map of a Morse-Smale C^∞ -vector field. In particular, $C(f)$ contains a subgroup isomorphic to \mathbb{R} .

Structure of the paper

In order to prove Theorem 0.3, it is enough to show that along the invariant manifolds of the periodic points, f satisfies an unbounded distortion property. This is discussed in Section 1. As a simpler setting, we also deal with contractions of \mathbb{R}^d whose unique periodic point is 0. In Section 2, we will see that C^1 -generic contraction of \mathbb{R}^d has the unbounded distortion property; this

can be generalized to the dynamics inside the invariant manifolds of the periodic points, since by Appendix A, any perturbation of the dynamics inside the stable manifold of a periodic point can be realized as a perturbation of the dynamics on M . Theorem 0.7 will be proved in Section 3.

1 The unbounded distortion property

Kopell's proof in [Ko] that $\mathcal{T}^r(S^1)$ is open-dense in $\text{Diff}^r(S^1)$ for $r \geq 2$ uses the fact that a C^2 diffeomorphism f of $[0, 1]$ without fixed points in $(0, 1)$ has *bounded distortion*, meaning: for any $x, y \in (0, 1)$, the ratio

$$\frac{|f^{n'}(x)|}{|f^{n'}(y)|} \tag{1}$$

is bounded, independent of n and uniformly for x, y lying in a compact set. A bounded distortion estimate lies behind many results about C^2 , hyperbolic diffeomorphisms of the circle and codimension-1 foliations.

Suppose that $r \geq 2$. Since Morse-Smale diffeomorphisms are open and dense in $\text{Diff}^r(S^1)$, the proof that $\mathcal{T}^r(S^1)$ is open-dense in $\text{Diff}^r(S^1)$ essentially reduces to showing that (C^r -open and densely) a C^r diffeomorphism $f : [0, 1] \rightarrow [0, 1]$ without fixed points in $(0, 1)$ has trivial centralizer. The bounded distortion of such an f forces its centralizer to embed simultaneously in two smooth flows containing f , one determined by the germ of f at 0, and the other by the germ at 1; for an open and dense set of $f \in \text{Diff}_+^r[0, 1]$, these flows agree only at the iterates of f . The $r \geq 2$ hypothesis is clearly necessary for bounded distortion.

The central observation and starting point of this paper is that the centralizer of a C^1 diffeomorphism of $[0, 1]$ with *unbounded* distortion is always trivial. We elaborate a bit on this. Notice that if x and y lie on the same f -orbit, then the ratio in (1) is bounded, independent of n . We show that, C^1 -generically among the diffeomorphisms of $[0, 1]$ without fixed points $(0, 1)$, the ratio (1) is uniformly bounded in n *only if* x and y lie on the same orbit; that is, for a residual set of f , and for all $x, y \in (0, 1)$, if $x \notin \mathcal{O}_f(y) = \{f^n(y) \mid n \in \mathbb{Z}\}$, then

$$\limsup_{n \rightarrow \infty} \frac{|f^{n'}(x)|}{|f^{n'}(y)|} = \infty. \tag{2}$$

Assume that this unbounded distortion property holds for f . Fix $x \in (0, 1)$. A simple application of the Chain Rule shows that if $gf = fg$, then the distortion in (1) between x and $y = g(x)$ is bounded; hence x and $g(x)$ must lie on the same f -orbit. From here, it is straightforward to show that $g = f^n$, for some n (see Lemma 1.2 and Corollary 1.3 below). As in [Ko], a small amount of additional work shows that a residual set in $\text{Diff}^1(S^1)$ has trivial

centralizer. The details of this argument we have just described for S^1 are contained in this section and Section 2.2.

The bulk of this paper is devoted to formulating and proving a higher-dimensional version of the argument we have just described. The interval is replaced by an invariant manifold (stable or unstable) of a periodic point. The derivative f' in (2) is replaced by the Jacobian of f along the invariant manifold.

1.1 Unbounded distortion along invariant manifolds

Let $f : M \rightarrow M$ be a C^1 diffeomorphism, and let $p \in M$ be a hyperbolic periodic point of f . For $x \in W^s(p)$ we denote by $\text{Jac}^s(f)(x)$ the Jacobian of the map induced by $T_x f$ between $T_x W^s(p)$ and $T_{f(x)} W^s(f(p))$.

Definition 1.1 *A hyperbolic periodic point $p \in M$ has the stable manifold distortion property if, for every $x, y \in W^s(p) \setminus \{p\}$ not in the same f -orbit,*

$$\limsup_{n \rightarrow \infty} \left| \frac{\text{Jac}^s(f^n)(x)}{\text{Jac}^s(f^n)(y)} \right| = \infty.$$

As mentioned in the previous subsection, unbounded distortion forces trivial centralizers:

Lemma 1.2 *Let p be a hyperbolic periodic point of period $k \in \mathbb{N}$ with the stable manifold distortion property, and let $g \in C(f)$. If $g(p) = p$, then there exists an $m \in \mathbb{Z}$ such that $g = f^{km}$ on $W^s(p)$.*

Proof of Lemma 1.2. We claim that for every $x \in W^s(p)$, g preserves the f^k -orbit of x . From this claim it follows that for every $x \in W^s(p)$, there exists an integer $m(x)$ such that $g(x) = f^{km(x)}(x)$, and there is a unique such $m(x)$ if $x \neq p$. Continuity of f implies that the function m is locally constant on $W^s(p) \setminus \{p\}$. If $\dim(W^s(p)) > 1$, then $W^s(p) \setminus \{p\}$ is connected, and m is constant. If $\dim(W^s(p)) = 1$, then m is constant on each of the connected components of $W^s(p) \setminus \{p\}$; in this case, since g is differentiable at p and $|f^{k^l}(p)| \neq 1$, the values of m on the two components must coincide.

It remains to prove the claim. We may assume that $x \neq p$. The relation $gf^n = f^n g$ implies that,

$$\text{Jac}^s(g)(f^n x) \text{Jac}^s(f^n)(x) = \text{Jac}^s(f^n)(gx) \text{Jac}^s(g)(x).$$

In particular, for all $m \geq 0$, we have

$$\left| \frac{\text{Jac}^s(g)(f^{mk} x)}{\text{Jac}^s(g)(x)} \right| = \left| \frac{\text{Jac}^s(f^{mk})(gx)}{\text{Jac}^s(f^{mk})(x)} \right|.$$

Since $f^{mk}(x)$ lies in a compact region of $W^s(p)$ for all $m \in \mathbb{N}$, the left hand side of this expression is uniformly bounded in m . On the other hand, it is easy to see that if f has the stable manifold distortion property, then so does f^k . This implies that the right hand side of the equation above is unbounded, a contradiction. This proves the claim. \square

Corollary 1.3 *Let p be a hyperbolic periodic point of period $k \in \mathbb{N}$ with the stable manifold distortion property, and let $g \in C(f)$. If $g(\mathcal{O}(p, f)) = \mathcal{O}(p, f)$, then there exists an $m \in \mathbb{Z}$ such that $g = f^m$ on $W^s(\mathcal{O}(p, f))$.*

Proof of Corollary 1.3. Since g preserves the f -orbit of p , we have $g(p) = f^j(p)$ for some integer j . Let $G = f^{-j}g$. Then it is easy to see that G commutes with f , and G fixes every point on the f -orbit of p . By Lemma 1.2, there is an integer m such that the restriction of G to $W^s(p)$ coincides with f^{mk} . Again, since f and G commute, the restriction of G to $W^s(f^i p)$ is conjugate by f^i to the restriction of G to $W^s(p)$. Consequently, G coincides with f^{mk} on the stable manifold $W^s(\mathcal{O}(p, f))$ of the orbit of p , and so g coincides with f^{mk+j} on $W^s(\mathcal{O}(p, f))$. \square

Note that for any integer $k \geq 1$, a C^1 -generic diffeomorphism f has only finitely periodic orbits of period k and, by transversality, that all these orbits have different exponents. In particular, any diffeomorphism $g \in C(f)$ preserves each of these orbits and satisfies the assumption of corollary 1.3.

To prove Theorem 0.3, we are thus reduced to proving:

Proposition 1.4 *There is a residual set $\mathcal{R} \subset \text{Diff}^1(M)$ such that, for any diffeomorphism $f \in \mathcal{R}$, every periodic point $x \in \text{Per}(f)$ is hyperbolic and has the stable manifold distortion property.*

There is a residual set $\mathcal{R}_{\text{symp}} \subset \text{Symp}^1(M)$ such that, for any diffeomorphism $f \in \mathcal{R}_{\text{symp}}$, every hyperbolic periodic point $x \in \text{Per}(f)$ has the stable manifold distortion property.

There is a residual set $\mathcal{R}_\mu \subset \text{Diff}_\mu^1(M)$ such that, for any diffeomorphism $f \in \mathcal{R}_\mu$, and any hyperbolic periodic point $x \in \text{Per}(f)$, if $W^s(x)$ has codimension at least $\dim(M)/2$, then x has the stable manifold distortion property.

1.2 Contractions of \mathbb{R}^d

Let B^d denotes the unit closed ball $\overline{B(0, 1)}$ of \mathbb{R}^d and consider the Banach space of C^1 maps $B^d \rightarrow \mathbb{R}^d$ that send 0 to 0, endowed with the C^1 -topology given by the C^1 -norm:

$$\|f - g\|_1 = \sup_{x \in B^d} \|f(x) - g(x)\| + \|D_x f - D_x g\|.$$

The set of embeddings $B^d \rightarrow \mathbb{R}^d$ fixing 0 defines an open subset that will be denoted by \mathcal{D}^d .

Remark 1.5 *Since the origin is fixed, the metric $\|\cdot\|_1$ is equivalent to the metric defined by*

$$\|f - g\|'_1 = \sup_{x \in B^d} \|D_x f - D_x g\|.$$

In general, we will prefer to work with this second one.

A *contraction* of \mathbb{R}^d is an element of \mathcal{D}^d that sends B^d into $B(0, 1)$, so that 0 is a (hyperbolic) sink that attracts all the points in B^d . The set of contractions of \mathbb{R}^d is an open subset $\mathcal{C}^d \subset \mathcal{D}^d$, hence a Baire space.

Let f be a diffeomorphism of a manifold M , p be a periodic point of f and n^s its stable dimension. A *stable chart* for p is a local chart $\psi: \mathbb{R}^d \rightarrow M$ such that if one denotes by π the projection of \mathbb{R}^d onto the n^s first coordinates we have the following properties.

- The domain $\psi(\mathbb{R}^d)$ contains p .
- In the chart ψ , the local stable manifold of p contains the graph of a C^1 map $g: \mathbb{R}^{n^s} \rightarrow \mathbb{R}^{d-n^s}$.
- Let v be equal to $\pi(\psi^{-1}(p))$ and let θ be the C^1 -map defined on a neighborhood of 0 by projecting on the space \mathbb{R}^{n^s} the dynamics of f in the local stable manifold of p :

$$\theta: x \mapsto \pi \circ \psi^{-1} \circ f \circ \psi(x + v, g(x + v)) - v,$$

then, θ belongs to \mathcal{C}^{n^s} .

Proposition 1.6 *Any hyperbolic periodic point p of a diffeomorphism f has a stable chart ψ . Moreover, for any diffeomorphism g in a C^1 -neighborhood \mathcal{U} of f , the continuation p_g of p also admits the chart ψ as a stable chart.*

The family of contractions θ_g associated to the periodic point and to the chart ψ induces a continuous map $\Theta: \mathcal{U} \rightarrow \mathcal{C}^{n^s}$. This map is open.

In the conservative setting, the same property holds.

Theorem 1.7 *Let $\Theta: \mathcal{U} \rightarrow \mathcal{C}^{n^s}$ be a family of contractions associated to a periodic point p and a stable chart ψ as in proposition 1.6. Then, the map $\Theta: \mathcal{U} \cap \text{Symp}^1(M) \rightarrow \mathcal{C}^{n^s}$ is open.*

If the dimension n^s of the stable space of p is larger or equal to $\dim(M)/2$, then, the map $\Theta: \mathcal{U} \cap \text{Diff}_\mu^1(M) \rightarrow \mathcal{C}^{n^s}$ is open.

This will be proved in Sections A.1 and A.2.

Proposition 1.8 *For any integer $n \geq 0$, there exists*

- a family \mathcal{P}_n of pairwise disjoint open subsets whose union is dense in $\text{Diff}^1(M)$,

- for each $\mathcal{U} \in \mathcal{P}_n$, finitely many charts $\psi_1, \dots, \psi_s: \mathbb{R}^d \rightarrow M$,

such that any diffeomorphism $f \in \mathcal{U}$ has the following properties:

- f has s periodic points of period less than n , all are hyperbolic. Each domain $\psi_i(\mathbb{R}^d)$ contains exactly one of them, it is called $p_{i,f}$ and its stable dimension is denoted by n_i^s .
- The chart ψ_i is a stable chart for $p_{i,f}$.

The major ingredient in the proof of Proposition 1.4 is the following.

Proposition 1.9 *There is a residual set $\mathcal{R}_0 \subset \mathcal{C}^d$ such that, for all $f \in \mathcal{R}_0$, if $x, y \in \overline{B(0,1)} \setminus \{x, y\}$ with $x \notin \mathcal{O}_f(y)$, then*

$$\limsup_{n \rightarrow \infty} \left| \frac{\text{Jac}(f^n)(x)}{\text{Jac}(f^n)(y)} \right| = \infty. \quad (3)$$

The proof that Proposition 1.9 implies Proposition 1.4 is quite immediate in the non-conservative case: the dynamics in any stable manifold is diffeomorphically conjugate to a contraction of $\mathbb{R}^{n_i^s}$; one concludes by noting that any perturbation of the dynamics inside the stable manifold extends to a perturbation of the dynamics on M . In the conservative case, this last property is much more delicate and its proof will be postponed until Appendix A.

1.3 K -distortion and the Baire argument

In this subsection, we explain how to reduce the unbounded distortion property (3) in Proposition 1.9 to a property satisfied in finite time, which we call the K -distortion property. Using a Baire argument, we then reformulate Proposition 1.9 in terms of this K -distortion property to obtain our main perturbation result (Theorem 1.13).

Definition 1.10 *Let $B \subset \mathbb{R}^d$ be a compact region and let $f : B \rightarrow B$ be an embedding. Given compact sets $\Lambda, \Delta \subset B$, we say that Λ and Δ are dynamically disjoint for f if $f^n(\Lambda) \cap f^m(\Delta) = \emptyset$ for any $n, m \in \mathbb{N}$.*

Definition 1.11 *Let $\Lambda, \Delta \subset B^d \setminus \{0\}$ be compact sets that are dynamically disjoint for $f \in \mathcal{C}^d$. We say that Λ and Δ satisfy the K -distortion property for f at time N if, for any $x \in \Lambda$, $y \in \Delta$ there exists $n \in \{0, \dots, N\}$ such that*

$$\left| \frac{\text{Jac } f^n(x)}{\text{Jac } f^n(y)} \right| > K.$$

More briefly, Λ, Δ have the K -distortion property for f if there exists an N so that they satisfy the K -distortion property for f at time N

The properties of dynamical disjointness and K -distortion persist under perturbations of both the diffeomorphism and the compact sets.

Proposition 1.12 *Let $\Lambda, \Delta \subset B^d \setminus \{0\}$ be dynamically disjoint for $f \in \mathcal{C}^d$.*

1. *There exist neighborhoods $\mathcal{U}_f \subset \mathcal{C}^d$ of f , U_Λ of Λ , and U_Δ of Δ such that all compact sets $\Lambda' \subset U_\Lambda$, $\Delta' \subset U_\Delta$ are dynamically disjoint for all $g \in \mathcal{U}_f$;*
2. *suppose that Λ and Δ satisfy the K -distortion property for f at time N . Then we can choose $\mathcal{U}_f, U_\Lambda, U_\Delta$ so that all compact sets $\Lambda' \subset U_\Lambda$, $\Delta' \subset U_\Delta$ satisfy the K -distortion property for $g \in \mathcal{U}_f$ at time N .*

Proof of Proposition 1.12. For the first item, it is enough to show that $g^n(\Lambda) \cap \Delta = \emptyset$ for $n \in \mathbb{N}$. One considers D , a neighborhood of 0 satisfying $f(\overline{D}) \subset D$ disjoint from Δ . Since Λ is compact, for large N we have $f^N(\Lambda) \subset D$. For g close enough to f , one gets $g^n(\Lambda) \subset D$ for each $n \geq N$, which implies the required property.

