Stable ergodicity of the time-one map of a geodesic flow

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April 18, 1997

Introduction

Let M be a closed, connected Riemannian manifold with volume form ω. A C^2, volume-preserving diffeomorphism $f : M \to M$ is *stably ergodic* if there is a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^2_\omega(M)$, the space of C^2, volume-preserving diffeomorphisms of M, such that every $g \in \mathcal{U}$ is ergodic. Grayson, Pugh and Shub showed that if $\varphi_t : T_1S \to T_1S$ is the geodesic flow on the unit tangent bundle of a closed surface S of constant negative curvature, then the time-one map φ_1 is stably ergodic ([5]). This is the first known example of a stably ergodic diffeomorphism that is not structurally stable. In this paper, we extend this result to variable negative curvature. More precisely, our Main Theorem states:

Main Theorem: If S is a closed, connected negatively-curved Riemannian surface, and if $\varphi_t : T_1S \to T_1S$ is the geodesic flow, then the time-one map φ_1 is stably ergodic.

This paper is meant to serve as a model for proving stable ergodicity for diffeomorphisms of a nonalgebraic origin. These diffeomorphisms have local behavior that is uniformly controlled and *qualitatively* looks the same at every point, but their *quantitative* local behavior varies from point to point (subject to the qualitative constraints).

For a closed manifold M, let $\mathrm{Diff}^k(M)$ denote the space of C^k diffeomorphisms of M in the C^k topology. If M is equipped with a smooth volume form ω, then denote by $\mathrm{Diff}^k_\omega(M)$ the subspace of $\mathrm{Diff}^k(M)$ consisting of those diffeomorphisms that preserve ω. How the ergodic diffeomorphisms "sit inside" the space $\mathrm{Diff}^k_\omega(M)$ is a fundamental question, without a complete answer at this date.[1] When M is a circle, ergodicity is determined by rotation number, and it is not hard to see that ergodic diffeomorphisms form a residual set in $\mathrm{Diff}^k_\omega(M)$, (for any k) whereas non-ergodic diffeomorphisms are dense. On the other hand, when M is a surface, and k is sufficiently large, KAM theory guarantees that there are open sets of nonergodic diffeomorphisms, the nonergodicity

[1] The question for $k = 0$, i.e. for the space of homeomorphisms $\mathrm{Homeo}_\mu(M)$, is settled by a classical result of Oxtoby and Ulam, which states that ergodic diffeomorphisms are residual.

manifested in a quite complicated dynamical picture. If M is a torus (for example) there are also the Anosov diffeomorphisms.

Volume-preserving Anosov diffeomorphisms are stable in at least two senses of the word. Being uniformly hyperbolic, they are structurally stable. Since all C^2 volume-preserving Anosov diffeomorphisms are ergodic ([1]) and the Anosov property is open in the C^k topology, Anosov diffeomorphisms are stably ergodic. On any manifold M supporting an Anosov diffeomorphism[2] there is then an open family in $\mathrm{Diff}^2_\omega(M)$ of ergodic diffeomorphisms, namely the Anosov diffeomorphisms.

Until recently, Anosov diffeomorphisms were the only class of diffeomorphisms known to be stably ergodic.

Question: Which manifolds support stably ergodic diffeomorphisms?

In [4], Brin and Pesin investigated the dynamical properties of diffeomorphisms like the time-one map of an Anosov flow, which they termed "partially hyperbolic." They showed that such maps have certain transitivity properties that are preserved under perturbations. Extending some of the techniques in [4], Grayson, Pugh and Shub showed in [5] that if S is a closed surface of constant negative curvature and φ is its geodesic flow, then the time-t map φ_t is is stably ergodic. This provided the first example of a stably ergodic diffeomorphism that is not an Anosov diffeomorphism, and hence the first example of a stably ergodic map that is not also structurally stable.

The results of [5] readily generalize to other algebraic examples of partially hyperbolic diffeomorphisms on three-manifolds [3] They do not readily generalize to non-algebraic examples. In this paper, we indicate how some of these obstacles can be overcome, focusing on the specific example of the geodesic flow for a surface of variable negative curvature.

We review some preliminary results and definitions in Section 1. In Section 2, we show that for a compact, negatively-curved surface S, the geodesic flow has the "stable/unstable accessibility" property described in [5]. We use the geometry of horospheres in the natural compactification of the universal cover $\tilde{S}$ to prove this.

In Section 3, we prove a regularity result about the strong stable and unstable foliations of the map φ_1. We show that these foliations are Hölder-continuous, with Hölder exponent arbitrarily close to 1. Further, we show that the Hölder constant of the foliation holonomy map is uniformly bounded in terms of the first two derivatives of φ_1. We use this to show that the Hölder continuity of the stable and unstable foliations is uniformly controlled under smooth perturbations of φ_1. The results in Section 3 are of independent interest and extend some of the results of Hirsch, Pugh and Shub in [9].

In Section 4, we use the results of the previous sections to prove the Main Theorem.

[2]All known examples are nilmanifolds or are finitely covered by nilmanifolds.

[3]See the footnote in Section 1.3.

0.1 Acknowledgment

The contents of this paper originally comprised a Ph.D. dissertation at the University of California, Berkeley. The author would like her advisor, Charles Pugh, for inspiration, guidance and support.

1 Preliminaries

1.1 Geodesic Flows

A Riemannian structure on a manifold S naturally gives rise to a flow on the tangent bundle TS. This flow is called the *geodesic flow* because its trajectories project to geodesics in S under the tangent bundle projection $\pi : TS \to S$. We will consider the induced flow on the compact submanifold T_1S, the unit tangent bundle. While this flow is defined on the unit tangent bundle of S, for the sake of brevity we will at times call it the geodesic flow *for* S. For a construction of the geodesic flow see, for example, [13].

The geodesic flow $\varphi : T_1S \times \mathbf{R} \to \mathbf{R}$ has several important properties. For any $v \in T_1S$, the unit-speed curve $\pi \circ \varphi_t(v) : \mathbf{R} \to S$ is a locally length-minimizing curve in S. The tangent bundle TS has a natural Riemannian metric (the "Sasake metric"), given by first decomposing the tangent space T_vTS at a point $v \in TS$ into its horizontal and vertical components H_v and V_v. In this metric, the vertical subspace V_v, which is the kernel of $T_v\pi$, inherits the inner product from $T_{\pi v}S$ via translation along v, and the horizontal space H_v, which is determined by the Riemannian connection, inherits the inner product from $T_{\pi v}S$ via pullback under $T_v\pi$. The metric on TS is then given by the direct sum of these two forms on $T(TS) = V \oplus H$. The geodesic flow preserves the Riemannian volume form ω given by this metric. Further, the flow preserves the trace of this metric on the submanifold T_1S. The measure m on T_1S induced by this metric is the Liouville measure. Locally, m is the product of the Riemannian measure on S with normalized Lebesgue measure on the fibers of TS.

The geodesic flow for a closed surface (or n-manifold) of negative sectional curvatures is ergodic with respect to m, and in fact Bernoulli. Ergodicity was proved first by Hedlund and Hopf in the constant curvature case (see [6] and [10]). In the same paper, Hopf also showed ergodicity of φ for a surface of finite area whose negative curvature and derivative of curvature are bounded, for instance, a closed negatively-curved surface. Anosov proved ergodicity in the general case case in [1], and Ornstein and Weiss proved in [15] that the flow is Bernoulli, meaning for every t the time - t diffeomorphism φ_t is conjugate to a Bernoulli shift. In particular, φ_t is ergodic. An alternate proof of this fact follows from the proof of Theorem 4.1.

1.2 Anosov Flows and Diffeomorphisms

A good reference for this section is [18]. A flow $\psi_t : M \to M$ is an *Anosov flow* if there are constants $C > 0$, $\lambda < 1$ and a splitting of the tangent bundle into Tf-invariant subbundles, called an *Anosov splitting:*

$$TM = H^u \oplus H^c \oplus H^s,$$

where H^c is spanned by the generating vector field $\dot{\psi}$ and:

$$\begin{aligned} \|T\psi_t(v)\| &\leq C\lambda^t \|v\| \quad \text{for all } t \geq 0, v \in H^s \\ \|T\psi_{-t}(v)\| &\leq C\lambda^t \|v\| \quad \text{for all } t \geq 0, v \in H^u \end{aligned}$$

The geodesic flow for a closed, negatively-curved manifold is an Anosov flow.

Theorem 1.1 *Let S be a complete Riemannian n-manifold of bounded negative sectional curvatures $-b^2 < K < -a^2 < 0$, with $a, b > 0$. Let $\varphi_t : T_1S \to T_1S$ be the geodesic flow. Then there is an Anosov splitting*

$$T(T_1S) = H^u \oplus H^c \oplus H^s$$

with $dim(H^u) = dim(H^s) = n - 1$ and

$$\begin{aligned} \|T\varphi_t(v)\| &\leq \tfrac{b}{a} e^{-at} \|v\| \quad \text{for all } t \geq 0, v \in H^s \\ \|T\varphi_{-t}(v)\| &\leq \tfrac{b}{a} e^{-at} \|v\| \quad \text{for all } t \geq 0, v \in H^u, \end{aligned}$$

with respect to the canonical Riemannian metric on T_1S (see Section 1.1).

A proof of Theorem 1.1 can be found in [8], [13], Chapter 3 or [2], Appendix 21.

For any Anosov flow, the distributions $H^s, H^u, H^c, H^u \oplus H^c, H^s \oplus H^c$ are uniquely integrable and integrate to give foliations $\mathcal{W}^u, \mathcal{W}^s, \mathcal{W}^c, \mathcal{W}^{cu}, \mathcal{W}^{cs}$ respectively ([9]). The foliation $\mathcal{W}^c$ is the *center* foliation, whose leaves consist of orbits of φ. The foliations $\mathcal{W}^u$ and $\mathcal{W}^s$ are called the *strong unstable* and *strong stable* foliations, respectively, and the foliations $\mathcal{W}^{cu}$ and $\mathcal{W}^{cs}$ are called the *center unstable* and *center stable* foliations. The latter are also called the *weak unstable* and *weak stable* foliations, respectively. The leaves of these foliations are injectively immersed, C^r submanifolds of M (if f is C^r). For geodesic flows, the strong stable and unstable foliations are also called the horocyclic foliations. We will denote them by $\mathcal{H}^u$ and $\mathcal{H}^s$. Some of their regularity properties are discussed in [8]. We discuss their geometric interpretation in Chapter 2.

For a closed, negatively-curved surface, the unit tangent bundle is three-dimensional, and it follows from, e.g. [18], Theorem 5.18, that the stable and unstable foliations are C^1. Hopf used this smoothness in a crucial way to prove ergodicity for these flows. Anosov showed in [1] that, while these foliations may fail to be smooth in higher dimensions, they do have the property of being absolutely continuous. He used this weaker property to prove ergodicity of the geodesic flow for any closed, negatively-curved manifold. This property will be used in our proof of the Main Theorem as well, so we review it here.

Let D_0 and D_1 be smooth, $n-s$-dimensional embedded disks transverse to the s-dimensional foliation $\mathcal{F} = \{F(p)\}_{p \in M}$. Let points $p \in D_0$ and $p' \in D_1$ and a path $\gamma : [0, 1] \to F(p)$ with $\gamma(0) = p$, $\gamma(1) = p'$ be given. Then there is a subdisk $D_0' \subset D_0$

such that the holonomy $h_\gamma : D_0' \to D_1$ is a homeomorphism onto its image. The foliation $\mathcal{F}$ is *absolutely continuous* if for every such pair of transversals and every such (leafwise) path γ, the holonomy h_γ is absolutely continuous; i.e. it takes sets of measure zero in D_0' to sets of measure zero in D_1. (The measure in question is the Riemann-Lebesgue measure of dimension $n-s$). Anosov proved in [1] that if ψ is a C^2, volume-preserving Anosov flow then the foliations $\mathcal{W}^u$ and $\mathcal{W}^s$ are absolutely continuous. Further, he showed that for every holonomy $h_\gamma : D_0' \to D_1$, there exists a continuous Jacobian $J : D_0' \to \mathbf{R}$ such that for every measurable subset $A \subset D_0'$,

$$\int_A J\, dm = m(h_\gamma(A)),$$

where m denotes Riemannian measure on D_0' or D_1. (A proof of this fact can be found in [16]).

Because φ_t restricted to a leaf of $\mathcal{W}^u$ is an expanding diffeomorphism, it follows from [17] that the leaves of $\mathcal{W}^u$ are diffeomorphic to $\mathbf{R}^{n-1}$ (as are the leaves of $\mathcal{W}^s$). This implies that the holonomy h_γ is determined by $\gamma(0)$ and $\gamma(1)$.