The second item is an easy continuity argument. □

We next reformulate Proposition 1.9 in terms of K -distortion.

Theorem 1.13 (Main perturbation result) *Let $\Lambda, \Delta \subset B^d \setminus \{0\}$ be compact sets that are dynamically disjoint for $f \in \mathcal{C}^d$. Then for every neighborhood \mathcal{U} of f in \mathcal{C}^d , and for every $K > 0$, there exist $g \in \mathcal{U}$ such that Λ and Δ satisfy the K -distortion property for g .*

To prove that Theorem 1.13 implies Proposition 1.9, we employ a standard Baire argument. Let \mathcal{U}_0 be a countable basis of (relatively) open balls for the topology on B^d , and for $U_1, U_2 \in \mathcal{U}$, let

$$O(U_1, U_2) = \{f \in \mathcal{D}^d \mid \overline{U}_1 \text{ and } \overline{U}_2 \text{ are dynamically disjoint for } f\}.$$

Proposition 1.12 implies that $O(U_1, U_2)$ is open in \mathcal{D}^d , and clearly:

$$\mathcal{D}^d = \bigcup_{U_1, U_2 \in \mathcal{U}} O(U_1, U_2). \quad (4)$$

Note that $O(U_1, U_2)$ is nonempty if and only if \overline{U}_1 and \overline{U}_2 are disjoint, and henceforth any two such sets we discuss will be assumed to be disjoint.

Given $U_1, U_2 \in \mathcal{U}$, open sets V_1, V_2 with $\overline{V}_1 \subset U_1$ and $\overline{V}_2 \subset U_2$, and $K > 0$, we define $O(U_1, U_2, V_1, V_2, K)$ be the set of all $f \in O(U_1, U_2)$ such that:

$$\overline{V}_1 \text{ and } \overline{V}_2 \text{ satisfy the } K\text{-distortion property at time } N,$$

for some $N > 0$.

Theorem 1.13 immediately implies:

Proposition 1.14 $O(U_1, U_2, V_1, V_2, K)$ is open and dense in $O(U_1, U_2)$.

For each $U \in \mathcal{U}$, let \mathcal{V}_U be a countable basis of open sets in U , consisting of sets whose closures are contained in U . Proposition 1.14 implies that

$$\mathcal{R}_{U_1, U_2} = \bigcap O(U_1, U_2, V_1, V_2, K)$$

is residual in $O(U_1, U_2)$, where the intersection is taken over all $V_1 \in \mathcal{V}_{U_1}, V_2 \in \mathcal{V}_{U_2}$ and $K \in \mathbb{N}$. Let

$$\mathcal{R}_0 = \bigcap_{U_1, U_2 \in \mathcal{U}} \left(\mathcal{R}_{U_1, U_2} \cup \left(\mathcal{D}^d \setminus O(U_1, U_2) \right) \right);$$

clearly \mathcal{R}_0 is residual in \mathcal{D}^d .

Suppose that $f \in \mathcal{R}_0$. Let $x, y \in B^d \setminus \{0\}$ such that $y \notin \mathcal{O}(x)$ and let $K > 0$ be given. We show that there exists an $n \in \mathbb{N}$ such that

$$\left| \frac{\text{Jac}(f^n)(x)}{\text{Jac}(f^n)(y)} \right| > K.$$

Since f is a contraction and $y \notin \mathcal{O}(x)$, there exist $U_1, U_2 \in \mathcal{U}_0$ such that $x \in U_1, y \in U_2$, and \bar{U}_1, \bar{U}_2 are dynamically disjoint for f . This means that $f \in O(U_1, U_2)$, and the definition of \mathcal{R}_0 then implies that $f \in \mathcal{R}_{U_1, U_2}$. Let $V_1 \in \mathcal{V}_{U_1}, V_2 \in \mathcal{V}_{U_2}$ be neighborhoods of x and y , respectively. Since $f \in O(U_1, U_2, V_1, V_2, K)$, we obtain that for some $n \in \mathbb{N}$

$$\left| \frac{\text{Jac}(f^n)(x)}{\text{Jac}(f^n)(y)} \right| > K,$$

which completes the proof of Proposition 1.9. □

2 Proof of Theorem 1.13 (Main perturbation result)

Before proving Theorem 1.13 we introduce notations and concepts that will be used in the whole section. We then isolate the proof in some particular cases. We do this for two reasons: first, it will allow us to illustrate some of the main ideas of the general case while avoiding serious technical issues, and second, these special cases will be ingredients in the proof of the general case.

2.1 Preliminaries

For our purposes, a map that is linear near the origin is especially easy to work with, because a linear map has constant Jacobian.

2.1.1 Linearization near the origin. A well-known feature of the C^1 topology is that a diffeomorphism may be C^1 approximated by its derivative in a neighborhood of a fixed point: we will say that $f \in \mathcal{C}^d$ has a *linear germ* if there exists a linear map $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and a neighborhood U of 0 in B^d with $f(U) \subset U$ such that $f|_U = A|_U$.

By Proposition 1.12, if the two compact sets $\Lambda, \Delta \subset B^d \setminus \{0\}$ are dynamically disjoint for some contraction f_0 , one can approximate f_0 by a contraction f that has a linear germ $A = D_0 f$ and such that Λ, Δ remain dynamically disjoint.

2.1.2 Bring Λ, Δ into the linearized region. Once one considers a contraction f having a linear germ, we show that one can reduce the proof of Theorem 1.13 to the case the contraction is a linear map A .

Let U be a forward invariant set where f coincides with a linear map A and choose some numbers $1 > r_2 > r_1 > 0$ satisfying $B(0, r_2) \subset U$. Since f is a contraction, there exists an integer $m > 0$ such that $f^m(\Lambda \cup \Delta) \subset B(0, r_1)$. We would like to work with the sets $f^m(\Lambda)$ and $f^m(\Delta)$ in place of Λ and Δ , and the following simple lemma allows us to do so.

Lemma 2.1 *Let $f \in \mathcal{C}^d$, and let Λ, Δ be dynamically disjoint for f . For every $K > 0$ and integer $m \geq 0$, there exists a neighborhood \mathcal{V} of f and $K' > 0$ such that, for all $g \in \mathcal{V}$, if $g^m(\Lambda)$ and $g^m(\Delta)$ have the K' -distortion property for g , then Λ and Δ have the K -distortion property for g .*

Note that changing the Riemannian metric on \mathbb{R}^d only affects our choice of K and the C^1 -size of the neighborhood \mathcal{U} in \mathcal{C}^d . Hence, one can assume that B^d is mapped into $B(0, 1)$ by A so that A also is a contraction.

Let us assume that Theorem 1.13 has been proven for the linear map. Then any small perturbation g of A in \mathcal{C}^d can be glued to f inside $B(0, r_2)$ thanks to the following standard lemma:

Lemma 2.2 *Given $f \in \mathcal{C}^d$, numbers $1 > r_2 > r_1 > 0$, and $\varepsilon > 0$, there exists $\varepsilon' > 0$ such that, for every embedding $g : B(0, r_2) \rightarrow \mathbb{R}^d$ satisfying*

$$\sup_{x \in B(0, r_2)} \|D_x g - D_x f\|_0 < \varepsilon',$$

and $g(0) = 0$, there exists a diffeomorphism $g' \in \mathcal{C}^d$ such that

1. $\|g' - f\|_1 < \varepsilon$
2. $g' = f$ on $B^d \setminus B(0, r_2)$
3. $g' = g$ on $B(0, r_1)$.

2.1.3 Localize the perturbations. We introduce some terminology that will be used in the rest of the paper.

Definition 2.3 Let $1 > r_2 > r_1 > 0$. The spherical shell in B^d of outer radius r_2 and inner radius r_1 is the set:

$$S(r_1, r_2) = B(0, r_2) \setminus B(0, r_1).$$

The modulus $m(S(r_1, r_2))$ of the shell $S(r_1, r_2)$ is defined by:

$$m(S(r_1, r_2)) = \log \left(\frac{r_2}{r_1} \right).$$

Let $S(r_1, r_2)$ be a spherical shell. We say that $x \in B^d$ is inside $S(r_1, r_2)$ if $\|x\| \leq r_1$ and outside $S(r_1, r_2)$ if $\|x\| \geq r_2$. The set of points inside of a spherical shell S is denoted by $I(S)$, and the set of points outside of S is denoted by $O(S)$.

If $S(r_3, r_4)$ is another spherical shell then we write $S(r_1, r_2) \prec S(r_3, r_4)$ if $r_2 \leq r_3$. We say that $x \in \mathbb{R}^d$ is in between $S(r_1, r_2)$ and $S(r_3, r_4)$ if x is outside $S(r_1, r_2)$ and inside $S(r_3, r_4)$; that is, if $r_2 \leq \|x\| \leq r_3$.

In this terminology, the map g' given by Lemma 2.2 coincides with g inside the shell $S(r_1, r_2)$ and coincides with f outside of $S(r_1, r_2)$.

2.1.4 Tidy perturbations. In order to compose several perturbations, we will often require the following property:

Definition 2.4 Let $B \subset \mathbf{R}^d$, let $f : B \rightarrow B$ be an embedding and let $X \subset B$. We say that an embedding $g : B \rightarrow B$ is tidy relative to f and X if,

1. $g(x) = f(x)$, for all $x \in B \setminus X$, and
2. $g^m(x) = f^m(x)$ for all $x \in B \setminus X$ and $m \geq 1$ such that $g^m(x) \in B \setminus X$.

Notice that if X is forward invariant by f (i.e. $f(X) \subseteq X$), then g is tidy relative to f and X if and only if $g = f$ on $B \setminus X$.

2.2 The conformal case

We prove here Theorem 1.13 in the particular case when $f = A$ is a conformal linear contraction: we have $A = \alpha I$ for some constant $\alpha \in (0, 1)$ and some isometry I . The main reason why the conformal case is simpler is that conjugacy by a linear conformal map preserves the C^1 -norm. Also note that when $d = 1$, this case is the general case. We will prove the following more precise statement.

Proposition 2.5 *Let $A \in \mathcal{C}^d$ be a linear conformal contraction and (Λ, Δ, r) be a triple such that $\Lambda, \Delta \subset B(0, 1)$ are two dynamically disjoint compact sets, disjoint from the ball $B(0, r)$. Fix constants $\varepsilon, K > 0$.*

Then there exists $\beta \in (0, 1)$ such that, for any $0 < s < r$, there exists $g \in \mathcal{C}^d$ with the following properties:

1. $\|Dg - A\|_0 < \varepsilon$;
2. Λ and Δ have the K -distortion property for g ;
3. g is tidy relative to A and the spherical shell $S(\beta s, s)$.

2.2.1 The case Δ is a small ball. Once again, it is instructive to consider a simple case; we assume first that Δ is a round ball D_1 , contained in a fundamental domain for the action of A , so that $A^i(D_1) \cap D_1 = \emptyset$, for all $i \neq 0$. Let $D_0 \subset G$ be another round ball with the same center as D_1 and containing $\overline{D_1}$ in its interior, chosen to be dynamically disjoint from Λ .

Let η be a diffeomorphism of B^d satisfying:

- P1. η is the identity map on $B^d \setminus D_0$;
- P2. the restriction of η to the ball D_1 is an affine conformal contraction whose fixed point is the center of D_0 and D_1 ; in particular, $\eta(D_1)$ is contained in the interior of D_1 , and the Jacobian of η in D_1 is a constant μ strictly less than 1.
- P3. $\|D(A \circ \eta) - A\|_0 < \varepsilon$, and $\|D(A \circ \eta^{-1}) - A\|_0 < \varepsilon$.

It is easy to see that the distortion for one or more iterates of $A \circ \eta$ between $x \in \Lambda$ and $y \in \Delta$ is equal to μ^{-1} . To get the distortion greater than K , we perform a sequence of such perturbations, each supported on a forward image $A^i(D_0)$.

To this end, choose $m > 0$ such that $\mu^{-m} > K$. For $i \geq 0$, the diffeomorphism $\eta_i = A^i \eta A^{-i}$ is supported on $A^i(D_0)$ and has distortion μ inside $A^i(D_0)$. Furthermore, since conjugacy by a linear conformal map preserves the C^1 norm, the C^1 -distance from $A \circ \eta_i$ to A is the same for all $i \geq 0$, and therefore less than ε . Now fix some integer $n \geq 0$ and let

$$g_{n,m}(x) = \begin{cases} A \circ \eta_{n+i}(x) & \text{if } x \in A^{n+i}(D_0), \text{ for some } i \in \{0, \dots, m-1\}, \\ A \circ \eta_{n+m+i}^{-1}(x) & \text{if } x \in A^{n+m+i}(D_0), \text{ for some } i \in \{0, \dots, m-1\}, \\ A(x) & \text{otherwise.} \end{cases}$$

It is straightforward to check that the diffeomorphism $g = g_{n,m}$ satisfies

$$\|Dg - A\|_0 = \sup_{i \in \{0, \dots, m-1\}} \|D(A \circ \eta_{n+i}) - A\|_0 < \varepsilon.$$

Moreover, for all $x \in \Lambda$, $y \in D_1 = \Delta$ we have

$$\left| \frac{\text{Jac } g^{n+m}(x)}{\text{Jac } g^{n+m}(y)} \right| > K.$$

Hence Λ, Δ have the K -distortion property for g .

Let us consider a point x on the outside of the shell $S = S(\alpha^{n+2m}r, \alpha^n r)$ and $i > 0$ such that $h^i(x)$ is inside the shell. If x does not belong to the orbit of D_0 for A , then $g_{n,m}$ coincides with A on the orbit of x . Let us assume now that x belongs to $A^k(D_0)$ for some $k < n$: a straightforward computation shows that $g_{n,m}^{(n-k)+2m}(x) = A^{(n-k)+2m}(x)$ which is the first point of the orbit of x that lies inside the shell S . Now, inside the shell $g_{n,m} = A$ and since the inside of S is forward-invariant under A , it follows that $g_{n,m}^i(x) = A^i(x)$. Hence, $g_{n,m}$ is tidy with respect to the shell S .

If $\beta = \alpha^{2m+1}$ and if n is the smallest integer such that $\alpha^n < s$, the map $g = g_{n,m}$ is tidy relative to A and the shell $S(\beta s, s)$. This ends the proof of Proposition 1.13 in this case.

2.2.2 Cover Δ with small balls. For arbitrary Λ, Δ , the strategy (to which we will return in later arguments) is to create distortion between Λ and Δ in small increments. Each increment will consist of a perturbation supported on a spherical shell and will produce distortion between Λ and a small piece of Δ .

Notice that for every $r \in (0, 1]$, the spherical shell $S(\alpha r, r)$ is a fundamental domain for the action of A ; to simplify notations, in this section it will be denoted by G_r . The following construction is an easy consequence of the fact that Λ and Δ are dynamically disjoint for A .

There is a family $\{(D_0^j, D_1^j, r_j)\}_{j \in \{1, \dots, k\}}$ with the following properties:

- *for every j , r_j is a number in $(0, 1)$, D_0^j is a round disk contained in the fundamental domain G_{r_j} , and D_1^j is a round disk, centered at the same point as D_0^j and contained in the interior of D_0^j ;*
- *Δ is contained in $\bigcup_{j=1}^k D_1^j$;*
- *for every j , Λ is disjoint from the orbit $\bigcup_{n \in \mathbb{Z}} h^n(D_0^j)$.*

2.2.3 Create distortion between Λ and balls in the cover of Δ .

We have seen that for any ball D_0^j in the cover of Δ , there is a perturbation g_j of A producing K -distortion between points in Λ and points in D_1^j . These perturbations can be chosen to be supported in disjoint spherical shells S_j . We will use the fact that g_j is tidy relative to A and S_j in order to ensure that these perturbations can be considered independently.

Consider an integer $\ell > 0$ such that the shell $S(\alpha^\ell, 1)$ (composed of ℓ successive fundamental domains of A) contains $\Lambda \cup \Delta$. Let β_j be the constants associated to the triples (D_0^j, D_1^j, r^j) by the proof of Proposition 2.5 given at Section 2.2.1. Fix now a sequence s_1, \dots, s_k such that $s_1 = \min(s, \alpha^\ell)$, and $s_{j+1} = \alpha^{2\ell+1}\beta_j s_j$, for every $j \in \{1, \dots, k-1\}$. Corresponding to this sequence of numbers is a sequence of shells $S_j = S(\beta_j s_j, s_j)$, nested as

$$S_k \prec S_{k-1} \prec \dots \prec S_1 \prec S(\alpha^\ell, 1).$$

By our choice of s_j , in between any two successive shells S_{j+1} and S_j , there are $2\ell + 1$ successive fundamental domains

$$G_{\alpha^{2\ell}\beta_j s_j} \prec \dots \prec G_{\alpha\beta_j s_j} \prec G_{\beta_j s_j}.$$

For each $j \in \{1, \dots, k\}$ we denote by g_j the diffeomorphism constructed at Section 2.2.1 for (D_0^j, D_1^j) and the shell S_j . Now let g be the map defined as follows:

$$g(x) = \begin{cases} g_j(x) & \text{if } x \in S_j, \text{ for some } j \in \{0, \dots, k\}, \\ A(x) & \text{otherwise.} \end{cases}$$

Then $\|Dg - A\|_0 < \varepsilon$. Moreover, one easily proves by induction on j the fact that if $x \in B^d \setminus B(0, s_1)$ and $A^n(x)$ is outside the shell S_1 , or between S_{j+1} and S_j , or inside the shell S_k , then $g^n(x) = A^n(x)$. In particular, g is tidy with respect to A and the shell $S(\beta_k s, s)$ where $\beta = \alpha^{(2\ell+1) \cdot k} \prod_{j=1}^k \beta_j$.

It remains to prove that Λ and Δ satisfy the K -distortion property for g . Recall that $\Lambda \cup \Delta \subset S(\alpha^\ell, 1)$ and that there are $2\ell + 1$ fundamental domains between any two shells S_j and S_{j-1} . Hence, for every $j \in \{2, \dots, k\}$ there exists $n_j > 0$ such that $A^{n_j}(\Lambda \cup \Delta)$ is between the shells S_j and S_{j-1} . In particular $g^{n_j} = A^{n_j}$ on $\Lambda \cup \Delta$. We also define $n_1 = 0$.

Consider $x \in \Lambda$ and $y \in \Delta$. Then there exists j such that $y \in D_1^j$. By assumption on g_j there exists $N_j \in \{n_j, \dots, n_{j+1}\}$ such that

$$\left| \frac{\text{Jac } A^{N_j}}{\text{Jac } g_j^{N_j}(y)} \right| > K.$$

Claim 1 *For every $m \in \{n_j, \dots, N_j\}$ one has $g^m = A^m$ in a neighborhood of x .*

Proof. The map g_j coincides with A on the complement of the orbit of D_0^j , which is disjoint from Λ . Hence, as long as $A^m(x)$ does not belong to S_{j+1} , one has $g^m = A^m$ in a neighborhood of x . It remains to show that $A^{N_j}(x)$ does not belong to the ball $B(0, s_{j+1})$. As there are $2\ell + 1$ fundamental domains between the shells S_{j+1} and S_j , if $A^{N_j}(x) \in B(0, s_{j+1})$, then the point $A^{N_j}(y)$ cannot belong to the shell S_j . As g_j is tidy with respect to S_j , this contradicts

the fact that $\text{Jac } g_j^{N_j}(y) \neq \text{Jac } A^{N_j}$. This contradiction concludes the proof of the claim. \square

As a direct consequence of this claim, we obtain:

$$\left| \frac{\text{Jac } g^{N_j}(x)}{\text{Jac } g^{N_j}(y)} \right| > K.$$

Hence Λ and Δ satisfy the K -distortion property for g .

2.3 The generic linear case, avoiding a codimension 1 or 2 submanifold

The arguments in the previous section do not generalize immediately to the case where the linear contraction A is not conformal, since conjugation by a nonconformal linear map does not preserve the C^1 norm. Nonetheless, by a small perturbation in \mathcal{C}^d , one can assume that the linear contraction A is *reduced*: it has simple spectrum, implying that \mathbb{R}^d splits into a direct sum of invariant subspaces on which A is conformal. We will denote by F the 1- or 2-dimensional invariant space of A corresponding to the eigenvalues of A of smallest modulus, and we will denote by E the sum of the other eigenspaces.

For such a reduced linear contraction, we can use an inductive argument and the result of Section 2.2. Difficulties arise because dynamical disjointness is not preserved by projection onto invariant subspaces. In this subsection, we treat another special case of Theorem 1.13: the linear contraction A is a reduced and the sets Λ, Δ avoid the codimension 1 or 2 submanifold E determined by the weakly contracting eigenspaces of A .

We write $A = (A_E, A_F)$, as a product of linear contractions of E and F : since A has simple spectrum, A_F is conformal. We denote by U_E and U_F the unit balls of the spaces E and F and by $\pi_F: \mathbb{R}^d \rightarrow F$ the linear projection of the product $E \times F$ on its second factor. Note that we can change the Euclidean norm on \mathbb{R}^d and assume that E and F are orthogonal.

2.3.1 The case $\pi_F(\Lambda), \pi_F(\Delta)$ are dynamically disjoint. In this case, since A_F is conformal, Proposition 2.5 implies that there is a perturbation $g_F \in \mathcal{C}^{\dim(F)}$ of A_F , tidy with respect to A_F and a shell $S_F \subset U_F$, so that $\pi_F(\Lambda)$ and $\pi_F(\Delta)$ have the K -distortion property for g_F . Then, since the linear map A_E has no distortion, the sets Λ and Δ also have the K distortion property for the product map $g = (A_E, g_F)$. The embedding g is tidy with respect to A and $U_E \times S_F$ and satisfies the conclusions of Theorem 1.13.

2.3.2 Decomposition of Λ and Δ . In general, even though Λ and Δ are dynamically disjoint for A , their projections $\pi_F(\Lambda)$ and $\pi_F(\Delta)$ might not be dynamically disjoint for A_F . A naïve way to try to fix the proof is to first

perturb $A|_{U_E \times U_F}$ so that the projections $\pi_F(\Lambda)$ and $\pi_F(\Delta)$ become dynamically disjoint. There are two problems with this approach:

1. the projections $\pi_F(\Lambda)$ and $\pi_F(\Delta)$ might in fact coincide, so that a very large perturbation of A would be required to disjoint them;
2. any perturbations that change the relative position of $\pi_F(\Lambda)$ and $\pi_F(\Delta)$ will destroy the invariance of the splitting $\mathbb{R}^d = E \oplus F$ in U , not to mention the linear conformality of the projection A_F .

To cope with these difficulties, we will cover Λ and Δ with finitely many compact, dynamically-defined pieces $\Lambda^1, \dots, \Lambda^{k_1}$ and $\Delta^1, \dots, \Delta^{k_2}$. We perform a sequence of perturbations; at each step we arrange for one of the pairs (Λ^i, Δ^j) to have the K -distortion property. Recall that in the conformal case, we chose pieces in our cover of Δ to be round balls; in this case, pieces in our covers of Λ and Δ will be of the form $\overline{D} \times \overline{S}$, where D is ball in U_E and S is a spherical shell in U_F .

We introduce more notation. Let α be the norm of the linear conformal contraction A_F of F . For $r \in (0, 1]$, we denote by G_r the shell $S(\alpha r, r) \subset U_F$, which is a fundamental domain for the action of A_F in U_F ; its modulus is $\mu = -\log \alpha$.

For $k \geq 1$, we define a family of spherical shells $S_{i,k} \subset U_F$, indexed by integers $i \geq 0$, by:

$$S_{i,k} = S(\alpha^{(i+1)/k}, \alpha^{i/k}).$$

Notice that $S_{i,k}$ has modulus μ/k , for all i, k , and that $G_1 = \bigcup_{i=0}^{k-1} S_{i,k}$. Notice also that $A_F^j(S_{i,k}) = S_{i+kj,k}$, and so the partition

$$U_F \setminus \{0\} = \bigcup_{i=0}^{\infty} S_{i,k}$$

is forward invariant under A_F .

The proof of Theorem 1.13 in the case we are considering here can be reduced to the following result.

Proposition 2.6 *Let A be a reduced linear contraction, $\mathbb{R}^d = E \oplus F$ its associated decomposition and a constant $\varepsilon > 0$. Then, there exists $k_0 \in \mathbb{N}$ such that the following holds.*

For some $u, w \in U_E$, $\delta > 0$, $k \geq k_0$ and $i, j \in \mathbb{N}$, we consider the sets

$$\Lambda = \overline{B_E(u, \delta) \times S_{i,k}} \quad \text{and} \quad \Delta = \overline{B_E(w, \delta) \times S_{j,k}},$$

and assume that $\hat{\Lambda} = \overline{B_E(u, 2\delta) \times S_{i,k}}$ and $\hat{\Delta} = \overline{B_E(w, 2\delta) \times S_{j,k}}$ are contained in $B^d \setminus \{0\}$ and are dynamically disjoint.

Then, for any $K > 0$, and any neighborhood V_F of 0 in F , there exists a shell $S \subset V_F$ and an embedding $g: U_E \times U_F \rightarrow U_E \times U_F$ such that:

1. $\|Dg - A\|_0 < \varepsilon$;
2. g is tidy with respect to A and $U_E \times S$;
3. Λ, Δ have the K -distortion property for A .

Let us show how this proposition implies Theorem 1.13. First, Proposition 2.6 associates to $\varepsilon > 0$ an integer k_0 . Using that Λ, Δ are dynamically disjoint and avoid the space E , one then obtains easily the following construction:

There exist $\delta > 0$ and two families $\{\Lambda^1, \dots, \Lambda^{k_1}\}$ and $\{\Delta^1, \dots, \Delta^{k_2}\}$ with the following properties:

- for every i, j , there exist $u_i, w_j \in E$ and $m_i, \ell_j \in \mathbb{N}$ such that

$$\Lambda^i = \overline{B_E(u_i, \delta) \times S_{m_i, k_0}} \quad \text{and} \quad \Delta^j = \overline{B_E(w_j, \delta) \times S_{\ell_j, k_0}}.$$

- Λ is contained in $\cup_{i=1}^{k_1} \Lambda^i$ and Δ in $\cup_{j=1}^{k_2} \Delta^j$;
- for every i, j , $\hat{\Lambda}^i = \overline{B_E(u_i, 2\delta) \times S_{m_i, k_0}}$ and $\hat{\Delta}^j = \overline{B_E(w_j, 2\delta) \times S_{\ell_j, k_0}}$ are contained in $B^d \setminus \{0\}$ and are dynamically disjoint.

Fix an integer $\ell > 0$ such that the shell $S(\alpha^\ell, 1)$ contains $\pi_F(\Lambda \cup \Delta)$. We order all the possible pairs (Λ^i, Δ^j) as a list $(P_1, \dots, P_{k_1 k_2})$ and apply Proposition 2.6 inductively for each pair P_m . We obtain a sequence of embeddings g_m which are tidy with respect to A and domains $U_E \times S_m$. Since the shells S_m can be chosen in arbitrarily small neighborhoods of 0, one can assume that they are nested as

$$S_{k_1 k_2} \prec S_{(k_1 k_2) - 1} \prec \dots \prec S_1 \prec S(\alpha^\ell, 1),$$

and that between any two successive shells S_{m+1} and S_m , there are $2\ell + 1$ successive fundamental domains.

One can define the contraction g and end the proof of Theorem 1.13 by gluing the g_m according to the domains $U_E \times S_m$ as in Section 2.2.3.

2.3.3 Move thin shells. Before proving Proposition 2.6, we need to prove the following lemma about perturbations inside the space F .

Lemma 2.7 *Given $\varepsilon_0 > 0$, there exists $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, and for every $i \geq 2k$, there is a diffeomorphism $\psi : U_F \rightarrow U_F$ such that:*

1. $\psi(\overline{S_{i,k}}) \cap (\overline{S_{i+1,k}} \cup \overline{S_{i,k}} \cup \overline{S_{i-1,k}}) = \emptyset$;
2. $\psi = Id$ in the complement of a fundamental domain G_r , which is a shell containing $S_{i+1,k} \cup \overline{S_{i,k}} \cup \overline{S_{i-1,k}}$;

$$3. \|D\psi - Id_F\|_0 < \varepsilon_0.$$

To construct ψ , we will use the following lemma, which will be used again in the next subsection.

Lemma 2.8 *Let $\mu_0 > 0$, $s_0 \geq 2$ and $\varepsilon_0 > 0$ be given. Then there exists $\xi > 0$ such that for any collection of conformal linear maps $f_0, f_2, \dots, f_{s_0} \in \mathcal{D}^d$ satisfying*

$$\sup_{i=1, \dots, s_0} \|f_i - f_0\| < \xi,$$

and any collection of spherical shells $S^1 \prec S^2 \prec \dots \prec S^{s_0} \subset B^d$ of modulus at least μ_0 , there exists an embedding $\psi \in \mathcal{D}^d$ such that:

$$1. \|D\psi - f_0\|_0 < \varepsilon_0,$$

$$2. \psi(x) = \begin{cases} f_0(x) & \text{if } x \text{ is inside } S_1, \\ f_i(x) & \text{if } x \text{ is between } S_i \text{ and } S_{i+1}, \text{ for } i = 1, \dots, s_0 - 1, \\ f_{s_0}(x) & \text{if } x \text{ is outside } S_{s_0}. \end{cases}$$

Proof. We prove it in the case $s_0 = 1$; the general case is obtained similarly. Let $\lambda = e^{\mu_0} > 1$ and consider a smooth function $\rho: [0, +\infty) \rightarrow [0, 1]$ which is 0 on $[0, 1]$ and 1 on $[\lambda, +\infty)$ and whose derivative is bounded by $2/(\lambda - 1)$.

Let $f_0, f_1 \in \mathcal{D}^d$ be two conformal linear maps. For $r \in (0, \lambda^{-1})$, consider $\psi \in \mathcal{D}^d$ defined by $x \mapsto (1 - \rho(\|x\|/r))f_0(x) + \rho(\|x\|/r)f_1(x)$; that is,

$$\psi(x) = f_0(x) + \rho\left(\frac{\|x\|}{r}\right)(f_1(x) - f_0(x)).$$

This map coincides with f_0 on the ball $B(0, r)$ and with f_1 on the complement of the ball $B(0, \lambda r)$. Since the modulus of S_1 is equal to $\mu_0 = \log(\lambda)$, one can choose r such that $S_1 = S(r, \lambda r)$. Moreover notice that

$$\|D\psi - f_0\|_0 \leq \left(1 + \frac{2\lambda}{\lambda - 1}\right) \|f_1 - f_0\|.$$

If one sets

$$\xi = \varepsilon_0 \left(1 + \frac{2\lambda}{\lambda - 1}\right)^{-1}$$

one thus gets $\|D\psi - f_0\|_0 < \varepsilon_0$ and this completes the proof. \square

We now give the proof of the first lemma.

Proof of Lemma 2.7. Observe that for any k , the conformal dilation $d_k: F \rightarrow F$ defined by $d_k(v) = \alpha^{-3/k}v$ moves the spherical shell $S_{i,k}$ to a shell

that is disjoint from the union $\overline{S}_{i+1,k} \cup \overline{S}_{i,k} \cup \overline{S}_{i-1,k}$. For a given i, k , the map ψ we construct will coincide with d_k on the set $\overline{S}_{i,k}$ and with the identity outside of a fundamental domain G_r containing $S_{i,k}$ in its interior. We will choose k_0 so that for $k \geq k_0$, the distance from d_k to the identity is small. To insure that the distance from ψ to the identity does not depend on i , we choose the fundamental domain G_r according to the following simple lemma.