1.3 Normal Hyperbolicity

Ignoring for a moment that φ_1 comes from a flow, we could alternately describe it as a diffeomorphism f with a Tf-invariant splitting $TM = H^u \oplus H^c \oplus H^s$ such that H^u and H^s are expanded and contracted exponentially under Tf and $\|Tf|_{H^c}\| = 1$. For much of the proof of Theorem 4.1 this is all we need to know about f. What restricts the generality of our result is that this description of φ_1 is not sufficient for the entire proof. The reason is that we cannot conclude from this information alone that the center distribution H^c is integrable. There *is* a center manifold theory for diffeomorphisms of this nature, and so we are assured the existence of local invariant center manifolds ([18], Theorem III.8). These manifolds are tangent to the center distribution at a single point, and are not in general unique. The assumption that $\|Tf|_{H^c}\|$ is close to 1 (which would be the case if f were a perturbation of φ_1), does not imply the existence of a smooth center manifold everywhere tangent to H^c.[4]

Since we will use center manifolds to carry out the proof of our Main Theorem, we need to use more information about φ_t. What we would like to say is that φ_t is hyperbolic "modulo" the foliation $\mathcal{H}^c$. The theory of normal hyperbolicity developed in [9] provides us with the tools.

Definition: (see [9], p. 116.) Let $\mathcal{F}$ be a continuous foliation whose leaves are C^1. A C^r diffeomorphism $f : M \to M$ is *r-normally hyperbolic to $\mathcal{F}$* if f permutes the leaves

[4]That is, there is a diffeomorphism f of a closed manifold M with (smooth) Tf-invariant splitting $TM = E^u \oplus E^c \oplus E^s$, and a $\rho > 1$ such that for all $p \in M$,

$$m(T_pf|_{E^u}) > \rho > \|T_pf|_{E^c}\| > m(T_pf|_{E^c}) > \frac{1}{\rho} > \|T_pf|_{E^s}\|$$

but the distribution E^c is not integrable (an example can be found by inspecting Smale's construction of an Anosov diffeomorphism on a six-dimensional nilmanifold in [19]).

of $\mathcal{F}$ and there is a Tf-invariant splitting:

$$TM = N^u \oplus T\mathcal{F} \oplus N^s$$

such that:

$$\begin{array}{rl} \inf m(T_pf|_{N^u}) > 1, & \sup \|T_pf|_{N^s}\| < 1 \\ \inf m(T_pf|_{N^u})\|T_pf|_{\mathcal{F}}\|^{-r} > 1, & \sup \|T_pf|_{N^s}\| m(T_pf|_{\mathcal{F}})^{-r} < 1, \end{array}$$

where $m(A) := \|A^{-1}\|^{-1}$ denotes the conorm of the operator A. An Anosov diffeomorphism is normally-hyperbolic to the foliation by points $\mathcal{F} = \{p\}_{p\in M}$. If ϕ is an Anosov flow, then ϕ_1 is normally hyperbolic to its orbit foliation. Incidentally, the converse does not hold: there are diffeomorphisms normally hyperbolic with respect to a smooth, 1-dimensional foliation that are not conjugate to the time-one map of any Anosov flow. [5]

Recall that an ϵ-pseudo orbit for a diffeomorphism f is a bi-infinite sequence $\{p_n\}_{n\in\mathbf{Z}}$ of points in M such that $d(f(p_n), p_{n+1}) \leq \epsilon$, for all $n \in \mathbf{Z}$. Suppose that the foliation $\mathcal{F}$ is given by a family of plaques $\mathcal{P}$ (see Chapter 3 for a definition). If $f : M \to M$ preserves the foliation $\mathcal{F}$, then the pseudo-orbit $\{p_n\}$ *respects* $\mathcal{P}$ if $f(p_n)$ and p_{n+1} lie in a common plaque of $\mathcal{P}$. We now generalize the notion of expansiveness to normally-hyperbolic diffeomorphisms.

Definition: ([9], p. 116) $f : M \to M$ is *plaque expansive* if there exists an $\epsilon > 0$ with the following property. If $\{p_n\}$ and $\{q_n\}$ are ϵ-pseudo orbits which respect $\mathcal{P}$ and if $d(p_n, q_n) \leq \epsilon$ for all n, then for each n, p_n and q_n lie in a common plaque.

Remark: If f is normally hyperbolic at $\mathcal{F}$ and $\mathcal{F}$ is a smooth foliation, then f is plaque expansive (see [9], p. 117). Hence if ψ_t is an Anosov flow, then ψ_1 is plaque-expansive at its orbit foliation.

Theorem 1.2 *(see [9], Theorem* 7.1, *p.* 117*) Let $\mathcal{F}_0$ be a continuous foliation of the closed manifold M whose leaves are C^r injectively immersed. Let f_0 be r-normally hyperbolic to $\mathcal{F}_0$. If f_0 is plaque expansive, then the pair $(f_0, \mathcal{F}_0)$ is* structurally stable. *That is, there is a neighborhood $\mathcal{U}$ of f_0 in Diff(M) such that, for every $f \in \mathcal{U}$, there is a foliation $\mathcal{F}$ and a homeomorphism h of M such that*

1. *h carries each leaf of $\mathcal{F}_0$ to a leaf of $\mathcal{F}$.*
2. *$hf_0(L) = fh(L)$, for every $L \in \mathcal{F}_0$.*
3. *f is r-normally hyperbolic and plaque expansive at $\mathcal{F}$.*

[5] For an example, one can take an automorphism of a compact quotient of the three-dimensional Heisenberg group with eigenvalues 1, $\lambda > 1$ and λ^{-1}. As a side note, these Heisenberg examples are in fact stably ergodic. We leave the proof of this fact as a lengthy exercise; the proof is almost identical to the proof in [5].

Theorem 1.2 implies structural stability for Anosov flows. Since the time-1 map of an Anosov flow is also normally hyperbolic and plaque-expansive with respect to the flow's orbit foliation, we also obtain the following.

Corollary 1.3 (Center manifolds persist under perturbations.) *Let ϕ be a C^1 Anosov flow on the closed manifold M. Then there is a neighborhood $\mathcal{U}$ of ϕ_1 in $\mathrm{Diff}^1(M)$ such that for every $f \in \mathcal{U}$, there is a unique Tf-invariant splitting $TM = E^u \oplus E^c \oplus E^s$ which approximates the Anosov splitting for φ. Tangent to this splitting are unique f-invariant foliations $\mathcal{W}^c(f)$, $\mathcal{W}^u(f)$, and $\mathcal{W}^s(f)$. The foliation $\mathcal{W}^c(f)$ is homeomorphic to the orbit foliation for ϕ.*

Strictly speaking, the existence of the foliations $\mathcal{W}^u(f)$ and $\mathcal{W}^s(f)$ is not a direct corollary to Theorem 1.2 but follows from the existence of an invariant splitting such that E^u is exponentially expanded, *etc.*, plus the strong stable manifold theorem for invariant sets (see, e.g. [18], p. 79 ff.).

Since ϕ_1 is normally hyperbolic (in fact, normally expanding) and plaque expansive at the weak-stable foliation (and ϕ_1^{-1} is normally hyperbolic and plaque expansive at $\mathcal{W}^{cu}$) we also obtain:

Corollary 1.4 (Weak unstable manifolds persist under perturbations.) *With ϕ as in Corollary 1.3, there is a neighborhood $\mathcal{U}'$ of ϕ_1 in $\mathrm{Diff}^1(M)$ such that for every $f \in \mathcal{U}'$, there are unique f-invariant foliations $\mathcal{W}^{cs}(f)$ and $\mathcal{W}^{cu}(f)$ tangent to $E^u \oplus E^c$ and $E^s \oplus E^c$, respectively.*

In Section 3 we show that if f is C^1-close to φ_1, then the $\mathcal{W}^u(f)$-holonomy maps between local $\mathcal{W}^{cs}(f)$-leaves are uniformly Hölder continuous (and similarly for the $\mathcal{W}^s(f)$-holonomy between local $\mathcal{W}^{cu}(f)$-leaves). In Section 4, we show that these holonomies are uniformly absolutely continuous, and C^1 and uniformly nearly isometric when restricted to $\mathcal{W}^c(f)$-leaves.

2 Accessibility

2.1 A Three-Legged Lemma

Let S be a negatively-curved surface, with geodesic flow $\varphi : T_1S \times \mathbf{R} \to T_1S$. As in the previous section, denote by $\mathcal{H}^u, \mathcal{H}^s$ the strong unstable and stable foliations for φ_t and by H^u and H^s the corresponding line-fields. Since φ_t is a geodesic flow, the plane field $H^u \oplus H^s$ it is *contact*: there is a smooth one form τ such that $H^u \oplus H^s = \ker(\tau)$ and $\tau \wedge d\tau$ is nonvanishing.[6] Hence the plane field $H^u \oplus H^s$ is totally non-integrable: the set of paths originating at a point p and everwhere tangent to $H^u \oplus H^s$ will fill a whole neighborhood of p. In this section we examine a coarse version of this infinitessimal property. We show that the set of all paths everywhere tangent either to H^u or to H^s fill up T_1S, uniformly, and in a way that persists under perturbations of φ_1.

[6] τ is the canonical contact one-form pdq. See [11], p. 230.

Define a $\mathcal{H}^{u,s}$*-path* to be a path $\psi : [0,1] \to T_1S$ consisting of a finite number of consecutive arcs, or *legs*, each arc everywhere tangent to either H^u or H^s; that is, lying wholly in a leaf of $\mathcal{H}^u$ or $\mathcal{H}^s$. Fix the convention that the first leg of a $\mathcal{H}^{u,s}$-path lies in $\mathcal{H}^s$. Similarly, for f close to φ_1 in the C^1 sense, define a $\mathcal{W}^{u,s}$-path to consist of consecutive $\mathcal{W}^u(f)$, $\mathcal{W}^s(f)$ legs, where $\mathcal{W}^u(f)$, $\mathcal{W}^s(f)$ are defined in Corollary 1.3. If such a path has n legs, then call it an n*-legged path.*

Lemma 2.1 *There is a neighborhood $\mathcal{V}$ of φ_1 in Diff$^2(T_1S)$ and a constant $N > 0$ such that for every $f \in \mathcal{V}$ and any two points $v_0, v_1 \in T_1S$, there is a three-legged $\mathcal{W}^{u,s}(f)$-path from v_0 to v_1 of length $\leq N$.*

Recently, Katok and Kononenko [12] have provided an alternate proof of a similar result, which they use to prove cocycle stability results for certain partially hyperbolic systems. They show that for a contact Anosov flow the stable and unstable foliations have this accessibility property and that this property is stable under smooth perturbations of the time-one map. Their proof has a different flavor and for completeness we describe it here.

The type of map studied in [12] is an Anosov flow φ that preserves a contact 1-form τ.[7] This class includes all geodesic flows for negatively-curved manifolds. The definition of an $\mathcal{H}^{u,s}$ - path for a general Anosov flow with Anosov splitting $TM = H^u \oplus H^c \oplus H^s$ is the same as in the specific case of a geodesic flow: it is a path consisting of finitely many smooth legs, each wholly tangent to either H^u or H^s. Since the time-one map φ_1 is normally hyperbolic to the orbit foliation, it is normally structurally stable, and any sufficiently C^1-close diffeomorphism f will have stable, unstable and center manifolds, by Theorem 1.2, and we define an n-legged $\mathcal{W}^{u,s}$ - path for such a perturbation in the obvious way.

Proposition 2.2 *([12],) Let $\varphi : M \to M$ be a contact Anosov flow on a closed manifold. Then there is a neighborhood $\mathcal{V}$ of φ_1 in Diff$^2(M)$ and an integer $j > 0$ such that for every $\epsilon > 0$ and every $f \in \mathcal{V}$ there exists a $\delta > 0$ such that for every $p, q \in M$ with $d(p,q) < \delta$, there exists a j-legged $\mathcal{W}^{u,s}$ - path from p to q of length less than ϵ.*

Roughly, the proof of Proposition [12] is as follows. Because $H^u \oplus H^s$ is a contact distribution, any two points $p, q \in M$ can be connected by a smooth path everywhere tangent to $H^u \oplus H^s$. First take p and q lying in the same center leaf. Using the transverse local product structure of $\mathcal{H}^{cu}$ and $\mathcal{H}^{cs}$, one can approximate this smooth path by an m-legged path, for some m, connecting p to a point q', with $d(q,q') << d(p,q)$. Then the codimension 1 manifold

$$C(q') := \{x \in \mathcal{H}^u_{loc}(y) \,|\, y \in \mathcal{H}^s_{loc}(q')\}$$

intersects the center manifold $\mathcal{H}^c(p)$ in a point p' distinct from p. This gives an $m+2$-legged path from p to p'. From here it is straightforward to construct a 1-parameter

[7] This means that $\tau(\dot{\varphi}) = 1$, $d\tau(\dot{\varphi}, \cdot) = 0$, and $\tau \wedge (d\tau)^{n-1}$ is nondegenerate.

family of $m+2$-legged paths connecting p to a neighborhood of p in $\mathcal{H}^c(p)$. Thus any point q in a neighborhood of p can be connected to p by an $m+4$-legged path: first connect q to $y \in \mathcal{H}^c(p) \cap C(q)$ by a 2-legged path and then connect y to p by a $m+2$-legged path. This proves accessibility for the unperturbed map φ_1.