Claim 2 *For $k \geq 5$ and $i \geq 2k$, there is a fundamental domain $G_r \subset B(0, \alpha) \subset U_F$ such that $\overline{S}_{i,k} \subset \text{int}(G_r)$, and the complement of $S_{i,k}$ in G_r is the union of 2 disjoint shells:*

$$G_r \setminus S_{i,k} = S^a \cup S^b,$$

where $S^a \prec S^b$, and setting $\mu = -\log \alpha$ we have

$$\min\{m(S^a), m(S^b)\} \geq \mu/3.$$

Proof. Fix $k \geq 5$ and let i_0 be an integer satisfying $4k/3 \leq i_0 \leq 5k/3 - 1$. The fundamental domain $G_\alpha \subset U_F$ contains $S_{i_0,k}$, and it is easily checked that the two shells $S^{a,k} \prec S^{b,k}$ defined by

$$G_\alpha \setminus S_{i_0,k} = S^{a,k} \cup S^{b,k}$$

have modulus at least $\mu/3$. For any $i \geq 2k$ we have $i \geq i_0$ and the shell $S_{i,k}$ is the image of $S_{i_0,k}$ under a linear conformal contraction; the images of $G_\alpha, S^{a,k}$ and $S^{b,k}$ under this conformal map satisfy the conclusions of the lemma for $S_{i,k}$. \square

Let $\xi > 0$ be the constant specified by Lemma 2.8, for $\mu_0 = \mu/3, s_0 = 2$ and ε_0 . Choose $k_0 \geq 5$ so that

$$\|d_{k_0} - Id\| = |\alpha^{-3/k_0} - 1| < \xi.$$

Let $i \geq 2k$, and let G_r be the fundamental domain given by the claim above. The complement of $S_{i,k}$ in G_r is the union of two spherical shells $S^a \prec S^b$ of modulus at least $\mu/3$. Applying Lemma 2.8 to the shells S^a and S^b and the maps $f_0 = f_2 = Id$ and $f_1 = d_{k_0}$, we obtain a map ψ that satisfies the conclusions of Lemma 2.7. \square

2.3.4 Separate Λ and Δ . We introduce a perturbation g_1 of A that will make some forward iterates of Λ and Δ have dynamically disjoint π_F -projections; this perturbation will take a special form, which will allow us to make the final perturbation of Proposition 2.6 tidy.

Lemma 2.9 *Let A be a reduced linear contraction, $\varepsilon > 0$ a constant and $\Lambda, \Delta \subset B^d \setminus \{0\}$ two sets satisfying the assumptions of Proposition 2.6.*

Then, for any neighborhood V_F of 0 in F , there exists a diffeomorphism $g_1 = A \circ \varphi$ of \mathbb{R}^d , a shell $S_1 \subset V_F$ of F which is a fundamental domain of A_F , and an integer $\ell_1 \geq 1$ such that:

1. *the map φ takes the form $\varphi(u, v) = (u, \theta(u, v))$, for $(u, v) \in E \times F$, and is supported in $U_E \times S_1$;*
2. *$\max(\|D\varphi - Id\|_0, \|D\varphi^{-1} - Id\|_0) < \frac{\varepsilon}{\|A\|}$, and thus $\|Dg_1 - A\|_0 < \varepsilon$;*
3. *$g_1^{\ell_1}(\hat{\Lambda} \cup \hat{\Delta})$ is contained in $U_E \times I(S_1)$, and $\pi_F(g_1^{\ell_1}(\hat{\Lambda}))$ and $\pi_F(g_1^{\ell_1}(\hat{\Delta}))$ are dynamically disjoint for the restriction of A_F to $I(S_1)$.*

Proof. Let $\varepsilon_0 = \frac{\varepsilon}{4\|A\|}$; Lemma 2.7 associates to this constant an integer k_0 . For some integer $k \geq k_0$, we consider two compact sets

$$\Lambda = \overline{B_E(u, \delta) \times S_{i,k}} \quad \text{and} \quad \Delta = \overline{B_E(w, \delta) \times S_{j,k}}.$$

The same formula with balls in E of radius 2δ defines the compact sets $\hat{\Lambda}, \hat{\Delta}$. Recall that these two sets are assumed to be contained in $B^d \setminus \{0\}$ and to be dynamically disjoint.

Let $\nu = \|A_E^{-1}\| \|A_F\|$; because F is the maximally contracted eigenspace of A , we have $\nu < 1$. Fix a large integer ℓ such that

$$i + k\ell \geq 2k \quad \text{and} \quad \nu^\ell < \frac{\delta\varepsilon_0}{4}.$$

If $\pi_F(A^\ell(\Lambda))$ and $\pi_F(A^\ell(\Delta))$ are dynamically disjoint for A_F , then there is nothing to prove (we choose S_1 to be any spherical shell, set $g_1 = A$ and choose $\ell_1 \geq \ell$ such that $g_1^{\ell_1}(\hat{\Lambda} \cup \hat{\Delta})$ is contained in $U_E \times I(S_1)$).

If, on the other hand, $\pi_F(A^\ell(\Lambda)) = \overline{S_{i+k\ell,k}}$ and $\pi_F(A^\ell(\Delta)) = \overline{S_{j+k\ell,k}}$ are not dynamically disjoint for A_F , then there exists $q_0 \in \mathbb{Z}$ such that

$$A_F^\ell(\overline{S_{i,k}}) \cap A_F^{q_0}(\overline{S_{j,k}}) \neq \emptyset.$$

This implies that for some $\kappa \in \{-1, 0, 1\}$ we have

$$A_F^{q_0}(\overline{S_{j,k}}) = \overline{S_{i+k\ell+\kappa,k}}.$$

Since $i + k\ell \geq 2k$, Lemma 2.7 for ε_0 and the shell $S_{i+k\ell,k}$ provides us with a fundamental domain $S_1 = G_r$ and a diffeomorphism $\psi \in \mathcal{D}^{\dim(F)}$ such that

1. $S_1 \cap \bigcup_{q \in \mathbb{N}} A_F^q(\overline{S_{j,k}}) = A_F^{q_0}(\overline{S_{j,k}}) = \overline{S_{i+k\ell+\kappa,k}}$,
2. $\psi(\overline{S_{i+k\ell,k}}) \cap (\overline{S_{i+k\ell+1,k}} \cup \overline{S_{i+k\ell,k}} \cup \overline{S_{i+k\ell-1,k}}) = \emptyset$,

3. $\psi = Id_F$ in the complement of S_1 , and

4. $\|D\psi - Id_F\|_0 < \varepsilon_0$.

Fix a smooth bump function $\rho_0 : U_E \rightarrow [0, 1]$ which is 1 on $B(u, \delta)$ and 0 outside $B(u, 2\delta)$, and whose derivative has a norm bounded by $2\delta^{-1}$. One then defines $\rho : A_E^\ell(U_E) \rightarrow [0, 1]$ and a map $\theta : A^\ell(\hat{\Lambda}) \rightarrow S_1$ by:

$$\begin{aligned}\rho(u) &= \rho_0(A_E^{-\ell}(u)), \\ \theta(u, v) &= \rho(u)\psi(v) + (1 - \rho(u))v.\end{aligned}$$

Finally, for $(u, v) \in A^\ell(\hat{\Lambda})$, let

$$\varphi(u, v) = (u, \theta(u, v));$$

we extend this definition by the identity to obtain a diffeomorphism $\varphi \in \mathcal{D}^d$. It is supported in $A^\ell(\hat{\Lambda}) \subset U_E \times S_1$ and coincides with $\text{Id} \times \psi$ on $A^\ell(\hat{\Lambda})$.

Note that the support of ψ is contained in the ball $A_F^\ell(U_F)$ with radius $\|A_F\|^\ell$. Hence, we have

$$\begin{aligned}\|D\varphi - Id\|_0 &\leq \|D\rho\|_0 \|\psi - Id_F\|_0 + \|\rho\|_0 \|D\psi - Id_F\|_0 \\ &< 2\delta^{-1} \|A_E^{-1}\|^\ell \cdot 2\|A_F\|^\ell + \varepsilon_0 \\ &< 4\delta^{-1} \nu^\ell + \varepsilon_0 \\ &< 2\varepsilon_0 = \frac{\varepsilon}{2\|A\|}.\end{aligned}$$

An elementary calculation shows that, for any diffeomorphism h of B^d , if $\|Dh - Id\|_0 \leq \frac{1}{2}$, then $\|Dh^{-1} - Id\|_0 \leq 2\|Dh - Id\|_0$. Hence, we get

$$\|D\varphi^{-1} - Id\|_0 \leq 2\|D\varphi - Id\|_0 \leq \frac{\varepsilon}{\|A\|}. \quad (5)$$

Let $g_1 = A \circ \varphi$ and choose $\ell_1 > \ell$ so that $g_1^{\ell_1}(\hat{\Lambda} \cup \hat{\Delta})$ is contained in $U_E \times I(S_1)$. Since φ is supported in $A^\ell(\hat{\Lambda})$, and since $\hat{\Lambda}$ and $\hat{\Delta}$ are dynamically disjoint for A , we have for each $k \geq 0$, $g_1^k(\Delta) = A^k(\Delta)$. Since

$$S_1 \cap \bigcup_{q \in \mathbb{Z}} \pi_F(A^q(\hat{\Delta})) = \pi_F(A^{q_0}(\hat{\Delta})) \subset \bar{S}_{i+k\ell+1, k} \cup \bar{S}_{i+k\ell, k} \cup \bar{S}_{i+k\ell-1, k},$$

and

$$\pi_F(A^\ell(\hat{\Lambda})) = \bar{S}_{i+k\ell, k},$$

it follows from the properties of ψ that $\psi(\pi_F(A^\ell(\hat{\Lambda})))$ and $\pi_F(A^{q_0}(\hat{\Delta}))$ are dynamically disjoint for A_F . Since $g_1 = A$ on $U_E \times S_1$ and $g_1 = A \circ \varphi$ on $A^\ell(\hat{\Lambda})$, this immediately implies that $\pi_F(g_1^{\ell_1}(\hat{\Lambda}))$ and $\pi_F(g_1^{\ell_1}(\hat{\Delta}))$ are dynamically disjoint for the restriction of A_F to $I(S_1)$. This completes the proof of Lemma 2.9. \square

2.3.5 Proof of Proposition 2.6. The desired perturbation g of A will be obtained in three steps g_1, g_2, g_3 . We first choose an integer $k_0 \in \mathbb{N}$ according to Lemma 2.7. We apply Lemma 2.9 and obtain the first perturbation g_1 supported on a set $U_E \times S_1$ where S_1 is a shell in F contained in an arbitrarily small neighborhood of 0. This also provides us with two dynamically disjoint sets $\pi_F(g_1^{\ell_1}(\Lambda))$ and $\pi_F(g_1^{\ell_1}(\Delta))$ for the dynamics of A_F .

We then apply the argument of Section 2.3.1: fixing a constant $K' > 0$, we obtain a perturbation $h = (A_E, g_F)$ of A and a shell $S_2 \prec S_1$ of F such that $g_1^{\ell_1}(\Lambda)$ and $g_1^{\ell_1}(\Delta)$ have the K' -distortion property for h . Moreover, h is tidy with respect to A and $U_E \times S_2$. We define g_2 as the map which coincides with g_1 on $B^1 \setminus (U_E \times S_2)$ and with h on $U_E \times S_2$. If K' has been chosen large enough, then (Λ, Δ) will have the K -distortion property at some time $N > \ell_1$ for g_2 .

In our final perturbation, we go from the untidy map g_2 to a map g_3 that is tidy with respect to a larger spherical shell containing S_1 and S_2 , while at the same time keeping the desired distortion properties of g_2 . We choose $\ell_2 > N$ so that $A_F^{\ell_2}(\overline{S_1}) \subset \text{int } I(S_2)$. Recall that there exists a diffeomorphism φ supported on $U_E \times S_1$ such that $g_1 = A \circ \varphi$. We define $g_3 = g_2 \circ \overline{\varphi}$ where,

$$\overline{\varphi} = A^{\ell_2} \circ \varphi^{-1} \circ A^{-\ell_2}.$$

If one sets $S_3 = A_F^{\ell_2}(S_1)$, one sees that the diffeomorphisms g_3 and g_2 coincide outside the set $U_E \times S_3$, which is contained in a small neighborhood of E , if ℓ_2 has been chosen large enough. In particular, the sets (Λ, Δ) have the K -distortion property for g_3 .

The maps g_3 and A coincide outside three domains. On $U_E \times S_1$ and $U_E \times S_2$, we have $\|Dg_3 - A\|_0 < \varepsilon$. On $U_E \times S_3$, the claim below gives

$$\|Dg_3 - A\|_0 < \|A\| \cdot \|D\overline{\varphi} - \text{Id}\|_0 < \|A\| \cdot \|D\varphi^{-1} - \text{Id}\|_0,$$

which is less than ε by Lemma 2.9.

Claim 3 *We have $\|D\overline{\varphi} - \text{Id}\|_0 < \|D\varphi^{-1} - \text{Id}\|_0$.*

Proof. Since φ^{-1} has the form $\varphi^{-1}(u, v) = (u, \overline{\theta}(u, v))$, we have,

$$D\varphi^{-1} = \begin{pmatrix} \text{Id}_E & 0 \\ \overline{\theta}_u & \overline{\theta}_v \end{pmatrix}.$$

This gives

$$D\overline{\varphi}^{-1} = \begin{pmatrix} \text{Id}_E & 0 \\ A_F^{\ell_2} \overline{\theta}_u A_E^{-\ell_2} & A_F^{\ell_2} \overline{\theta}_v A_F^{-\ell_2} \end{pmatrix}. \quad (6)$$

Since $\|A_F\| \|A_E^{-1}\| < 1$, we have $\|A_F^{\ell_2} \overline{\theta}_u A_E^{-\ell_2}\| \leq \|\overline{\theta}_u\|$. Using that A_F is conformal and since E, F were assumed to be orthogonal, one gets the announced inequality. \square

Let S be the smallest shell in F that contains S_1 , S_2 and S_3 . Note that all these shells can be constructed in any small neighborhood V_F of 0 in F . It remains to prove that $g = g_3$ is tidy with respect to A and $U_E \times S$. Consider a point $x \in U_E \times U_F$ and an integer m such that $\pi_F(x)$ is outside S and $\pi_F(g_3^m(x))$ is inside S . Since S_1 and S_3 are fundamental domains of A_F , there exist unique integers i_1 and i_3 such that $g_3^{i_1}(x) \in U_E \times S_1$ and $g_3^{i_3}(x) \in U_E \times S_3$; moreover $i_3 = i_1 + \ell_2$. Thus,

$$g_3^m(x) = A^{m-i_1-\ell_2} \bar{\varphi} h^{\ell_2} \varphi A^{i_1}(x).$$

Since h is tidy with respect to A and $U_E \times S_2$, and by definition of $\bar{\varphi}$, we have

$$g_3^m(x) = A^{m-i_1-\ell_2} A^{\ell_2} \varphi^{-1} A^{-\ell_2} A^{\ell_2} \varphi A^{i_1}(x) = A^m(x).$$

This ends the proof of Proposition 2.6.

2.4 The general case

We now prove Theorem 1.13; as we saw in Sections 2.1 and 2.3, we may assume without loss of generality that f is a reduced linear contraction A . The proof will be by induction on the dimension d . Note that the case $d = 1$ is a direct consequence of Proposition 2.5,

Let us suppose that Theorem 1.13 has been proved in any dimension $d' < d$ and let A be a reduced linear contraction of \mathbb{R}^d . As in Section 2.3, one introduces the associated decomposition $\mathbb{R}^d = E \oplus F$. Let $\Lambda, \Delta \subset B^d \setminus \{0\}$ be two compact sets that are dynamically disjoint for A , and fix a constant $K > 0$. The desired perturbation will be obtained in three steps g_1, g_2, g_3 : using the induction hypothesis, we will first obtain distortion between $\Lambda \cap E$ and $\Delta \cap E$; next, we will create the distortion property between $\Lambda \cap E$ and $\Delta \setminus W$ and between $\Lambda \setminus W$ and $\Delta \cap E$, where W is a small neighborhood of E in \mathbb{R}^d ; in the last step, we will use the results of Section 2.3 to complete the proof of the theorem, obtaining the distortion between $\Delta \setminus W$ and $\Lambda \setminus W$.

2.4.1 Distortion property on the weak-stable space: use of the induction hypothesis. We first consider the induced dynamics A_E of A on E : the compact sets $\Delta_E = \Delta \cap U_E$ and $\Lambda_E = \Lambda \cap U_E$ are dynamically disjoint for f_E .

By the induction hypothesis, there exists a contraction h_E of E , arbitrarily close to A_E such that Δ_E and Λ_E have the K -distortion property for h_E . Note that one can again perturb h_E in a small neighborhood of 0 and assume furthermore that h_E coincides with A_E near 0.

We define $g_1 = (h_E, A_F)$; by continuity, there exist compact neighborhoods $\hat{\Delta}_E$ and $\hat{\Lambda}_E$ of Δ_E and Λ_E , respectively, that satisfy the K -distortion property for any embedding close to g_1 .

2.4.2 Distortion between points in the weak stable space and points not in the weak stable space. Let $\Lambda' = \Lambda \setminus \text{int}(\hat{\Lambda}_E)$ and $\Delta' = \Delta \setminus \text{int}(\hat{\Delta}_E)$. We will obtain the K -distortion property between $\Lambda' \cup \Delta'$ and $\Delta_E \cup \Lambda_E$. Since these sets belongs to disjoint forward invariant regions, the construction will be much easier than in Section 2.3.

Fix a neighborhood V_1 of 0 that is forward invariant and where g_1 coincides with A . We choose an integer ℓ_1 such that $g_1^{\ell_1}(\Lambda' \cup \Delta')$ is contained in V_1 . Working in F , we note that the set $\Gamma = \pi_F(g_1^{\ell_1}(\Lambda' \cup \Delta'))$ is disjoint from a ball $B(0, r_0)$. We consider any triple of positive numbers (r_1, r_2, r_3) such that the shells $S_{r_i} = S(\frac{1}{2}r_i, r_i)$ are contained in $B(0, r_0)$ and satisfy $S_{r_3} \prec S_{r_2} \prec S_{r_1}$. Their modulus is equal to $\mu_0 = \log 2$. By Lemma 2.8, we associate to μ_0 , $s_0 = 4$ and a small ε_0 , a constant $\xi > 0$.

Choose a number $\beta > 1$ and conformal dilations f_1, f_2 of F such that

$$\|f_1 - A_F\| < \xi, \quad \|f_2 - A_F\| < \xi, \quad \left| \frac{\det f_1}{\det A_F} \right| > \beta, \quad \text{and} \quad \left| \frac{\det A_F}{\det f_2} \right| > \beta.$$

Let ψ be the map associated by Lemma 2.7 to the S_{r_i} , and the maps $f_0 = f_3 = A_F, f_1, f_2$. Then ψ is a contraction which is close to A_F if the constant ε_0 has been chosen small enough, that coincides with A_F outside $B(0, r_1)$ and in a neighborhood of 0, with f_1 between the shells S_{r_1} and S_{r_2} , and with f_2 between the shells S_{r_2} and S_{r_3} . The constants r_1, r_2, r_3 are now chosen according to the following property.

Claim 4 *For any constant $K_0 > 0$, and given any r_1 , there exists $C_1 > 0$ such that, for any r_2 in $(0, r_1/C_1)$, and for any r_3 , the map ψ satisfies*

$$\forall z \in \Gamma, \exists n > 0, \quad \left| \frac{\text{Jac } \psi^n(z)}{\text{Jac } \psi^n(0)} \right| > K_0.$$

Furthermore, given any r_1, r_2 , there exists $C_2 > 0$ such that, for any r_3 in $(0, r_2/C_2)$, the map ψ satisfies

$$\forall z \in \Gamma, \exists m > 0, \quad \left| \frac{\text{Jac } \psi^m(0)}{\text{Jac } \psi^m(z)} \right| > K_0.$$

Proof. When $\frac{r_1}{r_2}$ goes to infinity, the orbits of the points $z \in \Gamma$ spend an interval of times $\{n_1(z), \dots, n_2(z)\}$ between the shells S_{r_1} and S_{r_2} whose length $n_1(z) - n_2(z)$ goes to infinity; furthermore, $n_1(z)$ does not depend of r_2, r_3 and is uniformly bounded on Γ by some integer n_1 . At each iteration between S_{r_1} and S_{r_2} the distortion $\left| \frac{\text{Jac } \psi^n(z)}{\text{Jac } \psi^n(0)} \right| = \left| \frac{\text{Jac } \psi^n(z)}{\text{Jac } A_F^n} \right|$ increases by the factor β . Hence it is enough to choose r_2 such that $n = \inf_{z \in \Lambda} n_2(z) - n_1(z)$ satisfies

$$\beta^n \left| \frac{\text{Jac } \psi^{n_1}(z)}{\text{Jac } A_F^{n_1}} \right| > K_0.$$

The proof of the second part of the claim is analogous. \square

We now define $g_2 = (h_E, \psi)$. Then, for any $x \in g_2^{\ell_1}(\Lambda' \cup \Delta')$ and any $y \in g_2^{\ell_1}(\Delta_E \cup \Lambda_E)$, there exists an integer $n > 0$ such that

$$\frac{\text{Jac } g_2^n(x)}{\text{Jac } g_2^n(y)} = \frac{\text{Jac } \psi^n(\pi_F(x))}{\text{Jac } A_F^n(0)} > K_0,$$

and similarly there exists an integer $m > 0$ such that $\frac{\text{Jac } g_2^m(y)}{\text{Jac } g_2^m(x)} > K_0$. If K_0 has been chosen large enough (with respect to ℓ_1), one deduces that $\Lambda' \cup \Delta'$ and $\Delta_E \cup \Lambda_E$ have the K -distortion property for g_2 .

2.4.3 Distortion between points of Λ, Δ in the complement of the weak stable space. From the two previous steps, we have shown that the pairs (Λ_E, Δ) and (Λ, Δ_E) satisfy the K -distortion property for any embedding close to g_2 . Hence there are compact neighborhoods O_Λ and O_Δ of Λ_E and Δ_E respectively, such that the pairs (O_Λ, Δ) and (Λ, O_Δ) satisfy the K -distortion property for g_2 . Let $\Lambda'' = \Lambda \setminus \text{int}(O_\Lambda)$, and let $\Delta'' = \Delta \setminus \text{int}(O_\Delta)$. All that remains is to create distortion between Λ'' and Δ'' for an embedding g_3 close to g_2 .

By construction, there exists a neighborhood V_2 of 0 that is forward invariant and where g_2 coincides with A . We choose some integer ℓ_2 such that $g_2^{\ell_2}(\Lambda'' \cup \Delta'')$ is contained in V_2 . It remains to apply Theorem 1.13 as it was proved at Section 2.3, to the map A , the sets $g_2^{\ell_2}(\Lambda'')$, $g_2^{\ell_2}(\Delta'')$ and a constant K_0 large. We obtain an embedding $g = g_3$ such that Λ, Δ have the K -distortion property. The set where g and A differ has been constructed in an arbitrarily small neighborhood of the origin.

3 Large centralizer for a locally C^1 -dense set of diffeomorphisms

3.1 The case of the circle

Our aim in this subsection is to prove the first part of Theorem 0.7. The following lemma summarizes some very classical properties of diffeomorphisms of the circle.

Lemma 3.1 *Let $\mathcal{D}_0^1 \subset \text{Diff}^1(S^1)$ be the set of $f \in \mathcal{D}_0^1$ satisfying the following properties:*

- *f is a C^∞ Morse-Smale diffeomorphism (i.e. the non wandering set consists of finitely many hyperbolic periodic points, alternately attracting or repelling); and*

- for every periodic point $x \in \text{Per}(f)$, there is a neighborhood U_x of x such that the restriction $f|_{U_x}: U_x \rightarrow f(U_x)$ is an affine map (for the natural affine structure on $S^1 = \mathbb{R}/\mathbb{Z}$).

Then \mathcal{D}_0^1 is dense in $\text{Diff}^1(S^1)$.

For $\alpha > 1$ and $\beta \in (0, 1)$, one introduces the set $D_{\alpha, \beta}$ of orientation preserving C^∞ diffeomorphism of the interval $[0, 1]$ with the following properties:

- $\{0, 1\}$ is the set of fixed points of f , and $f(x) > x$ for $x \in (0, 1)$;
- $f(x) = \alpha x$ for small x and $f(x) = 1 + \beta(x - 1)$ for x close to 1.

Now item 1 of Theorem 0.7 is a consequence of the following proposition (see Section 3.1.3).

Proposition 3.2 *Let f be a diffeomorphism in $D_{\alpha, \beta}$. Then any C^1 neighborhood \mathcal{U} of f in $\text{Diff}^1([0, 1])$ contains a diffeomorphism g such that $g = f$ in a neighborhood of $\{0, 1\}$ and g is the time one map of a C^∞ -vector field on $[0, 1]$.*

3.1.1 Mather invariant

We recall here a construction introduced by J. Mather [Ma] which associates to any diffeomorphism $f \in D_{\alpha, \beta}$ a class of diffeomorphism of S^1 .

Let us fix $\alpha > 1$ and $\beta \in (0, 1)$ and introduce a C^∞ orientation preserving diffeomorphism $\varphi: (0, 1) \rightarrow \mathbf{R}$ such that $\varphi(x) = \frac{\ln x}{\ln \alpha}$ for x small and $\varphi(x) = \frac{\ln(1-x)}{\ln \beta}$ for x close to 1: there exists $K_0 > 0$ such that $\varphi^{-1}(x) = e^{\ln \alpha \cdot x}$ for $x < -K_0$ and $\varphi^{-1}(x) = 1 - e^{\ln \beta \cdot x}$ for $x > K_0$.

For any $f \in D_{\alpha, \beta}$ the conjugated diffeomorphism $\theta_f = \varphi \circ f \circ \varphi^{-1}$ of \mathbf{R} satisfies $\theta_f(x) > x$ for all x ; furthermore $\theta_f(x)$ coincides with $x + 1$ when $|x|$ is larger than a constant $K_f > K_0$.

The space \mathbf{R}/θ_f of the orbits of θ_f is a smooth circle S_f which has two natural identifications with the (affine) circle $S^1 = \mathbb{R}/\mathbb{Z}$: two points $x, y \in (-\infty, -K_f]$ (resp. $x, y \in [K_f, +\infty)$) are in the same orbit for θ_f if and only if they differ by an integer. This leads to two diffeomorphisms $\pi_+: S_f \rightarrow S^1$ and $\pi_-: S_f \rightarrow S^1$, respectively. Let $\Delta_{f, \varphi} = \pi_+ \circ \pi_-^{-1}: S^1 \rightarrow S^1$.

Lemma 3.3 *The diffeomorphism f is the time-one map of a C^1 vector field if and only if $\Delta_{f, \varphi}$ is a rotation.*

Proof. Note that $f \in D_{\alpha, \beta}$ coincides with the time one map of the vector field $X^- = \ln \alpha \cdot x \frac{\partial}{\partial x}$ in a neighborhood of 0 and with $X^+ = \ln \beta \cdot (x - 1) \frac{\partial}{\partial x}$ in a neighborhood of 1. Furthermore, if f is the time one map of a C^1 vector field X on $[0, 1]$ then $X = X^-$ in a neighborhood of 0 and $X = X^+$ in a neighborhood

of 1. The hypothesis on φ implies $\varphi_*(X_-) = \frac{\partial}{\partial x}$ on some interval $(-\infty, L_-)$ and $\varphi_*(X_+) = \frac{\partial}{\partial x}$ on an interval (L_+, ∞) .

Assume that $\Delta_{f,\varphi}$ is a rotation. Then we define a vector field Y on \mathbb{R} as follows: consider $n > 0$ such that $\theta_f^n(x) > K_f$. Now let $Y(x) = (D_x \theta_f^n)^{-1} (\frac{\partial}{\partial x})$. This vector does not depend of n (because θ_f is the translation $t \mapsto t + 1$ for $t \geq K_f$).

Claim 1 *if $x < -K_f$ then $Y(x) = \frac{\partial}{\partial x}$*

Proof. Consider the natural projection $\pi_f: \mathbb{R} \rightarrow S_f$ that maps each point to its orbit for θ_f . Since Y is invariant by θ_f , the vector field $(\pi_f)_*(Y)$ is well-defined. Since on $(K_f, +\infty)$ the vector $Y(x)$ is equal to $\frac{\partial}{\partial x}$, the map $\pi_+ \circ \pi_f$ coincides with the natural projection $\mathbb{R} \rightarrow S^1$ and we have $(\pi_+ \circ \pi_f)_*(Y(x)) = \frac{\partial}{\partial x}$. As $\Delta_{f,\varphi}$ is a rotation, and as the rotations preserve the vector field $\frac{\partial}{\partial x}$, we obtain that $(\pi_- \circ \pi_f)_*(Y(x)) = (\Delta_{f,\varphi}^{-1} \circ \pi_+ \circ \pi_f)_*(Y(x)) = \frac{\partial}{\partial x}$. As θ_f coincides with the translation $t \mapsto t+1$ on $(-\infty, -K_f]$, the projection $\pi_- \circ \pi_f$ coincides on $(-\infty, -K_f]$ with the natural projection $\mathbb{R} \mapsto \mathbb{R}/\mathbb{Z}$. Hence $(\pi_- \circ \pi_f)_*(Y(x)) = \frac{\partial}{\partial x}$ implies $Y(x) = \frac{\partial}{\partial x}$. \square

Notice that, by construction, the vector field Y is invariant by θ_f ; furthermore θ_f is the time one map of Y : this is true on a neighborhood of $\pm\infty$, and extends on \mathbb{R} because Y is θ_f -invariant.

Now, the vector field $X = \varphi_*^{-1}(Y)$, defined on $(0, 1)$, coincides with X_- and X_+ in a neighborhood of 0 and 1, respectively, hence induces a smooth vector field on $[0, 1]$. Finally, f is the time one map of X .

Conversely, if f is the time one map of a C^1 -vector field X on $[0, 1]$ then θ_f is the time one map of the vector field $Y = \varphi_*(X)$, which coincides with $\partial/\partial x$ in the neighborhood of $\pm\infty$ (because X coincides with X_- and X_+ in a neighborhood of 0 and 1, respectively). Hence the projections $(\pi_- \circ \pi_f)_*(Y)$ and $(\pi_+ \circ \pi_f)_*(Y)$ are both equal to the vector field $\partial/\partial x$ on S^1 . This implies that $(\Delta_{f,\varphi})_*(\partial/\partial x) = \partial/\partial x$, which implies that $\Delta_{f,\varphi}$ is a rotation. \square

Remark 3.4 *The function $\Delta_{f,\varphi}$ defined here seems to depend on the choice of φ . There is a more intrinsic way to define the diffeomorphism $\Delta_{f,\varphi}$ “up to composition by rotation”:*

The vector fields X_- and X_+ defined in a neighborhood of 0 and 1, respectively, are the unique vector fields such that f is the time one map of the corresponding flows, in the neighborhood of 0 and 1, respectively. Each of these vector fields induces a parametrization of the orbit space $(0, 1)/f = S_f$, that is, up to the choice of an origin, a diffeomorphism $\pi_f^\pm: S_f \rightarrow S^1$. The change of parametrization $\pi_f^+ \circ (\pi_f^-)^{-1}$ is well defined, up to the choice of an origin of the circle, i.e. up to multiplication, at the right and at the left, by rotations. This class of maps is called the Mather invariant of f .

3.1.2 Vanishing of the Mather invariant: proof of Proposition 3.2

Fix $f \in D_{\alpha,\beta}$ and $K_f > 0$ such that $\theta_f = \varphi \circ f \circ \varphi^{-1}$ coincides with $x \mapsto x + 1$ on $(-\infty, -K_f] \cup [K_f, +\infty)$.

Given a diffeomorphism $h: \mathbb{R} \rightarrow \mathbb{R}$, the *support of h* , denoted by $\text{supp}(h)$ is the closure of the set of points x such that $h(x) \neq x$.

Lemma 3.5 *Consider a number $a > K_f$ and a diffeomorphism $\tilde{\psi}: \mathbb{R} \rightarrow \mathbb{R}$ whose support is contained in $(a, a + 1)$. Let h denote the diffeomorphism $\varphi^{-1} \circ \tilde{\psi} \circ \varphi$, and let ψ denote the diffeomorphism of $S^1 \simeq [a, a + 1]/a \sim a + 1$ induced by $\tilde{\psi}$.*

Then the diffeomorphism $g = f \circ h$ belongs to $D_{\alpha,\beta}$, and $\Delta_{g,\varphi} = \psi \circ \Delta_{f,\varphi}$.

Proof. The diffeomorphism g coincides with f in neighborhoods of 0 and 1 proving that $g \in D_{\alpha,\beta}$. Furthermore, by construction, one may choose $K_g = a + 1$.