For f C^2-close to φ_1, the foliations $\mathcal{W}^a(f)$ ($a \in \{u, s, c, cs, cu\}$) locally uniformly approximate the corresponding foliations for φ_1, and so for p and q on the same center leaf, there is an $m+4$-legged path connecting p to a point q', with $d(q,q') << d(p,q)$. Following the argument above, we can conclude there is a uniform neighborhood of p covered by $m+8$-legged $\mathcal{W}^{u,s}$-paths.□

2.2 Hadamard Manifolds, Horospheres, and Horoballs

The proof of Lemma 2.1 uses several properties of connected, simply-connected complete Riemannian manifolds of nonpositive sectional curvature (called *Hadamard manifolds*) which we recall here (see [3]). Many of the properties of the standard hyperbolic plane (i.e. the Poincaré disk model) carry over to Hadamard manifolds. Most notably, just as the Poincaré disk has a compactification as a ball in $\mathbf{R}^2$, with boundary a circle (1-sphere), so does any Hadamard manifold X have a *sphere at infinity*. Say that two unit speed geodesics $\gamma_i : \mathbf{R} \to X$ are *asymptotic* if there is a constant c such that $d(\gamma_1(t), \gamma_2(t)) \leq c$, for all $t \geq 0$. The sphere at infinity, $X(\infty)$ is the set of equivalence classes of geodesics under this relation. The compactification $\overline{X} = X \cup X(\infty)$ is topologized as follows: first, given $x \in X$, $z_1, z_2 \in \overline{X}$, let

$$\angle_x(z_1, z_2) = \angle(\dot\gamma_1(0), \dot\gamma_2(0)),$$

where γ_i is a geodesic from x to z_i. For $x \in X, z \in X(\infty), \epsilon > 0$, let

$$C_x(z, \epsilon) = \{y \in \overline{X} \mid y \neq x \text{ and } \angle_x(z, y) < \epsilon\}.$$

The "cone topology" on $\overline{X}$ is generated by the topology on X and the cones $\{C_x(z,\epsilon)\}$. In the cone topology $X(\infty)$ is homeomorphic to a codimension-1 sphere.

In the hyperbolic plane $\mathbf{H}$ (of constant curvature -1) there are distinguished curves of constant geodesic curvature 1 called *horospheres*. In the Poincaré disk model, horospheres are represented by circles tangent to the boundary of the disk. For every point on the sphere at infinity there is a unique family of horospheres tangent to that point and every geodesic asymptotic to that point crosses every horosphere tangent to that point orthogonally.

If $\tilde\varphi : \mathbf{R} \times T_1\mathbf{H} \to T_1\mathbf{H}$ is the geodesic flow for $\mathbf{H}$ then for $v \in T_1\mathbf{H}$ the leaves $\tilde{\mathcal{H}}^s(v), \tilde{\mathcal{H}}^u(v)$ of the strong stable and unstable foliations for $\tilde\varphi$ project to horospheres in $\mathbf{H}$ through the basepoint of v, one tangent to the forward asymptote of v, and the other to the negative asymptote (see Figure 1). An analogous picture holds for any Hadamard manifold. For those manifolds with bounded negative curvature, horospheres are the traces in $\tilde S$ of the strong stable and unstable manifolds for the (lifted) geodesic flow.

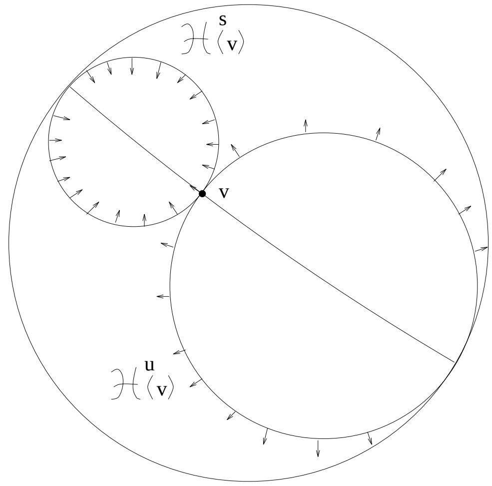

Figure 1: Strong stable and unstable leaves project to horospheres.

We proceed to definte horospheres for a Hadamard manifold X. Let $\gamma : \mathbf{R} \to X$ be a unit-speed geodesic with $\gamma(0) = x$. Define a function $h_{t,\gamma} : X \to \mathbf{R}$ as follows:

$$h_{t,\gamma}(y) = d(y, \gamma(t)) - t.$$

A function g on a Riemannian manifold X is *(strictly) convex* if for every geodesic $\gamma : \mathbf{R} \to X$, the function $g \circ \gamma$ is (strictly) convex.

Lemma 2.3 *The function $h_{t,\gamma}$ converges to a finite limit h_γ as $t \to +\infty$, and this limit is a C^1, convex function.*

For a proof of Lemma 2.3 see ([2], Appendix 21), or ([3], Lemma 3.4). The limiting function h_γ is called a *Busemann function* or a *horofunction*. In fact, Busemann functions for nonpositively curved manifolds are C^2 (for a proof, see [7], Prop. 3.1). A *horosphere* is a level set of the Busemann function h_γ.

Note that if $\gamma(0) = x$, then $x \in h_\gamma^{-1}(0)$. Each horosphere $h_\gamma^{-1}(c)$ bounds a convex region called a *horoball*, $h_\gamma^{-1}(-\infty, c]$. Each horosphere(/function/ball) has a unique *center* $z \in X(\infty)$ such that every geodesic that crosses the horosphere orthogonally is asymptotic to its center, and *vice versa*. When the curvature of X is strictly negative and bounded away from 0 and ∞, the unit-speed geodesics orthogonal to a given horosphere converge to each other uniformly exponentially in forward time and diverge exponentially in backward time. It follows that the strong stable and unstable foliations of the geodesic flow on the unit tangent bundle T_1X, project to foliations of X by horospheres. More precisely, a unit tangent vector $v \in T_1X$ is tangent to a unique directed unit-speed geodesic, $\tilde{\varphi}_t(v)$. The horospheres corresponding to the geodesics $\tilde{\varphi}_t(v)$ and $\tilde{\varphi}_{-t}(v)$ are called, respectively, the *positive* and *negative horospheres* through v, and denoted $H^+(v)$ and $H^-(v)$. The set of unit vectors orthogonal to $H^+(v)$ and pointing toward the center (resp. pointing away from the center of $H^-(v)$) forms the leaf through v of the strong stable (resp. unstable) foliation for the geodesic flow on T_1X

When the curvature of X is bounded away from zero, the horospheres in X have the following additional properties:

1. For every two different points $z_0, z_1 \in X(\infty)$ on the sphere at infinity, there is a unique geodesic $\gamma : \mathbf{R} \longrightarrow X$ with $\gamma(-\infty) = z_0$ and $\gamma(\infty) = z_1$.

2. Two horoballs with different centers intersect in a bounded region in X.

3. Given a horoball B and a horofunction h with a different center than B, there exists an $a \in \mathbf{R}$ such that $h^{-1}(-\infty, a] \cap B = \emptyset$.

4. Busemann functions (and hence, horoballs) are strictly convex. If a geodesic intersects a horosphere, it does so either tangentially, orthogonally, or transversely, in exactly two points.

Properties $1-3$ are equivalent for Hadamard manifolds (for a proof, see, [3]). Property 4 is stronger. It follows, e.g., from Lemma 4.3 in [7].

Properties $1-4$ imply that horospheres behave nicely with respect to each other.

Lemma 2.4 *Let X be a two-dimensional Hadamard space of bounded negative curvature $-b^2 < K < -a^2$. If H_1 is a horosphere and B_2 is a horoball in X, then*

1. *$H_1 \cap B_2$ is path-connected.*

2. *$H_1 \cap int(B_2)$ is path-connected.*

Corollary 2.5 *With X as above, two distinct, nondisjoint horospheres in X intersect tangentially in a single point or transversely in exactly two points.*

Remarks:

1. Lemma 2.4 does not follow from strict convexity alone, as two strictly convex planar curves can intersect in arbitrarily many points.

2. In general, although it is not proved here, two horospheres in an n-dimensional Hadamard manifold of strictly negative sectional curvatures intersect in a point or in a topological $n-2$-sphere.

Denote by B_1 the horoball bounded by H_1 and by H_2 the horosphere bounding B_2. The lemma holds trivially if H_1 and H_2 have the same center, so assume that their centers are distinct.

If H_1 and H_2 meet tangentially, then the interiors of B_1 and B_2 must be disjoint, since an outward normal direction uniquely determines a horosphere. From the convexity of $B_1 \cap B_2$, we must have card$(H_1 \cap H_2) = 1$. In this case (1) and (2) follow immediately.

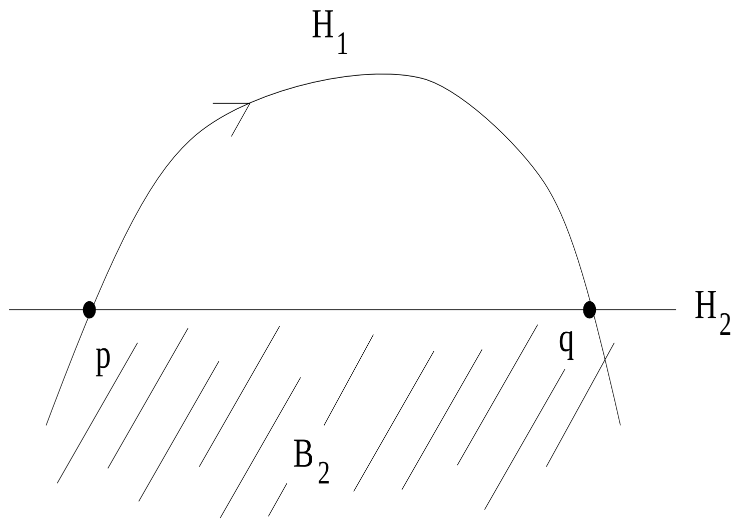

Figure 2: H_1 leaves and then re-enters B_2.

This leaves the case where H_1 and H_2 do not meet tangentially but cross transversely. Since $B_1 \cap B_2$ is bounded and convex, $H_1 \cap H_2$ must consist of a finite, even number of points and (1) and (2) are equivalent. If $H_1 \cap B_2$ is not path-connected then, as a parametrized simple curve, H_1 must leave and then re-enter B_2. We may assume that H_1 leaves B_2 at p and re-enters B_2 for the first time, to the right of p at q, as shown in Figure 2. We have not yet specified whether $\mathrm{int}(B_2)$ lies to the left or right of H_1. If $\mathrm{int}(B_1)$ lay to the left of H_1, then p and q could not be connected by a path in $B_1 \cap B_2$, violating the convexity of $B_1 \cap B_2$.

This leaves the possibility that $\mathrm{int}(B_1)$ lies to the right of H_1, as shown in Figure 3. Let h be the Busemann function that determines H_1, so that $B_1 = h^{-1}(-\infty, 0]$. By property 3 above, there exists an $a < 0$ such that $h^{-1}(-\infty, a] \cap B_2 = \emptyset$. This, together with the smoothness of h implies that there is a $b < 0$ and a point r lying on the arc of H_2 between p and q such that $h^{-1}(b)$ is tangent to H_2 at r and $h^{-1}(-\infty, b] \cap B_2 \neq \emptyset$. Since H_1 and H_2 have different centers, this gives a contradiction. $\square$

2.3 Proof of Lemma 2.1

Let $\tilde{S}$ be the universal cover of S equipped with the pulled-back Riemannian metric of S under the canonical projection $p : \tilde{S} \to S$. The Hadamard manifold $\tilde{S}$ has strictly negative curvature bounded, by compactness of S, between two constants: $-b^2 < K < -a^2$.

Choose $\tilde{v_0}, \tilde{v_1}$, lifts of v_0, v_1 under $Tp : T_1\tilde{S} \to T_1 S$ so that the directed geodesics in $\tilde{S}$ tangent to $\tilde{v_0}, \tilde{v_1}$ are *not* forward - asymptotic. This is possible because S is compact and so the limit set of the fundamental group of S acting on $\overline{\tilde{S}}$ is the entire sphere at

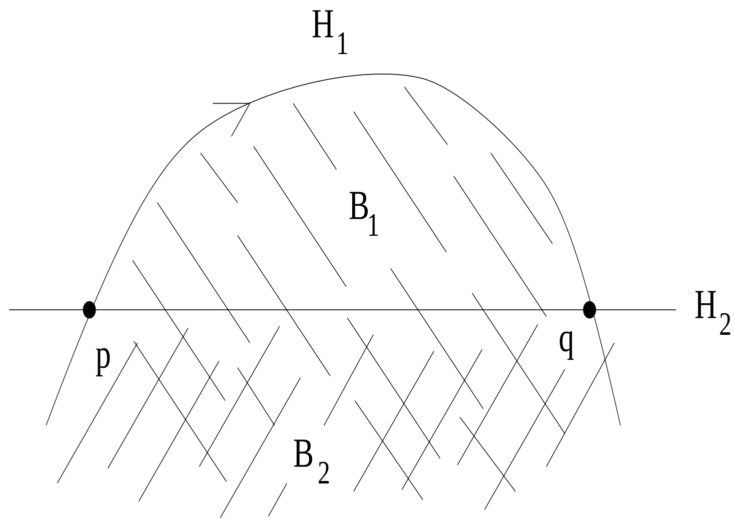

Figure 3: $\mathrm{int}(B_1)$ lies to the right of H_1.

infinity (see [13], Lemma 3.9.7, p. 370). Let

$$h_{v_i}(x) = \lim_{t\to\infty} d(x, \pi(\tilde{\varphi}_t(\tilde{v}_i))) - t$$

be the Busemann function associated with the vector $\tilde{v}_i$. By a similar argument, we may assume that the horospheres $H_i := h_{\tilde{v}_i}^{-1}(0)$ $(i = 0, 1)$ are disjoint (although this is not necessary for the argument, it simplifies the pictures). Let $c = d(H_0, H_1)$ and let $z_i \in \tilde{S}(\infty)$ be the center of the horosphere H_i.

For each $x \in H_0$, there is a unique unit-speed geodesic γ_x such that $\gamma_x(0) = x$ and $\gamma_x(-\infty) = z_0$. The horosphere $h_{\gamma_x}^{-1}(0)$ is tangent to the horosphere H_0, since the geodesic γ_x crosses H_0 orthogonally.

Define Δ, a continuous real-valued function on the horosphere H_0 as follows:

$$\Delta(x) = \inf_{y\in H_1} h_{\gamma_x}(y).$$

Because h_{γ_x} is a strictly convex function and H_1 bounds a strictly convex set, for a given x the infimum above is attained by a unique $y \in H_1$.

The function Δ measures the (signed) distance between the horospheres $H_x := h_{\gamma_x}^{-1}(0)$ and H_1. The distance between two horospheres is achieved along the geodesic connecting their centers (which intersects both horospheres orthogonally). If the interiors of the horoballs corresponding to $H_x := h_{\gamma_x}^{-1}(0)$ and H_1 are disjoint, then Δ is nonnegative. Otherwise, Δ is negative.

We are interested in Δ for the following reason: if $\Delta(x) = 0$, then, by Lemma 2.4 there exists a *unique* $y \in H_1$ such that $h_{\gamma_x}(y) = 0$, or in other words, the horosphere $h_{\gamma_x}^{-1}(0)$ is tangent to both H_0 and H_1. This tangent sphere provides a bridge between H_0 and H_1 from which we can construct an $\mathcal{H}^{u,s}$ path in $T_1\tilde{S}$ from $\tilde{v}_0$ to $\tilde{v}_1$.

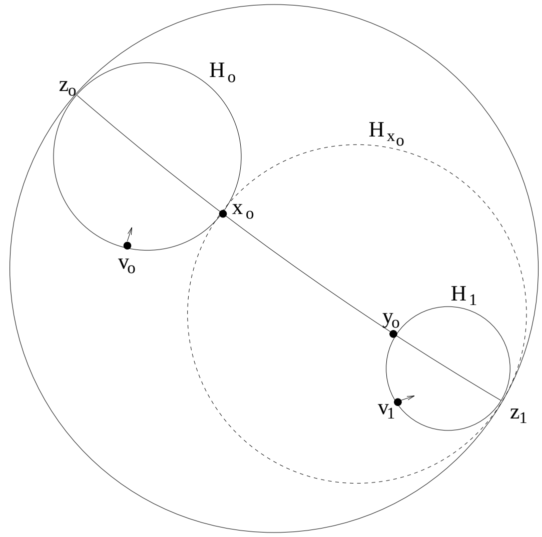

Figure 4: $\Delta(x_0) \leq 0$.

The horosphere H_0 is a one-dimensional subspace of $\tilde{S}$, homeomorphic to $\mathbf{R}$, so to show that Δ has a zero, it suffices to find $x_0, x_1 \in H_0$ such that $\Delta(x_0) \leq 0$ and $\Delta(x_1) \geq 0$.

Let γ_0 be the unique unit-speed geodesic in $\tilde{S}$ such that $\gamma_0(\infty) = z_0$ and $\gamma_0(-\infty) = z_1$. Let x_0 and y_0 be the (unique) intersections of the image of γ_0 with H_0 and H_1, respectively. Then, for all x sufficiently close to x_0, the horoballs corresponding to H_1 and H_x intersect, and $\Delta(x) \leq 0$. (In fact, $\Delta(x_0) = -\infty$). (See Figure 4).

To find x_1, we first find a geodesic γ_1 with $\gamma_1(-\infty) = z_1$ that intersects H_0 in 2 points. Up to parametrization, γ_0 is the unique geodesic from z_1 that intersects H_0 orthogonally. Pick $y_1 \in H_1$ near y_0 and let γ_1 be the unit-speed geodesic with $\gamma_1(-\infty) = z_1$ and $\gamma_1(0) = y_1$. Since horospheres and geodesics are smooth, if we pick y_1 sufficiently close to y_0, the intersection of γ_1 and H_0 will remain transverse, although no longer orthogonal. By property (4) above, this means that γ_1 intersects H_1 in exactly 2 points. Let $z_2 = \gamma_1(\infty)$. Let γ_2 be a geodesic with $\gamma_2(-\infty) = z_2$ and $\gamma_2(\infty) = z_0$ (See Figure 5). Let x_1 be the intersection of the image of γ_2 with H_0. We claim that $\Delta(x_1) \geq 0$.

Since H_{x_1} is tangent to H_0, the interiors of their associated horoballs are disjoint. The geodesic γ_1 crosses the horosphere H_{x_1} exactly once. Hence γ_1 crosses H_{x_1} *after* crossing H_0. This means that $\Delta(x_1) \geq d(H_0, H_1) = c > 0$.

Hence there exists a horosphere H_2 tangent to both H_0 and H_1. Note that the stable leaves $\mathcal{H}^s(\tilde{v}_0), \mathcal{H}^s(\tilde{v}_1)$ in $T_1\tilde{S}$ project under π to H_0, H_1 respectively. Let $x = H_0 \cap H_2$ and let $y = H_1 \cap H_2$. Because the intersections are tangential, they lift to intersections in $T_1\tilde{S}$: if H^u is the leaf of $\mathcal{H}^u$ in $\tilde{M}$ that projects to H_2, then $\pi(H^u \cap \mathcal{H}^s(\tilde{v}_0)) = x$ and $\pi(H^u \cap \mathcal{H}^s(\tilde{v}_1)) = y$. Thus there is an $\mathcal{H}^{u,s}$ path $\tilde{\psi}$ in $\tilde{S}$ defined by traversing first $\mathcal{H}^s(\tilde{v}_0)$, then H^u, and finally $\mathcal{H}^s(\tilde{v}_1)$. (See Figure 6).

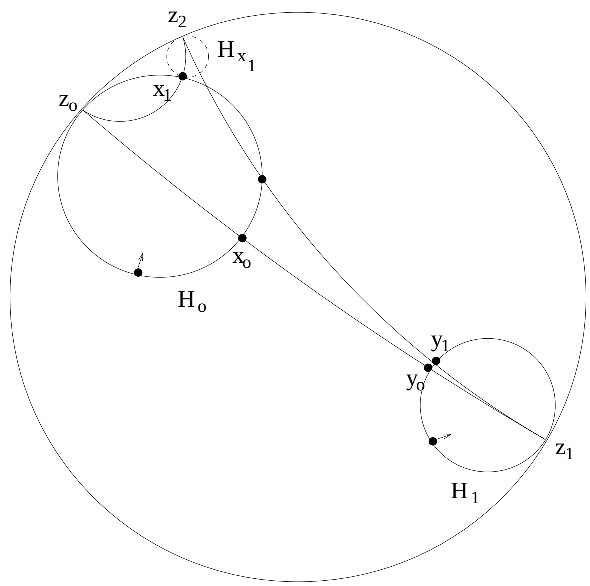

Figure 5: $\Delta(x_1) \geq 0$.

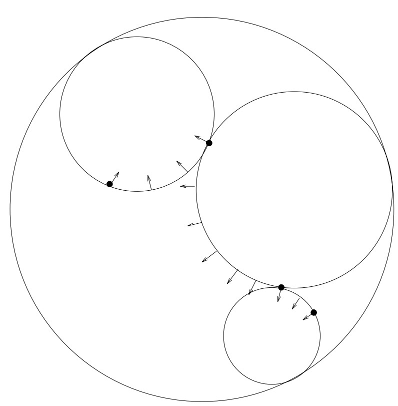

Figure 6: A three-legged $\mathcal{H}^{u,s}$ path in $T_1\tilde{S}$.

The path $\tilde{\psi}$ projects to a three-legged $\mathcal{H}^{u,s}$-path in T_1S. Hence any two points in T_1S can be connected by some three-legged $\mathcal{H}^{u,s}$-path.

The foliations $\mathcal{H}^s$ and $\mathcal{H}^u$ are at least C^1. Let ξ^s and ξ^u be unit stable and unstable vector fields tangent to the bundles H^u and H^s for the flow φ_t. These vector fields integrate to give C^1 flows ψ^+ and ψ^- on T_1S. The trajectories of ψ^+ and ψ^- lie along the leaves of the strong stable and unstable foliations $\mathcal{H}^s$ and $\mathcal{H}^u$, and are called the positive and negative horocyclic flows, respectively. (For a discussion of these flows and their ergodic properties, see [14]). We have shown that there exist $t_1, t_2, t_3 \in \mathbf{R}$, with $t_1 > 0$ such that $\tilde{v}_1 = \tilde{\psi}^+_{t_3} \circ \tilde{\psi}^-_{t_2} \circ \tilde{\psi}^+_{t_1}(\tilde{v}_0)$, where $\tilde{\psi}^\pm$ are the lifted positive and negative horocyclic flows on $T_1\tilde{S}$. These flows are smooth and hence their composition varies continuously in the parameter (t_1, t_2, t_3). We show now that the map $\tilde{\Gamma}(s_1, s_2, s_3) = \tilde{\psi}^+_{s_3} \circ \tilde{\psi}^-_{s_2} \circ \tilde{\psi}^+_{s_1}(\tilde{v}_0)$ is injective in a neighborhood of (t_1, t_2, t_3) in the positive half-space $t_1 > 0$ in $\mathbf{R}^3$. It follows from invariance of domain that $\tilde{\Gamma}$ is a local homeomorphism.

Let $x, y, \tilde{v}_0, \tilde{v}_1$ be as above, with x lying to the left of $\tilde{v}_0$ and y lying to the right of $\tilde{v}_1$ and x, y belonging to the same horosphere H. Suppose that there exist x', y' near x, y in H_0, H_1 and belonging to the horosphere H'. Assume that H and H' meet H_0, H_1 tangentially at these points.

Since the horocyclic foliations are orientable, the only possible configuration of x, y, x', y' (up to orientation) is shown in Figure 7. In theis configuration, x' lies to the left of x on H_0 and y' lies to the left of y in H_1 (this also follows from the fact that the horocyclic flows are orientation-preserving). Let B and B' be the horoballs corresponding to the horospheres H and H'. The points x and y must lie outside of the the horoball B' since H' intersects H_0 and H_1 each in exactly one point. Similarly x' and y' lie outside of B. It follows that the infinite segment of H to the right of x intersects the arc of H' between x' to y' in a point w_1. Likewise, the segment of H' to the left of y' intersects the segment of H between x and y in a point w_2. Finally, there is a third point of intersection w_3, where the H segment from x to y meets the H' segment from x' to y'. The three points w_1, w_2, w_3 are distinct, since H and H' are simple curves. This contradicts Corollary 2.5. This implies that $\tilde{\Gamma}$ is a local homeomorphism.

In Section 3 we show that under C^2 small perturbations of φ_1, the strong stable and unstable foliations persist and are transversally Hölder-continuous. The leaves of these foliations are unique integral curves of new vector fields, ξ_1' and ξ_2', which are C^0 perturbations of ξ_1 ξ_2. The perturbed vector fields ξ_1' and ξ_2' generate flows $\psi^{+'}$ and $\psi^{-'}$ that C^0-approximate the flows ψ^+ and ψ^- uniformly on compact time intervals.

Fix $v \in T_1S$. Let $\Gamma_v : \mathbf{R}^3 \to T_1S$ be defined by:

$$\Gamma_v(t_1, t_2, t_3) := \psi^+_{t_3} \circ \psi^-_{t_2} \circ \psi^+_{t_1}(v).$$

Since the flows $\psi^\pm$ lift to the flows $\tilde{\psi}^\pm$, the above arguments imply that, restricted to the half-space $D = \{(t_1, t_2, t_3) \in \mathbf{R}^3 \mid t_1 > 0\}$, Γ_v is surjective and a local homeomorphism. Thus for every $w \in T_1S$, there exist constants $r, \rho > 0$ and a point $s = (s_1, s_2, s_3) \in D$ such that $\Gamma_v(s_1, s_2, s_3) = w$ and

$$\text{index } (\Gamma_v|_{\partial B^3_r(s)}) = \pm 1,$$

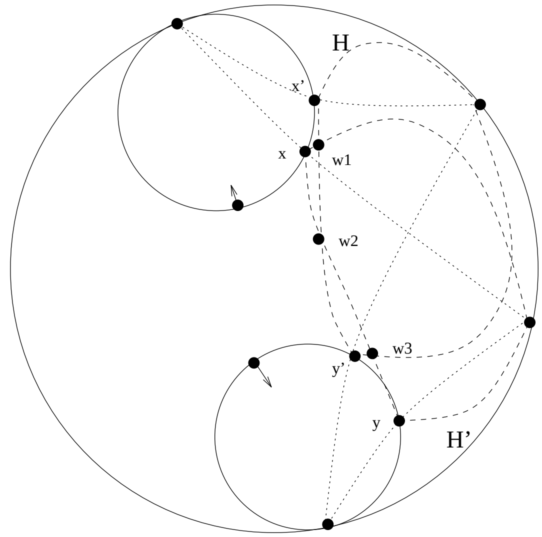

Figure 7: A possible configuration of x, y, x', y'.

with respect to every $w' \in B_\rho(w)$. The balls $\{B_\rho(w)\}_{w \in T_1 S}$ cover $T_1 S$. Pass to a finite subcover $\{B_{\rho_1}(w_1), B_{\rho_2}(w_2), \dots, B_{\rho_k}(w_k)\}$ and let r_i and $s^i \in D$ be such that

$$\text{index } (\Gamma_v|_{\partial B^3_{r_i}(s^i)}) = \pm 1,$$

with respect to every $w' \in B_\rho(w_i)$. Note that there exists an $R > 0$ such that for every $i \leq k$, $|s^i| \leq R$.