If $x < -a$, there is a (unique) integer such that $\theta_f^n(x) = \theta_g^n(x) \in [a, a + 1)$, and by construction of $\Delta_{f,\varphi}$, the projection of $\theta_f^n(x)$ on S^1 is $\Delta_{f,\varphi}(x)$. Now the projection on S^1 of $\theta_g^{n+1}(x) = \theta_f \circ \tilde{\psi} \circ \theta_f^n(x)$ is $\psi \circ \Delta_{f,\varphi}(x)$, by construction. As $\theta_g = \theta_f = y \mapsto y + 1$ for $y \geq a + 1$, one gets that the projection on S^1 of $\theta_g^{n+k}(x)$ is $\psi \circ \Delta_{f,\varphi}(x)$, for all $k > 0$; hence $\Delta_{g,\varphi} = \psi \circ \Delta_{f,\varphi}$. \square

Iterating the process described in Lemma 3.5 one gets:

Corollary 3.6 *Consider a finite sequence of numbers $a_i > K_f$, $i \in \{1, \dots, \ell\}$, such that $a_{i+1} > a_i + 1$ for all $i \in \{1, \dots, \ell - 1\}$. For every $i \in \{1, \dots, \ell\}$, fix a diffeomorphism $\tilde{\psi}_i: \mathbb{R} \rightarrow \mathbb{R}$ whose support is contained in $(a_i, a_i + 1)$. Let h_i denote the diffeomorphism $\varphi^{-1} \circ \tilde{\psi}_i \circ \varphi$, and let ψ_i denote the diffeomorphism of S^1 induced by $\tilde{\psi}_i$. (Note that the diffeomorphisms h_i have disjoint support, so that they are pairwise commuting.)*

Then the diffeomorphism $g = f \circ h_1 \circ h_2 \circ \dots \circ h_\ell$ belongs to $D_{\alpha,\beta}$, and we have:

$$\Delta_{g,\varphi} = \psi_\ell \circ \dots \circ \psi_1 \circ \Delta_{f,\varphi}.$$

Definition 3.7 *Let $a \in \mathbb{R}$, and let \bar{a} be its projection on $S^1 = \mathbb{R}/\mathbb{Z}$. Given a diffeomorphism $\psi: S^1 \rightarrow S^1$ with support in $S^1 \setminus \{\bar{a}\}$ we call the lift of ψ in $(a, a + 1)$ the diffeomorphism $\tilde{\psi}_a: \mathbb{R} \rightarrow \mathbb{R}$ with support in $(a, a + 1)$ such that for any $x \in (a, a + 1)$ the image $\psi_a(x)$ is the point of $(a, a + 1)$ which projects to $\psi(\bar{x})$ where \bar{x} is the projection of x .*

We denote by $\Theta_a(\psi)$ the diffeomorphism of $[0, 1]$ whose expression in $(0, 1)$ is $\Theta_a(\psi) = \varphi^{-1} \circ \tilde{\psi}_a \circ \varphi$.

Lemma 3.8 *For any C^1 neighborhood \mathcal{U} of f there is a neighborhood \mathcal{V} of $\text{Id}_{S^1} \in \text{Diff}^1(S^1)$ with the following property:*

Given any finite sequence $a_i > K_f$, $i \in \{1, \dots, \ell\}$, such that $a_{i+1} > a_i + 1$ for all $i \in \{1, \dots, \ell - 1\}$, we denote by \bar{a}_i the projection of a_i on S^1 . For

any i , let $\psi_i \in \mathcal{V}$ be a diffeomorphism of S^1 with support in $S^1 \setminus \{a_i\}$. Then the diffeomorphism $g = f \circ \Theta_{a_1}(\psi_1) \circ \cdots \circ \Theta_{a_\ell}(\psi_\ell)$ belongs to \mathcal{U} .

Proof. We fix a neighborhood \mathcal{U}_0 of the identity map of $[0, 1]$ such that, if $g_1, \dots, g_n \in \mathcal{U}_0$ and if the support of the g_i are pairwise disjoint, then $f \circ g_1 \circ \cdots \circ g_n$ belongs to \mathcal{U} . Now the lemma is a direct consequence of Lemma 3.9 below. \square

Lemma 3.9 *For any C^1 -neighborhood \mathcal{U}_0 of f there is a neighborhood \mathcal{V} of $Id_{S^1} \in \text{Diff}^1(S^1)$ with the following property:*

Consider any $a > K_f$, its the projection \bar{a} on S^1 and any diffeomorphism $\psi \in \mathcal{V}$ with support in $S^1 \setminus \{a\}$. Then the diffeomorphism $\Theta_a(\psi)$ belongs to \mathcal{U}_0 .

Proof. Notice that there exists $\varepsilon > 0$ such that \mathcal{U}_0 contains any diffeomorphism h of $[0, 1]$ with $\sup_{x \in [0, 1]} |D_x h - 1| < \varepsilon$.

Now consider $a > K_f$ and an integer $n > 0$. Then for any diffeomorphism ψ of S^1 with support in $S^1 \setminus \{\bar{a}\}$, the lifts ψ_a and ψ_{a+n} are conjugated by the translation $x \mapsto x + n$. As a consequence, $\Theta_{a+n}(\psi)$ is obtained from $\Theta_a(\psi)$ by the conjugacy by the homothety of ratio β^n . As a consequence one gets that $\sup_{x \in [0, 1]} |D_x \Theta_{a+n}(\psi) - 1| = \sup_{x \in [0, 1]} |D_x \Theta_a(\psi) - 1|$.

Hence one just has to prove the lemma for $a \in [K_f, K_f + 1]$. This is a direct consequence of the facts that the derivatives of φ and of φ^{-1} are bounded on $\varphi^{-1}([K_f, K_f + 2])$ and $[K_f, K_f + 2]$ respectively, and that for any ψ with support in $S^1 \setminus \{a\}$, one has:

$$\sup_{x \in [a, a+1]} |D_x \psi_a - 1| = \sup_{x \in S^1} |D_x \psi - 1|.$$

\square

Let us now recall a classical result which is the key point of our proof.

Theorem 3.10 *Let M be a closed Riemannian manifold, let $r > 0$ and let \mathcal{U} be a C^1 neighborhood of the identity map. Then for any smooth diffeomorphism f of M isotopic to the identity, there exist $k \geq 1$ and $g_1, \dots, g_k \in \mathcal{U}$ such that $g_i = id$ on the complement of a ball $B(x_i, r)$, and*

$$f = g_1 \circ \cdots \circ g_k.$$

Here we use Theorem 3.10 on the circle S^1 , where it is an easy consequence of the result, by M. Herman, that any smooth diffeomorphism is the product of a rotation by a diffeomorphism smoothly conjugate to a rotation. In Section 3.2, we will also use Theorem 3.10 on the torus T^2 .

Proof of Proposition 3.2. Given a C^1 -neighborhood \mathcal{U} of f , we choose a C^1 -neighborhood \mathcal{V} of the identity map of S^1 given by Lemma 3.8. Using

Theorem 3.10, we can write $\Delta_{f,\varphi}$ as a finite product $\Delta_{f,\varphi} = \psi_1^{-1} \circ \dots \circ \psi_\ell^{-1}$ such that $\psi_i \in \mathcal{V}$, and the support of ψ_i is contained in an interval of length $\frac{1}{2}$ in S^1 (and in particular is not all of S^1). Now we choose a finite sequence $a_i > K_k$ such that $a_{i+1} > a_i + 1$, and such that the projection \bar{a}_i does not belong to the support of ψ_i . Let $h_i = \Theta_{a_i}(\psi_i)$.

Applying Lemma 3.8, we obtain that the diffeomorphism

$$g = f \circ h_1 \circ h_2 \circ \dots \circ h_\ell$$

belongs to \mathcal{U} ; applying Corollary 3.6, we get that

$$\Delta_{g,\varphi} = \psi_\ell \circ \dots \circ \psi_1 \circ \Delta_{f,\varphi} = Id_{S^1}.$$

□

3.1.3 Proof of Theorem 0.7 on the circle

By Lemma 3.1, it is enough to consider $f \in \mathcal{D}_0^1$. The set $Per(f)$ is finite. Let \mathcal{I} be the set of segments joining two successive periodic points of f ; in other words, every element $I \in \mathcal{I}$ is the closure of a connected component of $S^1 \setminus Per(f)$. Notice that f induces a permutation on \mathcal{I} . Furthermore, all the elements of \mathcal{I} have the same period denoted by $k > 0$, under this action (this period is equal to 2 if f reverses the orientation, and is equal to the period of the periodic orbits in the orientation preserving case).

Consider a segment $I \in \mathcal{I}$. The endpoints of I are the fixed points of the restriction $f^k|_I$; moreover, one endpoint (denoted by a) is a repeller and the other (denoted by b) is an attractor. Let $h_I: I \rightarrow [0, 1]$ be the affine map such that $h_I(a) = 0$ and $h_I(b) = 1$ and let $\varphi_I: [0, 1] \rightarrow [0, 1]$ denote the diffeomorphism $h_I \circ f^k|_I \circ h_I^{-1}$.

According to Proposition 3.2, there is a sequence $(\psi_{I,n})_{n \in \mathbb{N}}$, of diffeomorphisms converging to φ_I in the C^1 -topology when $n \rightarrow +\infty$, and a sequence $(Y_{I,n})_{n \in \mathbb{N}}$ of C^∞ vector fields on $[0, 1]$ such that $\psi_{I,n}$ coincides with φ_I in a small neighborhood of $\{0, 1\}$ and is time one map of $Y_{I,n}$. One denotes $g_{I,n} = h_I^{-1} \circ \psi_{I,n} \circ h_I$. Notice that the diffeomorphism $g_{I,n}$ coincides with f^k in neighborhoods of the endpoints of I and converges to $f^k|_I$ when $n \rightarrow \infty$.

We now define a diffeomorphism $f_{I,n}$ of S^1 as follows:

$$f_{I,n} = \begin{cases} f & \text{on } S^1 \setminus f^{k-1}(I) \\ g_{I,n} \circ f^{-k+1} & \text{on } f^{k-1}(I). \end{cases}$$

This is a C^∞ diffeomorphism since it coincides with f in a neighborhood of the periodic orbits. Moreover, $(f_{I,n})$ converges to f when n goes to $+\infty$.

One denotes by $X_{I,n}$ the vector field, defined on the orbit $\bigcup_0^{k-1} f^i(I)$ of the segment I as follows:

- $X_{I,n} = (h_I^{-1})_*(Y_{I,n})$ on I ;
- for all $i \in \{1, \dots, k-1\}$ and all $x \in I$:

$$X_{I,n}(f^i(x)) = f_*^i(X_{I,n}(x)).$$

Finally, we fix a family $I_1, \dots, I_\ell \subset \mathcal{I}$ such that for $i \neq j$ the segments I_i and I_j have distinct orbits, and conversely every orbit of segment in \mathcal{I} contains one of the I_i .

We denote by f_n the diffeomorphism of S^1 coinciding with $f_{I_i,n}$ on the orbit of I_i for all $i \in \{1, \dots, \ell\}$. This diffeomorphism is well-defined because all the $f_{I_i,n}$ coincide with f in a small neighborhood of the periodic points (the endpoints of the segments in \mathcal{I}). We denote by X_n the vector field on S^1 that coincides with $X_{I_i,n}$ on the orbit of I_i , for all $i \in \{1, \dots, \ell\}$.

One easily verifies that X_n is a smooth vector field on S^1 , invariant by f_n , and such that f_n^k is the time one map of X_n : the unique difficulty consists in checking the continuity and smoothness of the vector field X at the periodic points. As f_n is affine in the neighborhood of the periodic orbits one verifies that, at both sides of a periodic point x , the vector field X is the affine vector field vanishing at x and whose eigenvalue at x is $\ln D_x f$.

Finally f_n converges to f in the C^1 topology, completing the proof of Theorem 0.7 on the circle.

3.2 The case of the sphere S^2

As in the one-dimensional case, the idea here is to measure how far certain diffeomorphisms of S^2 are from the time-one map of a vector field. One obtains in this way a generalization of the Mather invariant, which in this setting is a diffeomorphism of \mathbb{T}^2 . Such an invariant has already been constructed ¹ in [AY] by V. Afraimovich and T. Young, and we now have to show that by a C^1 -small perturbation of the dynamics, this invariant vanishes.

3.2.1 Preparation of diffeomorphisms in \mathcal{O}

Let S^2 be the unit sphere in \mathbb{R}^3 endowed with the coordinates (x, y, z) . We denote by $N = (0, 0, 1)$ and $S = (0, 0, -1)$ the north and the south poles of S^2 . Notice that the coordinates x, y define local coordinates of S^2 in local charts U_N and U_S in neighborhoods of N and S .

The following straightforward lemma asserts that one may assume that the fixed points of any diffeomorphism f in the open set \mathcal{O} are N and S and that the derivative at these points are conformal maps.

¹In [AY], the authors write that the Mather invariant for a diffeomorphism of S^2 is always isotopic to the identity, but this is not correct (their Proposition 1 is wrong). For this reason, we choose here to build in detail the construction of this invariant on the sphere.

Lemma 3.11 *Consider a diffeomorphism $f \in \mathcal{O}$. Then there is a smooth diffeomorphism $h: S^2 \rightarrow S^2$ such that $h(N_f) = N$, $h(S_f) = S$ are the fixed points of $g = hfh^{-1}$; furthermore, the derivatives $D_N g$ and $D_S g$ are conformal linear maps, i.e., each a composition of a rotation with a homothety of ratio $\alpha > 1$ and $\beta < 1$, respectively.*

Finally, any C^1 neighborhood of g contains a diffeomorphism \tilde{g} such that there are neighborhoods $V_N \subset U_N$ and $V_S \subset U_S$ of N and S , respectively, such that the expression of \tilde{g} in the coordinates (x, y) is $\tilde{g}(x, y) = D_N g(x, y)$ for $(x, y) \in V_N$ and $\tilde{g}(x, y) = D_S g(x, y)$ for $(x, y) \in V_S$.

3.2.2 Space of orbits of a conformal linear map

Let $A, B \in GL(\mathbb{R}, 2)$ be two conformal matrices of norm $\alpha > 1$ and $\beta < 1$, respectively. There exist $a, b \in [0, 2\pi)$ such that $A = R_a \circ h_\alpha$ and $B = R_b \circ h_\beta$, where R_a and R_b are the rotation of angles a and b , respectively, and h_α and h_β are the homotheties of ratio α and β , respectively. Notice that, for all $n \in \mathbb{Z}$, the linear map A is the time one map of the vector field

$$X_{A,n} = \ln \alpha \cdot \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) + (a + 2\pi n) \cdot \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right),$$

and B is the time one map of

$$X_{B,n} = \ln \beta \cdot \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) + (b + 2\pi n) \cdot \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right).$$

Notice that the orbit space $T_A = \mathbb{R}^2 \setminus \{0\} / A$ (of the action of A on $\mathbb{R}^2 \setminus \{0\}$) is a torus (diffeomorphic to $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$); we denote by π_A the canonical projection from $\mathbb{R}^2 \setminus \{0\}$ onto T_A . Moreover, the vector fields

$$Z = 2\pi \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$$

and $X_{A,n}$ project on T_A in pairwise transverse commuting vector fields, which we also denote by Z and $X_{A,n}$; the orbits of both flows are periodic of period 1. Hence, for any pair $(Z, X_{A,n})$ there is a diffeomorphism $\mathcal{L}_{A,n}: T_A \rightarrow T^2 = \mathbb{R}^2 / \mathbb{Z}^2$ which sends Z to $\partial / \partial x$ and $X_{A,n}$ to $\partial / \partial y$; this diffeomorphism is unique up to composition by a translation of T^2 . Furthermore, the diffeomorphisms $\mathcal{L}_{A,m} \circ \mathcal{L}_{A,n}^{-1}$ are affine maps of the torus T^2 , for all $n, m \in \mathbb{Z}$, so that T_A is endowed with a canonical affine structure (indeed the affine map $\mathcal{L}_{A,m} \circ \mathcal{L}_{A,n}^{-1}$ is isotopic to the map induced by the matrix $\begin{pmatrix} 1 & n - m \\ 0 & 1 \end{pmatrix}$).

Note that the orbits of Z correspond to the positive generator of the fundamental group of $\mathbb{R}^2 \setminus \{0\}$; we denote by σ the corresponding element of $\pi_1(T_A)$. Given any closed loop $\gamma: [0, 1] \rightarrow T_A$, and any point $x \in \mathbb{R}^2 \setminus \{0\}$ with

$\pi_A(x) = \gamma(0)$, there is a lift of γ to a path in $\mathbb{R}^2 \setminus \{0\}$ joining x to $A^k(x)$, where k is the algebraic intersection number of σ with γ . Finally, observe that the homotopy classes corresponding to the orbits of $X_{n,A}$, when $n \in \mathbb{Z}$ are precisely those whose intersection number with σ is 1: in other words, there is a basis of $\pi_1(T_A) = \mathbb{Z}^2$ such that $\sigma = (1, 0)$ and the orbits of $X_{A,n}$ are homotopic to $(n, 1)$.