Since the property of having index ± 1 is robust and Γ_v is continuous as a function of v, there is a neighborhood $B_{\rho'}(v)$ of v such that for every $v' \in B_{\rho'}(v)$,

$$\text{index } (\Gamma_{v'}|_{\partial B^3_{r_i}(s^i)}) = \pm 1,$$

with respect to every $w' \in B_\rho(w_i)$. For every $v \in T_1 S$ there is such an open neighborhood $B_{\rho'}(v)$, where of course ρ' depends on v.

The collection of open sets $\{B_{\rho'}(v)\}_{v \in T_1 S}$ covers $T_1 S$. Again, pass to a finite subcover by balls $\{B_{\rho'_1}(v_1), B_{\rho'_2}(v_2), \dots, B_{\rho'_m}(v_m)\}$. Re-indexing, we now have constants $N, r, \rho > 0$ and sets $\{v_1, \dots, v_m\}$, $\{w_{i,1}, \dots w_{i,k_i}\}_{i=1,k}$ such that for every $1 \leq i \leq m$ and every $1 \leq j \leq k_i$, there is a point $s^{i,j} \in D$, with $|s^{i,j}| \leq N$ such that $\Gamma_i(s^{i,j}) := \Gamma_{v_i}(s^{i,j}) = w_{i,j}$ and

$$\text{index } (\Gamma_i|_{\partial B^3_r(s^{i,j})}) = \pm 1,$$

with respect to every $w' \in B_\rho(w_{i,j})$.

For $1 \leq i \leq m$, and for f sufficiently close to φ_1, there is a map $\Gamma'_i : \mathbf{R}^3 \to T_1 S$ defined by:

$$\Gamma'_i(t_1, t_2, t_3) := \psi^{+'}_{t_3} \circ \psi^{-'}_{t_2} \circ \psi^{+'}_{t_1}(v_i),$$

where $\psi^{\pm'}$ are the positive and negative $\mathcal{W}^{u,s}(f)$- horocyclic flows. Then Γ_i' uniformly approximates Γ_i on the set $[-N,N]^3 \subset \mathbf{R}^3$. It follows that for f sufficiently close to φ_1,

$$\text{index } (\Gamma_i'|_{\partial B_r^3(s^{i,j})}) = \pm 1,$$

with respect to every $w' \in B_\rho(w^{i,j})$. The result follows.□

Remark: The proof of Lemma 2.1 generalizes to noncompact manifolds of bounded negative curvature, provided that M has at least two closed geodesics (equivalently, $\pi_1(M)$ does not have a global fixed point at infinity).

If $\pi_1(M)$ has a global fixed point at infinity, then Lemma 2.1 does not hold. In particular, if two points in $T_1\mathbf{H}$ lie in the same leaf of the weak-stable foliation, then there is no three-legged $\mathcal{H}^{u,s}$ in $T_1\mathbf{H}$ path connecting them.

3 Uniform Hölder-Continuity

If f is C^1 close to φ_1, then the estimates in Theorem 1.1 will hold; that is, we will have that $\|Tf|_{E^u}\|$ and $\|Tf|_{E^s}\|$ are bounded above and below by a function involving the bounds on the curvature $-b^2 < -a^2$. This information, combined with standard invariant manifold techniques (in, for example, [18]) implies that the splitting $E^u \oplus E^c \oplus E^s$ is α-Hölder continuous, with α determined by the ratio $\frac{a}{b}$. This is not enough for our purposes, unless the curvature is pinched close to one (i.e. a/b is close to 1). We would like to exploit the volume-preserving aspect of φ and use the fact that *pointwise* the expansion and contraction rates of $Tf|_{E^u}$ and $Tf|_{E^s}$ are nearly reciprocal to each other.

In this section, we prove some regularity results about the strong stable and unstable foliations and the invariant splitting of a C^2 perturbation of the time-one map φ_1. Recall that a function $s : X \to Y$ between compact metric spaces is said to be *Hölder-continuous* if there exist constants $0 < \alpha \leq 1$ and $H > 0$ such that:

$$d_Y(s(x), s(x')) \leq H \cdot d_X(x, x')^\alpha,$$

for all $x, x' \in X$. In the expression above, α is called the Hölder *exponent* and H the Hölder *constant*.

3.1 Hölder-Continuity of the Strong Unstable Foliation.

Definition: Let M be a closed manifold. A C^r *pre-lamination* of M is a continuous choice of C^r-embedded disk $W(p)$ through each $p \in M$. The choice of $W(p)$ is given by a covering of M by *plaque charts* (U_j, σ_j), such that:

- $\sigma_j : U_j \to \text{Emb}^r(D^k, M)$ is a continuous section of the bundle of C^r embeddings of the k-disk $D^k = (-1,1)^k$ into M.

- For $p \in U_j$, $\sigma_j(p)(0) = p$ and $\sigma_j(p)(D^k) = W(p)$.

Let $\mathcal{F} := \{F(p)\}_{p\in M}$ be a foliation of M and let $\mathcal{T} = \{T(p)\}_{p\in M}$ be a pre-lamination of M by C^1 disks transverse to $\mathcal{F}$ and of complementary dimension. Recall that if $\gamma : [0,1] \to F(p)$ is a leafwise path with $\gamma(0) = p$ and $\gamma(1) = q$, then on a sufficiently small subdisk $T(p)' \subset T(p)$ there is a well-defined holonomy map

$$h_\gamma : T(p)' \to T(q),$$

such that $h_\gamma(r) = F(r) \cap T(q)$, which is a homeomorphism onto its image. Let $d_{T(\cdot)}$ denote the induced Riemannian path-metric on the plaque $T(\cdot)$.

Definition: With notation as above, the foliation $\mathcal{F}$ is *θ-Hölder continuous with respect to $\mathcal{T}$* if for every $R > 0$, there is a constant $H(R) > 0$ and an $\epsilon > 0$ such that for every $p, q \in M$ and for every leafwise path γ from p to q of length $l(\gamma) \leq R$,

$$d_{T(q)}(h_\gamma(r), h_\gamma(r')) \leq H(R) \cdot d_{T(p)}(r, r')^\theta,$$

for all $r, r' \in B_\epsilon(p) \cap T(p)$. The function $R \mapsto H(R)$ for $R \geq 0$ is called a *θ-Hölder constant assignment* of $\mathcal{F}$ (relative to $\mathcal{T}$).

Remark: Changing the Riemannian metric on M has the effect of changing the constant assignment H in the above definition; that is, the property of a foliation being θ-Hölder continuous with respect to a transverse family is independent of metric. In the following proposition, the actual values of the constants H_0 and $H(R)$ are not important; the content of the statement is that they can be chosen uniformly over perturbations of φ_1.

Proposition 3.1 *Let S be a closed, negatively-curved surface and let $\varphi_1 : T_1S \to T_1S$ be the time-one map of the geodesic flow φ_t on the unit tangent bundle T_1S of S. Then for every $\theta < 1$ there exists an $H_0 > 0$, a function $H : \mathbf{R}_+ \to \mathbf{R}_+$, and a neighborhood $\mathcal{U}$ of φ_1 in Diff$^2(T_1S)$, such that for every $f \in \mathcal{U}$,*

1. *The Tf-invariant splitting $TM = E^u \oplus E^c \oplus E^s$ obtained in Corollary 1.3 is θ-Hölder continuous, with Hölder constant H_0.*

2. *The strong unstable foliation $\mathcal{W}^u(f)$ is θ-Hölder continuous with respect to $\mathcal{W}^{cs}_{loc}(f)$, with θ-Hölder constant assignment $R \mapsto H(R)$.*

3. *The strong stable foliation $\mathcal{W}^s(f)$ is θ-Hölder continuous with respect to $\mathcal{W}^{cu}_{loc}(f)$ with θ- Hölder constant assignment $R \mapsto H(R)$.*

Remark: At first glance, it might appear that the Hölder-continuity of $\mathcal{W}^u(f)$ follows from the fact that the leaves of $\mathcal{W}^u(f)$ are tangent to a Hölder-continuous line-field E^u, just as a Lipschitz vector field generates a Lipschitz flow. In the Appendix, we show that condition 2 does not follow from condition 1. In other words, a Hölder-continuous vector field, even if it is uniquely integrable, does not necessarily integrate to a Hölder-continuous flow.

This section is organized as follows. In Section 3.2, we prove a general theorem about the Hölder-continuity of an invariant section of a fiber-bundle map. Theorem 3.3 may be used wherever the Hölder Section Theorem in ([18]: Theorem 5.18(c)) can be applied. In this section, we use Theorem 3.3 to prove Proposition 3.1. Because the proof of Proposition 3.1 is lengthy, we postpone the verification of one of the hypotheses of Theorem 3.3 to the later Section 3.3.

Proof of Proposition 3.1: Let $M := T_1S$ and let $TM = H^u \oplus H^c \oplus H^s$ be the Anosov splitting for φ_t. The arguments below are for unstable bundles and foliations; the same results follow for stable bundles and foliations by considering the inverse φ_{-1}.

Let η be the (unit) vector field generating φ_t and choose vector fields ξ_1, ξ_2 spanning H^u and H^s, respectively, so that $\{\eta, \xi_1, \xi_2\}$ is a unit volume frame for M. The bundles H^u and H^s are C^1 by [8]. The original metric is C^∞ and so consequently, the vector fields ξ_1 and ξ_2 can be chosen to be C^1. Since φ_t preserves the volume form and $T_p\varphi_t(\eta(p)) = \eta(\varphi_t(p))$, it follows that there is a function $\lambda : M \to (1, \infty)$ such that

$$T_p\varphi_1(\xi_1(p)) = \lambda(p) \cdot \xi_1(\varphi_1(p)),$$

and

$$T_p\varphi_1(\xi_2(p)) = \lambda(p)^{-1} \cdot \xi_2(\varphi_1(p)),$$

for all $p \in M$. These vector fields define a new C^1 Riemannian structure on M in which $\{\eta, \xi_1, \xi_2\}$ form an orthonormal frame. With respect to this metric,

$$\|T_p\varphi_1|_{H^u}\| = \lambda(p), \quad \|T_p\varphi_1|_{H^s}\| = \lambda(p)^{-1}, \quad \|T_p\varphi_1|_{H^c}\| = 1.$$

All calculations that follow are done with respect to the new Riemannian structure on M.

These vector fields uniquely trivialize the bundles H^u, H^c,H^s, H^{cs} and H^{cu}. Let $\alpha : M \times \mathbf{R} \to H^u$ be a C^∞ trivializing map for H^u such that for $p \in M$, $\alpha_p := \alpha(p, \cdot) : \mathbf{R} \to H^u(p)$ is the linear isometry with $\alpha_p(1) = \xi_1(p)$. Similarly, let β be the trivializing map for H^{cs} such that $\beta_p := \beta(p, \cdot) : \mathbf{R}^2 \to H^{cs}(p)$ is orthonormal, with $\beta_p((1, 0)) = \xi_2(p)$ and $\beta_p((0, 1)) = \eta(p)$.

Now suppose $f \in \mathrm{Diff}^2(M)$. With respect to the splitting $TM = H^u \oplus H^{cs}$, write:

$$T_pf = \begin{pmatrix} A_p & B_p \\ C_p & K_p \end{pmatrix},$$

where A_p, B_p, C_p, K_p are linear maps. If the approximation of f to φ_1 is close in the C^1 sense, then

$$\begin{aligned}
&\|B_p\|, \|C_p\| \doteq 0, \\
&m(A_p) = \|A_p\| \doteq \|T_p\varphi_1|_{H^u}\|, \ \|K_p\| \doteq \|T_p\varphi_1|_{H^c}\| \\
&\text{and } m(K_p) \doteq \|T_p\varphi_1|_{H^s}\|,
\end{aligned}$$

uniformly in $p \in M$. (Note that A_p is a 1×1 matrix, so its norm and conorm are automatically equal).

The proof of 1. follows from standard graph-transform techniques and the Pointwise Hölder Section Theorem (Theorem 3.3). We prove Hölder-continuity of the splitting $E^u \oplus E^c \oplus E^s$ in two steps. First we first show directly that the bundles E^u and E^s are Hölder-continuous. Then, to show that E^c is Hölder-continuous, we show that the weak stable and unstable bundles E^{cu} and E^{cs} are Hölder-continuous and then intersect them to obtain E^c.