In the same way $T_B = \mathbb{R}^2 \setminus \{0\}/B$ is a torus endowed with the vector fields obtained by projection of Z and $X_{B,n}$ and we denote by $\mathcal{L}_{B,n}: T_B \rightarrow T^2$ a diffeomorphism that sends Z to $\partial/\partial x$ and $X_{B,n}$ to $\partial/\partial y$.

3.2.3 Mather invariant for diffeomorphisms of S^2

Denote by $D_{A,B} \subset \mathcal{O}$ the set of diffeomorphisms $f \in \mathcal{O}$ whose expression in the coordinates (x, y) coincides with A in a neighborhood U_f^N of N and with B in a neighborhood U_f^S of S . The aim of this part is to build a Mather invariant for diffeomorphisms in $D_{A,B}$.

We retain the notation of the previous subsection. Consider $f \in D_{A,B}$. The orbit space $(S^2 \setminus \{N, S\})/f$ is a torus T_f and we denote by $\pi_f: S^2 \setminus \{N, S\} \rightarrow T_f$ the natural projection. Furthermore, as f coincides with A on U_f^N , the torus T_f may be identified with the torus T_A by a diffeomorphism $\pi_N: T_f \rightarrow T_A$, and in the same way, the fact that f coincides with B in a neighborhood of S induces a diffeomorphism $\pi_S: T_f \rightarrow T_B$.

Notice that the morphisms $\pi_{N*}: H_1(T_f, \mathbb{Z}) \rightarrow H_1(T_A, \mathbb{Z})$ and $\pi_{S*}: H_1(T_f) \rightarrow H_1(T_B)$ preserve the homology class of σ (corresponding to the positive homology generator of $S^2 \setminus \{N, S\}$ or of $\mathbb{R}^2 \setminus \{0\}$), and the homology intersection form with σ .

Consequently, for any $f \in D_{A,B}$, there is an integer $n(f)$ such that the map $\Delta_{f,0,0} = \mathcal{L}_{B,0} \circ \pi_S \circ \pi_N^{-1} \circ \mathcal{L}_{A,0}^{-1}$ is isotopic to the linear map of T^2 induced by the matrix $\begin{pmatrix} 1 & n(f) \\ 0 & 1 \end{pmatrix}$.

Lemma 3.12 *For any $f \in D_{A,B}$ there is a C^1 -neighborhood \mathcal{U} of f in $\text{Diff}^1(S^2)$ such that for any $g \in \mathcal{U} \cap D_{A,B}$ one has $n(f) = n(g)$.*

Proof. We can choose a neighborhood \mathcal{U} such that, if $g \in \mathcal{U}$ then the map

$$f_t(x) = \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}$$

is a smooth isotopy between f and g . Furthermore, by shrinking \mathcal{U} if necessary, for any $g \in \mathcal{U}$, the isotopy f_t belongs to \mathcal{O} (that is $\Omega(g) = \{N_g, S_g\}$).

If $g \in \mathcal{U} \cap D_{A,B}$ then there are discs D^N and D^S centered on N and S , respectively, such that $f_t = A$ on D^N and $f_t = B$ on D^S so that $f_t \in D_{A,B}$. In particular $f_t(D^S) \subset D^S$, and $f_t^{-1}(D^N) \subset D^N$. Furthermore, there exists $\ell > 0$ such that for any $x \in S^2 \setminus (D^N \cup D^S)$, $f_t^\ell(x) \in D^S$ and $f_t^{-\ell}(x) \in D^N$.

Let $x \in D^N$ such that $A(x) = f_t(x) \in D^N$ and $A^2(x) \notin D^N$. Hence $y_t = f_t^{\ell+2}(x) \in D^S$ and $f_t(y_t) = B(y_t) \in D^S$. Let γ be the segment of orbit of $X_{A,0}$ joining x to $A(x) = f_t(x)$, and let $\gamma_t = f_t^{\ell+2}(\gamma)$. For every t , γ_t is homotopic (relative to $\{y_t, B(y_t)\}$ in $S^2 \setminus \{N, S\}$) to a segment of orbit of $X_{B,n(f_t)}$. As a consequence, $n(f_t)$ varies continuously with t as t varies from 0 to 1. Hence $n(f_t)$ is constant; that is, $n(g) = n(f)$. \square

Hence there is a partition of $D_{A,B}$ into open subsets $D_{A,B,n}$ such that $n(f) = n$ for $f \in D_{A,B,n}$. For $f \in D_{A,B,n}$, we define:

$$\Delta_f = \mathcal{L}_{B,n} \circ \pi_S \circ \pi_N^{-1} \circ \mathcal{L}_{A,0}.$$

Then Δ_f is a diffeomorphism of T^2 , isotopic to the identity.

Theorem 3.13 below justifies calling Δ_f *the Mather invariant of f* .

Theorem 3.13 *Let $f \in D_{A,B,n}$ be a smooth diffeomorphism such that Δ_f is a translation of the torus T^2 . Then f leaves invariant two transverse commuting vector fields Z_f and X_f on S^2 such that $Z_f = Z$ in a neighborhood of $\{N, S\}$, $X_f = X_{A,S}$ in a neighborhood of N and $X_f = X_{B,S}$ in a neighborhood of S .*

As a consequence the centralizer of f is isomorphic to $S^1 \times \mathbb{R}$.

Proof. Fix two discs D^N and D^S centered at N and S , respectively, in which f coincides with A and B , respectively.

For any $x \neq S$ there exists $m(x) < 0$ such that $f^{m(x)} \in D^N$. One defines $Z_f(x) = f_*^{-m(x)}(Z(f^{m(x)}))$ and $X_f(x) = f_*^{-m(x)}(X_{A,N}(f^{m(x)}))$. As Z and $X_{A,N}$ are invariant by A , one proves that the vectors $Z_f(x)$ and $X_f(x)$ are independent of the choice of $m(x)$. As a consequence, one deduces that they depend smoothly on $x \in S^2 \setminus \{S\}$ and that they commute on $S^2 \setminus \{S\}$. Furthermore the restrictions of Z_f and X_f to D^S are invariant by f , and hence by B , so that they induce two vector fields on T_B whose images by $\mathcal{L}_{B,n}$ are $\Delta_f(\frac{\partial}{\partial x}) = \frac{\partial}{\partial x}$ and $\Delta_f(\frac{\partial}{\partial y}) = \frac{\partial}{\partial y}$, respectively; that is, they coincide with the projections of the restrictions Z and $X_{B,S}$ to D^S . Thus $Z_f = Z$ and $X_f = X_{B,S}$ on D^S , ending the proof. \square

3.3 Vanishing of the Mather invariant

This part is now very close to the 1-dimensional case.

For any $f \in D_{A,B,n}$ we denote by D_f^S a disk centered on S on which $f = B$.

Let $h: S^2 \rightarrow S^2$ be a diffeomorphism whose support is contained in a disk $D \subset D_f^S$, disjoint from all $B^m(D)$ for $m > 0$. The disk D projects homeomorphically onto a disk $D' \subset T_B$, and finally onto a disk $\tilde{D} = \mathcal{L}_{B,n}(D') \subset T^2$. Let ψ be the diffeomorphism of T^2 with support in \tilde{D} whose restriction to \tilde{D} is the projection of h . We say that ψ is the projection of h on T^2 and conversely, that h is the lift of ψ with support in D .

Fix $k > 0$ such that D is disjoint from $B^k(D_f^S)$.

Lemma 3.14 *With the notation above, the composition $f \circ h$ is a diffeomorphism in $D_{A,B,n}$ with $B^k(D_f^S) \subset D_{f \circ h}^S$, and whose Mather invariant is*

$$\Delta_{f \circ h} = \psi \circ \Delta_f.$$

Corollary 3.15 *Let $D_0, \dots, D_\ell \subset D_f^S$ be a finite sequence of disks such that*

- *for every i , D_i is disjoint from $B^k(D_i)$ for $k > 0$;*
- *for all $i < j$ the disk D_i is disjoint from $B^k(D_j)$, $k \geq 0$*

For every i , let h_i be a diffeomorphism of S^2 with support in D_i , and let ψ_i be the projection of h_i on T^2 (by $\mathcal{L}_{B,n} \circ \pi_S \circ \pi_f$).

Then the Mather invariant of $f \circ h_0 \circ \dots \circ h_\ell$ is

$$\Delta_{f \circ h_0 \circ \dots \circ h_\ell} = \psi_\ell \circ \dots \circ \psi_0 \circ \Delta_f.$$

Observe that for any disk $\tilde{D} \subset T^2$ with diameter strictly less than 1, each connected component of $(\mathcal{L}_{B,n} \circ \pi_S \circ \pi_f)^{-1}(\tilde{D})$ projects diffeomorphically onto \tilde{D} , and f induces a permutation of these components. For $i > 0$, let D_i denote the (unique) component of $(\mathcal{L}_{B,n} \circ \pi_S \circ \pi_f)^{-1}(\tilde{D})$ such that $f^{-i}(D_i) \subset D_f^S$ but $f^{-(i+1)}(D_i)$ is not contained in D_f^S .

For any diffeomorphism ψ with support in \tilde{D} we will denote by $\theta_i(\psi): S^2 \rightarrow S^2$ the lift of ψ with support in D_i .

The next lemma is the unique reason we required that the derivative of f at N, S be complex, hence conjugate to conformal linear maps:

Lemma 3.16 *Let $\tilde{D} \subset T^2$ be a disk with diameter strictly less than 1 and let $i, j \in \mathbb{N}$. Then:*

$$\sup_{x \in S^2} \|D_x \theta_i(\psi) - Id\| = \sup_{x \in S^2} \|D_x \theta_j(\psi) - Id\|.$$

Proof. $\theta_i(\psi)$ is conjugated to $\theta_j(\psi)$ by B^{j-i} which is the composition of a homothety by a rotation; the C^1 norm is preserved by conjugacy by an isometry, and also by conjugacy by a homothety, hence is preserved by the conjugacy by B^{j-i} . \square

Corollary 3.17 *For any $\varepsilon > 0$ there is a C^1 -neighborhood $\mathcal{V}_\varepsilon \subset \text{Diff}(T^2)$ of the identity map such that for any diffeomorphism $\psi \in \mathcal{V}_\varepsilon$ with support in a disk $\tilde{D} \subset T^2$ with diameter strictly less than 1, and for any $i \geq 0$, the lift $\theta_i(\psi)$ satisfies :*

$$\sup_{x \in S^2} \|D_x \theta_i(\psi) - Id\| < \varepsilon.$$

Definition 3.18 Let ψ_1, \dots, ψ_ℓ be ℓ diffeomorphisms of T^2 such that the support of every ψ_i is contained in a disk \tilde{D}_i with diameter strictly less than 1; a lift of the sequence ψ_1, \dots, ψ_ℓ is a sequence of lifts $h_1 = \theta_{i_1}(\psi_1), \dots, h_\ell = \theta_{i_\ell}(\psi_\ell)$ such that, for every $i < j$ the support of h_i is disjoint from all the iterates $B^k(\text{supp}(h_j))$, for $k \geq 0$.

It is easy to check that, for any sequence ψ_1, \dots, ψ_ℓ of diffeomorphisms of T^2 such that the support of every ψ_i is contained in a disk \tilde{D}_i with diameter strictly less than 1, the sequence $h_i = \theta_i(\psi_i)$ is a lift.

Proof of Theorem 0.7 on the sphere S^2 . Consider $f \in D_{A,B,n}$ and a C^1 -neighborhood \mathcal{U} of f . Fix $\varepsilon > 0$ such that, if g_1, \dots, g_m , $m > 0$, are diffeomorphisms of S^2 with pairwise disjoint supports in $S^2 \setminus \{N, S\}$, and such that $\sup_{x \in S^2} \|Dg_i(x) - Id\| < \varepsilon$, then $f \circ g_1 \circ \dots \circ g_m \in \mathcal{U}$. Let \mathcal{V}_ε be the C^1 -neighborhood of the identity map of T^2 given by Corollary 3.17.

Using Theorem 3.10, we write

$$\Delta_f = \psi_1^{-1} \circ \dots \circ \psi_\ell^{-1},$$

for some $\ell > 0$, where $\psi_i \in \mathcal{V}_\varepsilon$, and the support of ψ_i is contained in a disk \tilde{D}_i with diameter strictly less than 1. Let (h_1, \dots, h_ℓ) be a lift of the sequence $(\psi_1, \dots, \psi_\ell)$; the h_i satisfy

$$\sup_{x \in S^2} \|D_x h_i - Id\| < \varepsilon,$$

by our choice of \mathcal{V}_ε .

Our choice $\varepsilon > 0$ implies that $g = f \circ h_1 \circ \dots \circ h_\ell$ is a diffeomorphism belonging to $D_{A,B,n} \cap \mathcal{U}$. Furthermore, its Mather invariant is $\Delta_g = \psi_\ell \circ \dots \circ \psi_1 \circ \Delta_f = Id$.

We have just shown that any $f \in D_{A,B}$ is the C^1 -limit of a sequence $g_k \in D_{A,B}$ whose Mather invariant is the identity map; in particular, the centralizer of g_k is isomorphic to $\mathbb{R} \times S^1$.

Since by Lemma 3.11, \mathcal{O} contains a dense set of diffeomorphisms smoothly conjugate to elements of $D_{A,B,n}$, any diffeomorphism in \mathcal{O} is the limit of diffeomorphisms g_k that are the time 1 map of Morse-Smale vector fields, ending the proof of Theorem 0.7. \square

A Conservative extension results

We explain in this appendix how a perturbation of a conservative diffeomorphism along a submanifold W can be extended as a conservative perturbation on the whole manifold M .

This allows to obtain Proposition 1.4 from Proposition 1.9: the results proven in this section will be applied to the case W is an invariant manifold of a hyperbolic periodic point p . In the volume-preserving case, one will assume that $\dim(W) \leq \frac{1}{2} \dim(M)$ (note that this hypothesis is always satisfied either by the stable or by the unstable manifold of p). In the symplectic case, there is no additional hypothesis, but we use the following well-known fact.

Lemma A.1 *Let $f \in \text{Symp}^1(M)$ and let p be a hyperbolic periodic point for f . Then $W^s(p)$ and $W^u(p)$ are Lagrangian submanifolds of M .*

Proof. Let $x \in W^s(p)$, and let $v, w \in T_x W^s(p)$ be tangent vectors to $W^s(p)$. On the one hand, since f is a symplectomorphism, we have

$$\omega(D_x f^k(v), D_x f^k(w)) = \omega(v, w),$$

for all $k \in \mathbb{Z}$. On the other hand, as $k \rightarrow +\infty$, we have

$$\omega(D_x f^k(v), D_x f^k(w)) \rightarrow 0.$$

Hence ω vanishes identically on $W^s(p)$. The same is true for $W^u(p)$. Since $W^s(p)$ and $W^u(p)$ have complementary dimension and ω is nondegenerate, they must have the same dimension. Hence, both are Lagrangian submanifolds of M . \square

A.1 The symplectic case

Proposition A.2 *Let M be a symplectic manifold and z a point contained in a C^1 Lagrangian submanifold $W \subset M$. Then there exists in W a disk $D = \overline{B}_W(z, r_0)$ centered at z such that, for every neighborhood $U \subset M$ of D and every $\varepsilon > 0$, there exists $\delta > 0$ with the following property.*

For every C^1 diffeomorphism $\psi : D \rightarrow D$ satisfying:

- a. $\psi = \text{Id}$ on a neighborhood of ∂D , and*
- b. $d_{C^1}(\psi, \text{Id}) < \delta$,*

there exists $\varphi \in \text{Symp}^1(M)$ such that:

- 1. $\varphi = \text{Id}$ on $M \setminus U$,*
- 2. $\varphi = \psi$ on D , and*
- 3. $d_{C^1}(\varphi, \text{Id}) < \varepsilon$.*

Proof. The basic strategy is first to symplectically embed the disk D as the 0-section of its cotangent bundle T^*D . On T^*D , the symplectic form is $\omega = d\alpha$, where α is the canonical one-form on T^*D . Any diffeomorphism $\psi : D \rightarrow D$ lifts to a canonical symplectomorphism $\psi^* : T^*D \rightarrow T^*D$; namely the pull-back map $(\psi, D\psi^{-1})$. The natural thing to try to do is to set $\varphi = \psi^*$ in a neighborhood of the 0-section, symplectically interpolating between ψ^* and Id using a generating function. This simple approach fails, however, because ψ is only C^1 , and so ψ^* is merely continuous. (Even assuming that ψ is C^2 does not help: in order to control the C^1 size of such a map, it is necessary to have some control on the C^2 size of ψ , and we cannot assume any such control). Using a convolution product, it is possible to overcome this problem. This approach mirrors that in [BGV], but in the symplectic setting.