Consider the bundle $\mathcal{L} \to M$ with fiber $L(H^u(p), H^{cs}(p))$ over p, where $L(H^u(p), H^{cs}(p))$ is the space of linear maps from $H^u(p)$ to $H^{cs}(p)$, with the operator norm. Note that this bundle is trivial, since H^u and H^{cs} are trivial; the trivialization is given by

$$\kappa : M \times L(\mathbf{R}, \mathbf{R}^2) \to \mathcal{L}$$

$$\kappa(p, P) = \beta_p \circ P \circ \alpha_p^{-1}.$$

In fact $\mathcal{L}$ is metrically trivial: $\kappa(p, \cdot)$ is an isometry with respect to the operator norms on $L(\mathbf{R}, \mathbf{R}^2)$ and $L(H^u(p), H^{cs}(p))$. Let $\mathcal{L}(1)$ be the compact subbundle of $\mathcal{L}$ whose fibers are the linear maps of operator norm ≤ 1. κ induces a trivialization of $\mathcal{L}(1)$, as well.

We then define a bundle map $\Gamma_{Tf} : \mathcal{L}(1) \to \mathcal{L}(1)$ covering f, which on each fiber $\{P \in L(H^u(p), H^{cs}(p)) \mid \|P\| \leq 1\}$ is the linear graph transform:

$$\Gamma_{Tf}(p, P) := (f(p), (C_p + K_p \circ P) \circ (A_p + B_p \circ P)^{-1}).$$

If f is sufficiently C^1-close to φ_1, then Γ_{Tf} will satisfy the hypotheses of Theorem 3.3. That is, in the notation of Section 3.2, there is an $L > 0$ such that if $f \doteq \varphi_1$, and $\delta > 0$ is sufficiently small, then Γ_{Tf} is Lipschitz with $\mathrm{Lip}(\Gamma_{Tf}) \leq L$, and Γ_{Tf} has pointwise fiber constant k_p (depending on f) and base constant μ_p (depending on f and δ) given by:

$$k_p \doteq \frac{\|K_p\|}{\|A_p\|} \doteq \frac{1}{\|T_p\varphi_1|_{H^u}\|} \qquad \text{(since } f \doteq \varphi_1\text{)}$$

and

$$\begin{aligned} \mu_p &= \mathrm{Lip}(f^{-1}|_{f(B_\delta(p))})^{-1} \\ &\doteq m(T_p f) \qquad \text{(since } \delta \doteq 0\text{)} \\ &\doteq \|T_p\varphi_1|_{H^s}\| \qquad \text{(since } f \doteq \varphi_1\text{)} \end{aligned}$$

(where $B_\delta(p)$ denotes the ball in M centered at p of radius δ). But for each $\theta < 1$, $\|T_p\varphi_1|_{H^u}\| \cdot \|T_p\varphi_1|_{H^s}\|^\theta > 1$. Thus, if we initially fix $\delta > 0$ small enough, then there is a neighborhood $\mathcal{U}_0$ of φ_1 in $\mathrm{Diff}^1(M)$, such that for $f \in \mathcal{U}_0$,

$$k_p \cdot \mu_p^{-\theta} < 1.$$

Then by Theorem 3.3 there is is an $H_0 > 0$ such that, for $f \in \mathcal{U}_0$, the unique Γ_{Tf}-invariant section $s : M \to \mathcal{L}$ of the bundle $\mathcal{L}$ is θ-Hölder continuous, with constant H_0.

This invariant section gives the unstable bundle E^u for f by: $E^u(p) = \{(x, s(p,x)) \mid x \in H^u(p)\}$, and so E^u is uniformly θ-Hölder continuous.

Similarly, we can find E^{cu} as the graph of an invariant section of a bundle over M, this time with fiber $L(H^{cu}(p), H^s(p))$. In this case we also have $k_p \doteq \|T_p\varphi_1|_{H^u}\|^{-1}$ and $\mu_p \doteq \|T_p\varphi_1|_{H^s}\|$, so it follows that for f sufficiently C^1-close to φ_1, the bundle E^{cu} is θ-Hölder continuous, with constant H_0, and the same is true for E^{cs}. Since the transverse intersection of θ-Hölder bundles is θ-Hölder, it follows that E^c is θ-Hölder and the proof of 1. is complete.

To prove 2., we will construct a nonlinear version of the bundle $\mathcal{L}$. For $\nu > 0$, denote by $B^k_\nu(x)$ the closed ball of radius ν, with respect to the box-norm, centered at x in $\mathbf{R}^k$. Define the map

$$\omega : M \times \mathbf{R}^3 \to M$$

by:

$$\omega_p(t_1, t_2, t_3) := \exp_p((\alpha_p(t_1) + \beta_p(t_2, t_3)),$$

where exp is the C^∞ exponential map with respect to the C^∞ canonical (Sasake) metric on M. Note that $T_0\omega_p$ is orthonormal with respect to the standard metric on $\mathbf{R}^3$ and the adapted C^1 metric on M defined above. For ν_0 sufficiently small, ω is a (C^1) diffeomorphism from $B^3_{\nu_0}(0)$ onto its image in M. In fact more is true: fixing a value of p, the functions $\omega_p(x)$ and $\partial\omega_p(x)/\partial p$ are C^∞ in x. A function with this property is said to be of class $C^{1\times\infty}$.

There is a well-defined bundle map

$$F : M \times B^3_{\nu_0}(0) \to M \times \mathbf{R}^3,$$

covering $f : M \to M$. It is given by:

$$F(p, x) = (f(p), F_p(x)),$$

where

$$F_p(x) := \omega^{-1}_{f(p)} \circ f \circ \omega_p(x),$$

for $p \in M$, $x \in B^3_{\nu_0}(0)$.

Then

$$D_0F_p = T_{f(p)}\omega^{-1}_{f(p)} \circ T_pf \circ T_0\omega_p.$$

Since $T_0\omega_p$ maps the standard basis of $\mathbf{R}^3$ to the orthonormal basis $\{\xi_1(p), \xi_2(p), \eta(p)\}$ of T_pM, we identify D_0F_p with T_pf so that by a slight abuse of notation:

$$D_0F_p = \begin{pmatrix} A_p & B_p \\ C_p & K_p \end{pmatrix},$$

Then F_p takes the form:

$$F_p(x, y) = (A_px + B_py, C_px + K_py) + r_p(x, y), \tag{1}$$
for $p \in M$, $x \in B^1_{\nu_0}(0), y \in B^2_{\nu_0}(0)$

where the remainder $r : M \times B^3_{\nu_0}(0) \to M \times \mathbf{R}^3$ is a $C^{1\times\infty}$ function. Since $D_0 r_p = 0$, for all $p \in M$, we may choose $\nu_0 > 0$ small enough so that $\mathrm{Lip}(r(p,\cdot)) \doteq 0$, uniformly in $p \in M$.

For $\nu > 0$ and $g : B^1_\nu(0) \to \mathbf{R}^2$ a continuous function with $g(0) = 0$, define the special norm:

$$\|g\|_* := \sup_{x \in B^1_\nu(0)} \frac{|g(x)|}{|x|}.$$

(Naturally, this and all similar suprema are taken over $x \neq 0$). With the norm $\|\ \|_*$, the space

$$\mathcal{G}_{*\nu} := \{g : B^1_\nu(0) \to \mathbf{R}^2 \mid g(0) = 0, \|g\|_* < \infty\}$$

is a Banach space ([18], p. 62).

Now consider the subset $\mathcal{G}_\nu \subset \mathcal{G}_{*\nu}$ consisting of Lipschitz functions of Lipschitz norm ≤ 1:

$$\mathcal{G}_\nu := \{g : B^1_\nu(0) \to \mathbf{R}^2 \mid g(0) = 0, \mathrm{Lip}(g) \leq 1\}.$$

With respect to the metric given by $\|\ \|_*$, $\mathcal{G}_\nu$ is a closed subset of the unit ball in $\mathcal{G}_{*\nu}$. We will view the graph of an element of $\mathcal{G}_{*\nu}$ as a candidate for a local stable manifold. A choice of such a function $g_p \in \mathcal{G}_\nu$ at each point $p \in M$ is a section of the fiber bundle $M \times \mathcal{G}_\nu$.

For $\nu_1 < \nu_0$ sufficiently small and or f sufficiently C^1-close to φ_1, the restriction of F_p to $B^1_{\nu_1}(0)$ satisfies the hypotheses of the pseudo-hyperbolic stable manifold theorem in [18]. (An outline of some the discussion that follows, on the level of the fibers $\mathcal{G}_{*\nu}$ is in [18], pages 61ff). Thus F induces a graph-transform bundle map $F_\sharp : M \times \mathcal{G}_{\nu_1} \to M \times \mathcal{G}_{\nu_1}$, given by

$$F_\sharp(p, g) := (f(p), F_{\sharp_p}(g)),$$

where:

- For $(p, g) \in M \times \mathcal{G}_{\nu_1}$, $\mathrm{graph}(F_{\sharp_p}(g)) = F_p(\mathrm{graph}(g)) \cap B^3_{\nu_1}(0)$.

- $\mathrm{Lip}(F_{\sharp_p}(\cdot)) \doteq \|K_p\| \cdot m(A_p)^{-1}$ (since $\nu_1 \doteq 0$) $\doteq \|T_p\varphi_1|_{H^u}\|^{-1}$ (since $f \doteq \varphi_1$).

Proposition 3.4 shows that there is an $L > 0$ and a neighborhood $\mathcal{U}_1$ of φ_1 in $\mathrm{Diff}^2(M)$ such that for $f \in \mathcal{U}_1$, $F_\sharp$ is Lipschitz, with Lipschitz norm L; that is, for $g \in \mathcal{G}_{\nu_1}$ and $q, q' \in M$,

$$\|F_{\sharp_q}(g) - F_{\sharp_{q'}}(g)\|_* \leq L \cdot d(q, q').$$

We now show that $F_\sharp$ satisfies the hypotheses of the Pointwise Hölder Section Theorem (Theorem 3.3). First, $F_\sharp$ is a Lipschitz bundle map, covering a overflowing bilipschitz homeomorphism f. For $p \in M$, $F_\sharp$ has Lipschitz fiber constant $k_p < 1$ (depending on f and ν_1) and base constant $\mu_p > 0$ (depending on f and δ) defined as follows. As in the linear case, we have:

$$\mu_p = \mathrm{Lip}(f^{-1}|_{f(B_\delta(p))})^{-1} \doteq m(T_p f) \doteq \|T_p\varphi_1|_{H^s}\|,$$

and by Lemma III.6 in [18],

$$k_p = \mathrm{Lip}(F_{\sharp_p}(\cdot)) \doteq \frac{1}{\|T_p\varphi_1|_{H^u}\|} < 1.$$

It follows that if δ and ν_1 are chosen initially to be sufficiently small, then there is a neighborhood of φ_1 in $\mathrm{Diff}^1(M)$ where the relation

$$\sup_{p\in M} k_p \cdot \mu_p^{-\theta} < 1 \qquad (*)$$

will hold for $F_\sharp$.

Given $\theta < 1$, and $\delta, \nu_1 > 0$ small enough, let $\mathcal{U} \subset \mathcal{U}_0 \cap \mathcal{U}_1$ be a neighborhood of φ_1 in $\mathrm{Diff}^2(M)$ such that relation $(*)$ is satisfied, for every $f \in \mathcal{U}$. By Theorem 3.3, for $f \in \mathcal{U}$ there is a unique continuous function $s_f : M \to \mathcal{G}_{\nu_1}$ such that:

$$F_\sharp(p, s_f(p)) = (f(p), s_f(f(p))).$$

Further, there is an $H_1 > 0$ such that for all $f \in \mathcal{U}$ and for every $p, q, \in M$,

$$\|s_f(p) - s_f(q)\|_* \leq H_1 \cdot d(p,q)^\theta.$$

Fix $f \in \mathcal{U}$ and write "s" for "s_f". The graph of $s(p) : B^1_{\nu_1}(0) \to \mathbf{R}^2$ in $B^3_{\nu_1}(0)$ maps, under ω_p, to the local strong unstable manifold for f at p. For $p \in M$, let $\sigma_p : B^1_{\nu_1}(0) \to M$ be given by:

$$\sigma_p(t) := \omega_p(t, s(p)(t)).$$

Then $\sigma_p(B^1_{\nu_1}(0)) = W^u_{loc}(p)$.