The problem is local and one can work in \mathbb{R}^{2n} endowed with the standard symplectic form $\omega = \sum_i du_i \wedge dv_i$ where $u = (u_1, \dots, u_n), v = (v_1, \dots, v_n)$. By a symplectic change of coordinates, we may assume that the disk D lies inside a disk $\{(u, v), \|u\| \leq R, v = 0\}$. We define ψ using a generating function S .

We first recall the definition and properties of generating functions. Suppose that $h : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is a C^r symplectomorphism, taking the form:

$$h(u, v) = (\xi(u, v), \eta(u, v)),$$

with $\xi, \eta : \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ and $h(0, 0) = (0, 0)$. Let us assume that the partial derivative matrix $\frac{\partial}{\partial v}\eta(u, v)$ is invertible (this is the case for instance if h preserves $\mathbb{R}^n \times \{0\}$). We can solve for $\eta = \eta(u, v)$ to obtain new coordinates (u, η) on a small neighborhood of $(0, 0)$ in \mathbb{R}^{2n} . Since h is symplectic, the 1-form $\alpha = \sum_i v_i du_i + \xi_i d\eta_i$ is closed, and hence, exact. Thus there exists a C^{r+1} function $S = S(u, \eta)$, unique up to adding a constant, defined in a neighborhood of $(0, 0)$, such that $dS = \alpha$. The function S is called a *generating function for h* .

On the other hand, any C^{r+1} function $S = S(u, \eta)$ satisfying the non-degeneracy condition that $\frac{\partial^2}{\partial u \partial \eta} S$ is everywhere nonsingular is the generating function of a C^r symplectic diffeomorphism. Solving for α in the equation

$$dS = \frac{\partial S}{\partial u} du + \frac{\partial S}{\partial \eta} d\eta = \alpha = v du + \xi d\eta,$$

we obtain the system:

$$\frac{\partial S}{\partial u} = v; \quad \frac{\partial S}{\partial \eta} = \xi.$$

The nondegeneracy condition implies that this system can be solved implicitly for a C^r function $\eta = \eta(u, v)$. We then obtain a C^r symplectomorphism:

$$h(u, v) = \left(\frac{\partial S}{\partial \eta}(u, \eta(u, v)), \eta(u, v) \right),$$

and S is a generating function for h .

It is easy to see that the generating function for the identity map is

$$S_0(u, \eta) = u \cdot \eta = \sum_{i=1}^n u_i \eta_i.$$

Claim 5 *For every $\varepsilon > 0$, there exists $\delta > 0$ such that, if $d_{C^2}(S, S_0) < \delta$ then $d_{C^1}(h, Id) < \varepsilon$.*

Proof. This follows from the implicit function theorem, and the details are omitted. \square

Returning to the proof of Proposition A.2, assume that $\psi : D \rightarrow D$ is written in u -coordinates as

$$\psi(u_1, \dots, u_n) = (\psi_1(u_1, \dots, u_n), \dots, \psi_n(u_1, \dots, u_n)).$$

We may assume that the domain of ψ has been extended to \mathbb{R}^n . To prove Proposition A.2, it suffices to prove the following lemma. \square

Lemma A.3 *Given a disk $D \subset \mathbb{R}^n$, and a neighborhood U of $D \times \{0\}$ in \mathbb{R}^{2n} , there exists $C > 0$ with the following property.*

For every C^1 diffeomorphism $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$, equal to the identity on a neighborhood of ∂D , there is a C^2 function $S : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ such that:

1. $d_{C^2}(S_0, S) \leq C d_{C^1}(\psi, Id)$,
2. $S = S_0$ outside of U ,
3. $\frac{\partial S}{\partial u}(u, 0) = 0$ for all $u \in \mathbb{R}$ and
4. $\frac{\partial S}{\partial \eta}(u, 0) = \psi(u)$ for all $u \in D$.

Note that condition 1. implies that S is nondegenerate, provided that $d_{C^1}(\psi, Id)$ is sufficiently small.

Proof of Lemma A.3. To illustrate the argument in a simple case, we first prove the lemma for $n = 1$. The proof of the general case is very similar. Let

$$a(u) = \psi'(u) - 1.$$

Note that a is a continuous map, $\|a\|_\infty \leq d_{C^1}(\psi, Id)$, and $a(u) = 0$ if $u \notin \text{int}(D)$. Let $\Phi : \mathbb{R} \rightarrow [0, 1]$ be a C^∞ function satisfying:

- $\Phi(0) = 1$ and $\Phi = 0$ outside of $(-1, 1)$,
- $\Phi^{(k)}(0) = 0$, for all $k \geq 1$,

- $\int_{\mathbb{R}} \Phi(w) dw = 1.$

Fix a point $u_* \in \partial D$, so that $\psi(u_*) = u_*$. For $(u, \eta) \in \mathbb{R}^2$, $\eta \neq 0$, let:

$$Q(u, \eta) = \eta \int_{u_*}^u \int_{\mathbb{R}} \Phi(w) a(x - w\eta) dw dx.$$

For $\eta \neq 0$, one can make the change of variables $w' = x - w\eta$ and get

$$Q(u, \eta) = \text{Sign}(\eta) \int_{u_*}^u \int_{\mathbb{R}} \Phi\left(\frac{x - w'}{\eta}\right) a(w') dw' dx.$$

Let $\rho : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C^∞ bump function identically equal to 1 on a neighborhood of $D \times \{0\}$ and vanishing outside of U . Consider

$$S = S_0 + \rho Q.$$

Lemma A.3 in the case $n = 1$ is a direct consequence of:

Claim 2 *The map $Q : \mathbb{R}^2 \rightarrow \mathbb{R}$ is C^2 and there is $C = C(U) > 0$ such that:*

1. $\|Q|_{\bar{U}}\|_{C^2} \leq C\|a\|_\infty,$
2. $\frac{\partial Q}{\partial u}(u, 0) = 0,$ for all $u \in \mathbb{R},$ and
3. $\frac{\partial Q}{\partial \eta}(u, 0) = \int_{u_*}^u a(x) dx = \psi(u) - u,$ for all $u \in \mathbb{R}.$

Proof. We derive explicitly the formulas:

$$\begin{aligned} \frac{\partial Q}{\partial u} &= \eta \int_{\mathbb{R}} \Phi(w) a(u - w\eta) dw \\ &= \text{Sign}(\eta) \int_{\mathbb{R}} \Phi\left(\frac{u - w'}{\eta}\right) a(w') dw', \\ \frac{\partial Q}{\partial \eta} &= \frac{-\text{Sign}(\eta)}{\eta^2} \int_{u_*}^u \int_{\mathbb{R}} \Phi'\left(\frac{x - w'}{\eta}\right) (x - w') a(w') dw' dx \\ &= - \int_{u_*}^u \int_{\mathbb{R}} \Phi'(w) w a(x - w\eta) dw dx \\ &= - \int_{\mathbb{R}} \Phi'(w) w \int_{u_* - w\eta}^{u - w\eta} a(x') dx' dw, \end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 Q}{\partial \eta \partial u} &= - \int_{\mathbb{R}} \Phi'(w) w a(u - w\eta) dw, \\
\frac{\partial^2 Q}{\partial u^2} &= \frac{1}{|\eta|} \int_{\mathbb{R}} \Phi' \left(\frac{u - w'}{\eta} \right) a(w') dw' \\
&= \int_{\mathbb{R}} \Phi'(w) a(u - w\eta) dw,
\end{aligned}$$

and finally:

$$\frac{\partial^2 Q}{\partial \eta^2} = \int_{\mathbb{R}} \Phi'(w) w^2 (a(u - w\eta) - a(u_* - w\eta)) dw.$$

Properties 1. and 2. follow immediately from these formulas. To see 3., note that

$$\begin{aligned}
\frac{\partial Q}{\partial \eta} \Big|_{\eta=0} &= - \left(\int_{u_*}^u a(x) dx \right) \left(\int_{\mathbb{R}} \Phi'(w) w dw \right) \\
&= - \left(\int_{u_*}^u a(x) dx \right) \left(- \int_{\mathbb{R}} \Phi(w) dw \right) \\
&= \int_{u_*}^u a(x) dx.
\end{aligned}$$

□

We now turn to the case $n \geq 1$ in Lemma A.3. For $i = 1, \dots, n$, let α_i be the continuous 1-form defined by

$$\alpha_i = d(\psi_i - \pi_i),$$

where $\pi_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is the projection onto the i th coordinate. As above, fix a point $u_* \in \partial D$, so that $\psi(u_*) = u_*$. Then we have the formula:

$$\psi_i(u_1, \dots, u_n) - u_i = \int_{u_*}^u \alpha_i,$$

where the right-hand side is a path integral evaluated on any path from u_* to $u = (u_1, \dots, u_n)$. Furthermore, we have $\|\alpha_i\|_{\infty} \leq d_{C^1}(\psi, Id)$, for all i . When $n = 1$, the 1-form α_1 is just $\alpha_1 = a(u) du$, where $a(u) = \psi'(u) - 1$, as above.

Let $\Phi_n : \mathbb{R}^n \rightarrow \mathbb{R}$ be an n -dimensional bell function:

$$\Phi_n(x_1, \dots, x_n) = \Phi(x_1) \cdots \Phi(x_n).$$

For each 1-form α , and $t \in \mathbb{R}$, we define a new 1-form α_i^{*t} on \mathbb{R}^n by taking the convolution:

$$\alpha^{*t}(u) = t \int_{\mathbb{R}^n} \Phi_n(w) \alpha(u - tw) dw.$$

We integrate along any path from u_* to u and set

$$Q(u, \eta) = \int_{u_*}^u \sum_{i=1}^n \alpha_i^{*\eta_i} = \sum_{i=1}^n \eta_i \int_{\mathbb{R}^n} \Phi_n(w) \left(\int_{u_*}^u \alpha_i(u - tw) \right) dw.$$

This is well-defined since $\int_{u_*}^u \alpha_i(u - tw)$ is independent of choice of path.

Let $\rho_n : \mathbf{R}^{2n} \rightarrow [0, 1]$ be a C^∞ bump function vanishing identically outside of U and equal to 1 on a neighborhood of D . As before, the map $S = S_0 + \rho_n Q$ satisfies the conclusions of Lemma A.3 provided the following claim holds.

Claim 3 *The map $Q : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ is C^2 and there is $C = C(U) > 0$ such that:*

1. $\|Q|_{\bar{U}}\|_{C^2} \leq C \max_i \|\alpha_i\|_\infty$,
2. $\frac{\partial Q}{\partial u}(u, 0) = 0$, for all $u \in \mathbb{R}^n$, and
3. $\frac{\partial Q}{\partial \eta_i}(u, 0) = \int_{u_*}^u \alpha_i = \psi_i(u) - u_i$, for all $1 \leq i \leq n$ and $u \in \mathbb{R}$.

Proof. We repeat the calculations from the proof of Lemma 2 in the general setting. When $t \neq 0$, the change of variable $w' = u - tw$ gives

$$\begin{aligned} \alpha^{*t}(u) &= \text{Sign}(t) \int_{\mathbb{R}^n} \Phi_n(u - tw') \alpha(w') dw', \\ \frac{d}{dt} \alpha^{*t}(u) &= - \int_{\mathbb{R}^n} (d\Phi_n(w).w + (n-1)\Phi_n(w)) \alpha(u - tw) dw. \end{aligned}$$

One deduces:

$$\begin{aligned} \frac{\partial Q}{\partial u} &= \sum_{i=1}^n \eta_i \int_{\mathbb{R}^n} \Phi_n(w) \alpha_i(u - tw) dw \\ &= \sum_{i=1}^n \frac{\eta_i}{|\eta_i|^n} \int_{\mathbb{R}^n} \Phi_n\left(\frac{u - w'}{\eta_i}\right) \alpha_i(w') dw', \\ \frac{\partial Q}{\partial \eta_i} &= \int_{u_*}^u \frac{d}{d\eta_i} \alpha_i^{*\eta_i} \\ &= - \int_{\mathbb{R}^n} (d\Phi_n(w).w + (n-1)\Phi_n(w)) \left(\int_{x=u_* - \eta_i w}^{x=u - \eta_i w} \alpha_i(w) \right) dw, \\ \frac{\partial^2 Q}{\partial u \partial \eta_i} &= - \int_{\mathbb{R}^n} (d\Phi_n(w).w + (n-1)\Phi_n(w)) \alpha_i(u - \eta_i w) dw, \\ \frac{\partial^2 Q}{\partial u^2} &= \sum_{i=1}^n \int_{\mathbb{R}^n} d\Phi_n(w) \alpha_i(u - \eta_i w) dw, \end{aligned}$$

and finally:

$$\frac{\partial^2 Q}{\partial \eta_i \partial \eta_j} = \delta_{i,j} \int_{\mathbb{R}^n} (d\Phi_n(w).w + (n-1)\Phi_n(w)) [\alpha_i(x - \eta_i w).w]_{x=u_*}^{x=u} dw.$$

It is not difficult to verify that 1.–3. hold. □

The proof of Lemma A.3 is now complete. □

A.2 The volume-preserving case

Proposition A.4 *Let M be a Riemannian manifold endowed with a volume form μ and W be a C^1 submanifold satisfying*

$$\dim(W) \leq \operatorname{codim}(W).$$

Centered at any point $z \in W$, there exists a disk $D = \overline{B_W(z, r_0)}$ of W such that, for every neighborhood $U \subset M$ containing D and every $\varepsilon > 0$, there exists $\delta > 0$ with the following property.

For every C^1 diffeomorphism $\psi : D \rightarrow D$ satisfying:

- a. $\psi = \operatorname{Id}$ on a neighborhood of ∂D , and
- b. $d_{C^1}(\psi, \operatorname{Id}) < \delta$,

there exists $\varphi \in \operatorname{Diff}_\mu^1(M)$ such that:

1. $\varphi = \operatorname{Id}$ on $M \setminus U$
2. $\varphi = \psi$ on D , and
3. $d_{C^1}(\varphi, \operatorname{Id}) < \varepsilon$.

Proof. Let $n = \dim(M)$. By a local change of coordinates, we may assume that μ is the standard volume form $dx_1 \wedge \cdots \wedge dx_n$ on a neighborhood of the origin in \mathbb{R}^n . By composing these coordinates with an isometry of \mathbb{R}^n , we may further assume that D is the graph of a C^1 -function $h : \mathbb{R}^k \rightarrow \mathbb{R}^{n-k}$, where $k \leq n/2$. The final change of coordinates

$$(x_1, \dots, x_n) \mapsto ((x_1, \dots, x_k), (x_{k+1}, \dots, x_n) - h(x_1, \dots, x_k))$$

preserves volume. Applying this change of coordinates, we may assume that D lies in the coordinate plane $\{x_{k+1} = x_{k+2} = \cdots = x_n = 0\} \simeq \mathbb{R}^k$. Now we apply the symplectic perturbation result (Proposition A.2) inside the space $\{x_{2k+1} = \cdots = x_n = 0\} \simeq \mathbb{R}^{2k}$ to obtain a local C^1 symplectomorphism φ_0 of $\{x_{2k+1} = \cdots = x_n = 0\}$ that agrees with ψ on D . This symplectomorphism is C^1 -isotopic to the identity through symplectomorphisms $\{\varphi_t\}_{t \in [0,1]}$, where $\varphi_1 = \operatorname{Id}$ (to obtain this isotopy, just choose a smooth isotopy of the generating function for ψ to the generating function for the identity).

Now we extend φ_0 to \mathbb{R}^n using this isotopy to obtain a locally-supported volume-preserving diffeomorphism that agrees with ψ on D . More precisely, choose an appropriate C^∞ bump function $\rho : \mathbb{R}^{n-2k} \rightarrow [0, 1]$, and set

$$\varphi(x_1, \dots, x_n) = (\varphi_{\rho(\|(x_{2k+1}, \dots, x_n)\|)}(x_1, \dots, x_{2k}), x_{2k+1}, \dots, x_n).$$

This is the desired map φ . □

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