We show now that the Hölder-continuity of s implies the Hölder-continuity of the foliation $\mathcal{W}^u$ with respect to plaque family $\mathcal{W}^{cs}_{loc}$. For $\nu_2 < \nu_1$ sufficiently small, the Riemannian path - metric on $\omega_p(B^3_{\nu_2}(0))$ is uniformly close to the Euclidean metric on $B^3_{\nu_2}(0)$. In local ω_p-coordinates, let $T_p(q)$ be the plane through $q \in \omega_p(B^3_{\nu_2}(0))$ given by:

$$T_p(q) := \{\omega_p(\omega_p^{-1}(q), t_1, t_2) \mid (t_1, t_2) \in B^2_{\nu_2}(0)\}.$$

First consider the plaque family of transversals $\mathcal{T} = \{T(p)\}_{p\in M}$ given by:

$$T(p) := T_p(p) = \{\omega_p(0, t_1, t_2) \mid (t_1, t_2) \in B^2_{\nu_2}(0)\}.$$

The change of coordinates

$$\Omega_{p,q} : B_{\nu_2}(p) \cap B_{\nu_2}(q) \to B_{\nu_2}(q)$$

given by $\Omega_{p,q}(x) = \omega_q \circ \omega_p^{-1}(x)$ is uniformly $C^{1\times 1\times\infty}$ in the variables (p, q, x) and as $d(p,q) \to 0$, $\Omega_{p,q} \rightrightarrows Id$. It follows that as $d(p,q) \to 0$,

$$T(p) \quad \rightrightarrows \quad T(q)$$

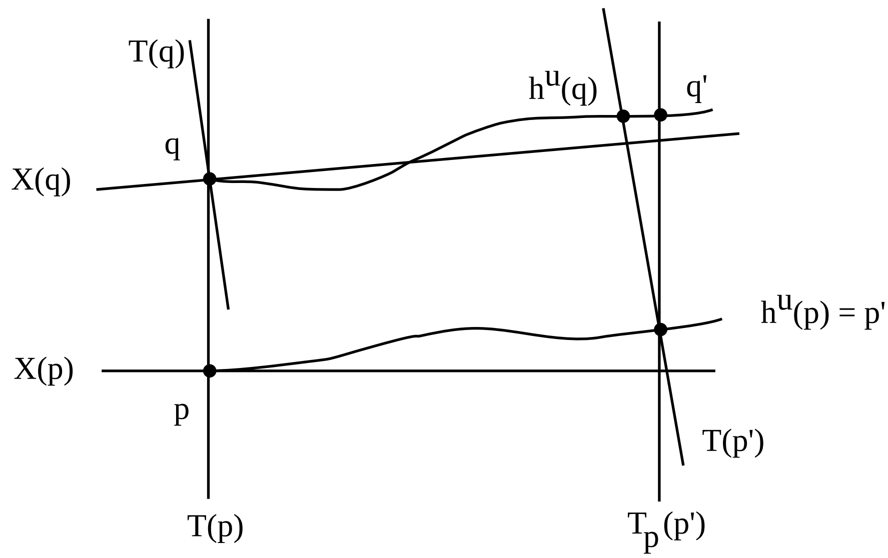

Figure 8: The $\mathcal{W}^u$-holonomy between $T(p)$ and $T(p')$.

and that the angle between $T_p(q)$ and $T(q)$ is bounded by $c_0 \cdot d(p,q)$, where c_0 is a uniform constant. In what follows we write $T(p)(\epsilon) := T(p) \cap B_\epsilon(p)$. Similarly define the "$x$-axis" for the ω_p-coordinate system by:

$$X(q) := \{\omega_q(t,0,0) \mid t \in B^1_{\nu_1}(0)\}.$$

Fix p for now and suppose $p' \in W^u_{\nu_2}(p)$. Then $p' = \sigma_p(x)$, for some $x \in B^1_{\nu_2}(0)$. Consider the $\mathcal{W}^u$-holonomy between the transversals $T(p)$ and $T(p')$:

$$h^u : T(p)(\nu_2) \to T(p').$$

$T(p')$ is nearly parallel to $T_p(p')$ (the angle between them is $\leq c_0 \cdot d(p,p')$) and so there exists a uniform constant $c_1 > 0$ such that:

$$d_{T(p')}(h^u(p), h^u(q)) = d_{T(p')}(p', h^u(q)) \leq c_1 \cdot d_{T_p(p')}(p', q'),$$

where $q' = W^u_{\nu_2}(q) \cap T_p(p')$ (See Figure 8).

The x-axes $X(p)$ and $X(q)$ are nearly (in a Lipschitz sense) parallel, so there exists a uniform constant $c_2 > 0$ such that

$$d_{T_p(p')}(p', q') \leq c_2 \cdot d(\sigma_p(x), \sigma_q(x)).$$

Finally, since $T(p)$ and $T(q)$ are nearly parallel, there exists $c_3 > 0$ such that

$$\begin{aligned} d(\sigma_p(x), \sigma_q(x)) &\leq c_3 \cdot [d_{T(p)}(p,q) + |s_p(x) - s_q(x)|] \\ &\leq c_3 \cdot [d_{T(p)}(p,q) + |x| \cdot \|s_p - s_q\|_*] \\ &\leq c_3 \cdot [d_{T(p)}(p,q) + |x| \cdot H_1 \cdot d(p,q)^\theta], \end{aligned}$$

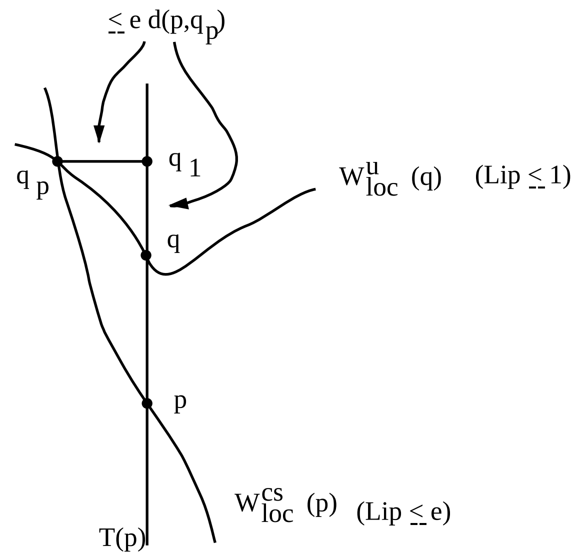

Figure 9: $d(p,q) \leq c_5 \cdot d(p,q_p)$.

and so there exists a $c_4 > 0$ such that:

$$d_{T(p')}(h^u(p), h^u(q)) \leq c_4 \cdot d_{T(p)}(p,q)^\theta.$$

Now consider the center-stable plaque family $\mathcal{W}^{cs}_{\nu_2}(f) = \{W^{cs}_{\nu_2}(p)\}$. This family is invariant under f and coherent: the plaques patch together to give a foliation of M. This property will be used in what follows. As $f \to \varphi_1$ in the C^1 sense,

$$\mathcal{W}^{cs}_{\nu_2}(f) \quad \rightrightarrows \quad \mathcal{H}^{cs}_{\nu_2},$$

where $\mathcal{H}^{cs}_{\nu_2}$ is there local center-stable plaque family for φ_1. The plaque $\mathcal{H}^{cs}_{\nu_2}(p)$ is tangent to $T(p)$. Locally, $W^{cs}(p)$ is the graph of a smooth function:

$$g^{cs}(p) : T(p) \to X(p).$$

(Where here we really mean that $W^{cs}(p) = \exp_p(\text{graph}(g^{cs\prime}(p)))$, where $g^{cs\prime}(p) : H^{cs}(p) \to H^u(p)$). Further, there exists a small constant $e < 1$ such that if $f \doteq \varphi_1$ in the C^1 sense, then $\text{Lip}(g^{cs}(p)) \leq e$, for all $p \in M$. For $q \in T(p)$, let $q_p = W^{cs}_{\nu_2}(p) \cap W^u_{\nu_2}(q)$.
Claim: Let $c_5 = (1+e) \cdot (1-e)^{-1}$. Then:

$$c_5^{-1} \cdot d(p,q_p) \leq d(p,q) \leq c_5 \cdot d(p,q_p).$$

Proof of Claim: See Figures 9 and 10. Let q_1 denote the intersection of $X(q)$ with $T(p)$. We use the facts that $\text{Lip}(s(q)) \leq 1$ and $\text{Lip}(g^{cs\prime}(p)) \leq e$.

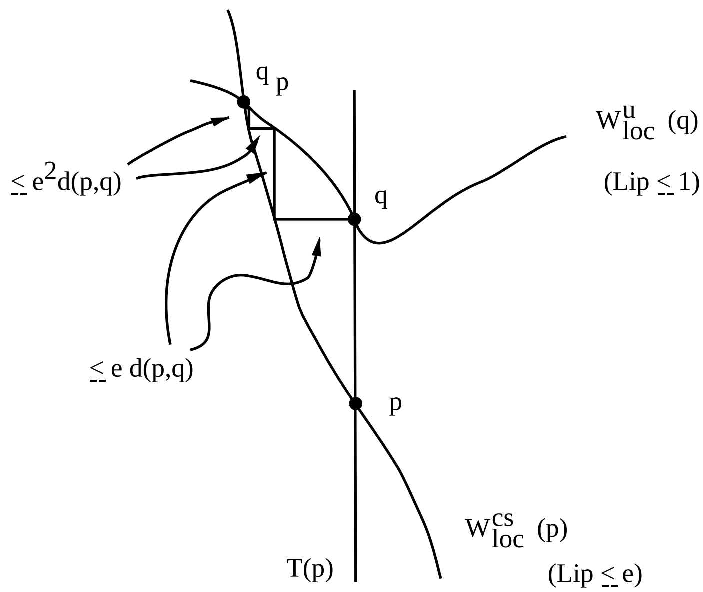

Figure 10: $d(p, q_p) \leq c_5 \cdot d(p, q)$.

On the one hand,

$$\begin{aligned} d(p,q) &\leq d(p,q_1) + d(q_1,q) \\ &\leq d(p,q_1) + e \cdot d(p,q_1) \\ &\leq (1+e) \cdot d(p,q_p) \\ &\leq c_5 \cdot d(p,q_p), \end{aligned}$$

since $d(p, q_1) \leq d(p, q_p)$ and $d(q, q_1) < e \cdot d(p, q_1)$.

On the other hand, by inspecting Figure 10, we see that

$$\begin{aligned} d(p,q_p) &\leq (1 + 2(e + e^2 + e^3 + \cdots)) \cdot d(p,q) \\ &= (1 + \frac{2e}{1-e}) \cdot d(p,q) \\ &= c_5 \cdot d(p,q), \end{aligned}$$

and the claim is proved.

To complete the proof of Proposition 3.1, let $p \in M$, $q \in W^{cs}_{\nu_2}(p)$, and $p' \in W^{u}_{\nu_2}(p)$ and $q' \in W^{cs}_{\nu_2}(p') \cap W^{u}_{\nu_2}(q)$. Then

$$\begin{aligned} d_{W^{cs}(p')}(p',q') &\leq c_5 \cdot d_{T(p')}(p', {q'}_{p'}) \\ &\leq c_5 \cdot c_4 \cdot d_{T(p)}(p,q_p)^{\theta} \\ &\leq c_5 \cdot c_4 \cdot c_5^{\theta} \cdot d_{W^{cs}(p)}(p,q)^{\theta}. \end{aligned}$$

Thus for $c = c_5 \cdot c_4 \cdot c_5^\theta$, the $\mathcal{W}^u$-holonomy between $\mathcal{W}^{cs}_{loc}$-transversals of distance $\leq \nu_2$ apart is θ-Hölder continuous, with constant c.

Fix $R \geq 0$ and suppose $p' \in W^u(p)$ and $\gamma : [0,1] \to M$, is a leafwise path from p to p' of length $l(\gamma) \leq R$. Since $\mathcal{W}^u(f)$-leaves contract uniformly exponentially under iterates of f^{-1}, there exists a uniform (over $\mathcal{U}$) $k \in \mathbf{Z}_+$ such that $d(f^{-k}(p), f^{-k}(p')) < \nu_2$. There also exists an $\epsilon > 0$ such that if $d(p,q) < \epsilon$, then for all $f \in \mathcal{U}$, $d(f^{-k}(p), f^{-k}(q)) < \nu_2$. Further, there is a $D > 0$ such that:

$$D^{-1} \cdot d(p,q) \leq d(f^{-k}(p), f^{-k}(q)) \leq D \cdot d(p,q),$$

for all $f \in \mathcal{U}$. Consider the holonomy

$$h_\gamma : W^{cs}_\epsilon(p) \to W^{cs}(p').$$

From the above considerations plus the invariance of $\mathcal{W}^{cs}(f)$ under f, we have

$$\begin{aligned} d_{W^{cs}(p')}(h_\gamma(q_1), h_\gamma(q_2)) &\leq D \cdot d_{W^{cs}(f^{-k}(p'))}(h_{f^{-k}\circ\gamma}(f^{-k}q_1), h_{f^{-k}\circ\gamma}(f^{-k}(q_2))) \\ &\leq D \cdot c \cdot d_{W^{cs}(f^{-k}(p))}(f^{-k}(q_1), f^{-k}(q_2))^\theta \\ &\leq D \cdot D^\theta \cdot c \cdot d_{W^{cs}(p)}(q_1, q_2)^\theta. \end{aligned}$$

Setting $H(R) = D \cdot D^\theta \cdot c$, the proposition is proved.□

3.2 The Pointwise Hölder Section Theorem

Let (X, d) be a complete metric space and let $Y \subset E$ be a closed bounded subset of diameter D in the Banach space E. We shall be concerned with sections of the trivial bundle $X \times Y$ over X. A section of such a bundle can naturally be regarded as a map $s : X \to Y$. Given constants $0 < \delta, \theta < 1$, and $H > 0$, we define the section spaces:

$$\begin{aligned} &\Gamma_{\delta,\theta,H} := \{s : X \to Y \mid d(x,x') \leq \delta \Rightarrow |s(x) - s(x')| \leq H d(x,x')^\theta\} \\ &\Gamma_{\theta,H} := \{s : X \to Y \mid |s(x) - s(x')| \leq H d(x,x')^\theta\}. \end{aligned}$$

Endow both spaces with the uniform topology. The second space $\Gamma_{\theta,H}$ is the space of all θ-Hölder-continuous maps from X to Y with Hölder constant $\leq H$.

Lemma 3.2 *Given θ, δ as above. If $H > D \cdot \delta^{-\theta}$ then $\Gamma_{\delta,\theta,H} = \Gamma_{\theta,H}$.*

Proof of Lemma 3.2: Evidently $\Gamma_{\theta,H} \subset \Gamma_{\delta,\theta,H}$. Suppose that $s \in \Gamma_{\delta,\theta,H}$. We want to show that:

$$\frac{|s(x) - s(x')|}{d(x,x')^\theta} \leq H.$$

This is true for $d(x,x') \leq \delta$ because s belongs to $\Gamma_{\delta,\theta,H}$. For $d(x,x') \geq \delta$,

$$\frac{|s(x) - s(x')|}{d(x,x')^\theta} \leq \frac{\text{diam}(Y)}{\delta^\theta} = \frac{D}{\delta^\theta} < H.$$

$\square$

Now suppose that $F : X \times Y \to X \times Y$ is a homeomorphism that preserves the set of fibers $\{\{x\} \times Y\}$. Then F takes the form:

$$F(x, y) = (h(x), v(x, y)).$$

We assume the following hypotheses on F:

1. The map $h : X \to X$ is an overflowing bilipschitz homeomorphism, and there exists a $\delta > 0$ such that for all $x \in X$
$$\inf_{x' \neq x} \{\frac{(h(x), h(x'))}{d(x, x')} \text{ s.t. } d(x, x') < \delta\} = \mu_x > 0,$$
and $\mu := \inf \mu_x > 0$.

2. There exists a constant $L \geq 1$ such that for all $x, x' \in X$ and all $y \in Y$,
$$|v(x, y) - v(x', y)| \leq Ld(x, x')^\theta.$$

3. For each $x \in X$,
$$\inf \frac{|v(x, y) - v(x, y')|}{|y - y'|} = k_x < 1,$$
and $\sup_{x \in X} k_x = k < 1$.

The map F induces a map $\tilde{F}$ on sections of the bundle $X \times Y \to X$ by:

$$\tilde{F}s(h(x)) = v(x, s(x)).$$

Then $\tilde{F}$ carries continuous sections into continuous sections and contracts fibers of $X \times Y$ by a definite amount $k < 1$. Since the space sections is complete there exists a unique invariant section $\tilde{s} : X \to Y$ satisfying: $\tilde{s}(x) = \tilde{F}\tilde{s}(x) = v(h^{-1}(x), \tilde{s}(h^{-1}(x)))$. This section is continuous. The following theorem gives sufficient conditions for the invariant section to be Hölder (compare with [18], Theorem 5.18).

Theorem 3.3 (Pointwise Hölder Section Theorem) *Given F as above. Suppose that*

$$\sup_{x \in X} k_x \mu_x^{-\theta} = \eta < 1$$

Then the unique $\tilde{F}$-invariant section $\tilde{s}$ is θ-Hölder continuous. The θ-Hölder constant H may be chosen to be no smaller than

$$\frac{LD}{\mu\delta^\theta(1 - \eta)}.$$

Proof of Theorem 3.3: Let $H = LD/(\mu\delta^\theta(1-\eta))$. We show that $\Gamma_{\theta,H}$ is mapped into itself by $\tilde{F}$. The uniqueness of $\tilde{s}$ in the space of continuous sections then implies the result.

Because H is not less than $D\delta^{-\theta}$, we only need to show that $\Gamma_{\delta,\theta,H}$ is carried into itself by $\tilde{F}$ (Lemma 3.2). So pick an arbitrary $s \in \Gamma_{\delta,\theta,H}$. Then $\tilde{F}s(h(x)) = v(x, s(x))$. We want to show that $\tilde{F}s \in \Gamma_{\delta,\theta,H}$; in other words, that

$$d(h(x), h(x')) \le \delta \Rightarrow |v(x, s(x)) - v(x', s(x'))| \le Hd(h(x), h(x'))^\theta.$$

So suppose that $d(h(x), h(x')) \le \delta$.

Case 1: $d(x, x') \ge \delta$. Then

$$\begin{aligned} \frac{|v(x, s(x)) - v(x', s(x'))|}{d(h(x), h(x'))^\theta} &\le \frac{\mathrm{Diam}(Y)}{(\mu d(x, x'))^\theta} \\ &\le \frac{D}{(\mu\delta)^\theta} \le H. \end{aligned}$$

Case 2: $d(x, x') < \delta$. Then

$$\begin{aligned} |v(x, s(x)) - v(x', s(x'))| &\le |v(x, s(x)) - v(x, s(x'))| + |v(x, s(x')) - v(x', s(x'))| \\ &\le k_x|s(x) - s(x')| + Ld(x, x')^\theta \\ &\le k_x H d(x, x')^\theta + Ld(x, x')^\theta \end{aligned}$$

(Since $d(x, x') \le \delta \Rightarrow |s(x) - s(x')| \le Hd(x, x')^\theta$).

$$\begin{aligned} &= (k_x H + L)d(x, x')^\theta \\ &\le (k_x H + L)(\frac{1}{\mu_x} d(h(x), h(x')))^\theta \end{aligned}$$

(Since $d(x, x') \le \delta \Rightarrow d(h(x), h(x')) \ge \mu_x d(x, x')$).

$$\begin{aligned} &\le (H\frac{k_x}{\mu_x^\theta} + \frac{L}{\mu_x})d(h(x), h(x'))^\theta \\ &\le (H\eta + \frac{L}{\mu})d(h(x), h(x')))^\theta \\ &\le Hd(h(x), h(x')))^\theta, \end{aligned}$$

since $H \ge L/\mu(1-\eta)$.$\square$

3.3 The Bundle Map $F_\sharp$ is Lipschitz

Let $F_\sharp$ be the bundle map defined in the proof of Proposition 3.1. By Lemma III.6 in [18], it is Lipschitz when restricted to fibers. In this section, we show that $F_\sharp$ is globally Lipschitz. In particular, we have:

Proposition 3.4 ($F_\sharp$ is uniformly Lipschitz on constant sections) *There exists an $L > 0$ and a neighborhood $\mathcal{U}_1$ of φ_1 in Diff$^2(M)$ such that for all $f \in \mathcal{U}_1$,*

$$\|F_{\sharp_q}(g) - F_{\sharp_{q'}}(g)\|_* \leq L \cdot d(q, q'),$$

for all $q, q' \in M$, and $g \in \mathcal{G}_{\nu_1}$.

Proposition 3.4 in turn follows from Lemma 3.5:

Lemma 3.5 *There exists $C > 0$ and a neighborhood $\mathcal{U}_1$ of φ_1 in Diff$^2(M)$ such that for all $f \in \mathcal{U}_1$,*

$$|F_q(x) - F_{q'}(x)| \leq C \cdot d(q, q') \cdot |x|,$$

for all $q, q' \in M$, and $x \in B^3_{\nu_1}(0)$.

Proof of Lemma 3.5: Recall that $F_p = \omega^{-1}_{f(p)} \circ f \circ \omega_p$. The map ω is $C^{1\times\infty}$, and f is C^2, so $F_p(x)$ is $C^{1\times\infty}$ in (p, x). In particular, if f is sufficiently close to φ_1 in the C^2 sense, for a fixed $p \in M$, the Lipschitz norm of $\frac{\partial F_p}{\partial p}(x)$ as a function of x is bounded by a uniform constant C_1. Note also that for every $p \in M$, F_p fixes the origin of $\mathbf{R}^3$, so in local coordinates,

$$\frac{\partial F_p}{\partial p}(0) = 0.$$

Since $\frac{\partial F_p}{\partial p}(x)$ is C^∞ as a function of x, with Lipschitz norm bounded by C_1, we have for $x \in B^3_{\nu_1}(0)$,

$$\frac{\partial F_p}{\partial p}(x)|_{p=q} \leq C_1|x|,$$

for all $q \in M$. But F is C^1 as a function of p, so by the Mean Value Theorem, in local linear coordinates we have:

$$|F(q, x) - F(q', x)| = |\int_0^1 \frac{\partial F_p}{\partial p}(x)|_{p=tq+(1-t)q'} dt| \cdot |q - q'| \leq C_1 \cdot |x| \cdot |q - q'|,$$

and the result follows. □

Proof of Proposition 3.4: Let $\mathcal{U}_1$ be the neighborhood specified in Lemma 3.5 and let $f \in \mathcal{U}_1$. For $x_1 \in B^1_{\nu_1}(0), x_2 \in B^2_{\nu_1}(0)$, and $q \in M$, write $F_q(x_1, x_2) = (F_{1,q}(x_1, x_2), F_{2,q}(x_1, x_2))$. Then, by definition of the graph-transform, for $g \in \mathcal{G}_{\nu_1}$ we have ,

$$F_{\sharp_q}(g)(F_{1,q}(x, g(x))) = F_{2,q}(x, g(x)),$$

or in other words,

$$F_{\sharp_q}(g)(x) = (F_{2,q} \circ G) \circ (F_{1,q} \circ G)^{-1}(x),$$

where $G := (id, g) : B^1_{\nu_1}(0) \to \mathbf{R}^3$. Now, for $q, q' \in M$, $g \in \mathcal{G}_{\nu_1}$ and $x \in B^1_{\nu_1}(0)$, we have:

$$\begin{aligned}
|F_{\sharp_q}(g)(x) - F_{\sharp_{q'}}(g)(x)| &= |F_{2,q} \circ G \circ (F_{1,q} \circ G)^{-1}(x) - F_{2,q'} \circ G \circ (F_{1,q'} \circ G)^{-1}(x)| \\
&\leq |F_{2,q} \circ G \circ (F_{1,q} \circ G)^{-1}(x) - F_{2,q} \circ G \circ (F_{1,q'} \circ G)^{-1}(x)| + \\
&\quad |F_{2,q} \circ G \circ (F_{1,q'} \circ G)^{-1}(x) - F_{2,q'} \circ G \circ (F_{1,q'} \circ G)^{-1}(x)| \\
&\leq \mathrm{Lip}(F_{2,q} \circ G)|(F_{1,q} \circ G)^{-1}(x) - (F_{1,q'} \circ G)^{-1}(x)| + \\
&\quad C \cdot d(q, q')|G \circ (F_{1,q'} \circ G)^{-1}(x)| \\
&\leq C_2 \cdot |(F_{1,q} \circ G)^{-1} \circ (F_{1,q'} \circ G) \circ (F_{1,q'} \circ G)^{-1}(x) - \\
&\quad (F_{1,q} \circ G)^{-1} \circ (F_{1,q} \circ G) \circ (F_{1,q'} \circ G)^{-1}(x)| + C_3 \cdot d(q, q')|x| \\
&\leq C_2 \cdot \mathrm{Lip}((F_{1,q} \circ G)^{-1})|(F_{1,q'} \circ G) \circ (F_{1,q'} \circ G)^{-1}(x) - \\
&\quad (F_{1,q} \circ G) \circ (F_{1,q'} \circ G)^{-1}(x)| + C_3 \cdot d(q, q')|x| \\
&\leq C_4 \cdot C \cdot d(q, q')|(F_{1,q'} \circ G)^{-1}(x)| + C_3 \cdot d(q, q')|x| \\
&\leq C_5 \cdot d(q, q')|x| + C_3 \cdot d(q, q')|x| \\
&= L \cdot d(q, q')|x|,
\end{aligned}$$

and so

$$\begin{aligned}
\|F_{\sharp_q}(g) - F_{\sharp_{q'}}(g)\|_* &= \sup_{x \in B^1_\gamma(0)} \frac{|F_{\sharp_q}(g)(x) - F_{\sharp_{q'}}(g)(x)|}{|x|} \\
&\leq L \cdot d(q, q').
\end{aligned}$$

□

4 Stable Ergodicity

In this section, we develop the remaining tools necessary to prove the Main Theorem and then carry out the proof. We restate it here for convenience:

Theorem 4.1 (Main Theorem) *If S is a closed, connected negatively-curved Riemannian surface, and if $\varphi_t : T_1S \to T_1S$ is the geodesic flow, then the time-one map φ_1 is stably ergodic.*

Here is an outline of the proof. The strategy from this point on is similar in spirit to that in [5], but the estimates there need refinement to work in a setting where the local structure varies from point to point. Let $M = T_1S$ and suppose that $f \in \mathrm{Diff}^2_\omega(M)$. If f is sufficiently C^1 close to φ_1, then Corollary 1.3 and 1.4 apply, and there is a Tf-invariant splitting:

$$TM = E^u \oplus E^c \oplus E^s$$

and invariant foliations $\mathcal{W}^u$, $\mathcal{W}^c$, $\mathcal{W}^s$, $\mathcal{W}^{cu}$, and $\mathcal{W}^{cs}$ tangent to E^u, E^c, E^s, $E^u \oplus E^c$, and $E^s \oplus E^c$. The foliations $\mathcal{W}^u$ and $\mathcal{W}^s$ are dynamically-defined; that is, for $q \in W^s(p)$,

$$d(f^k(p), f^k(q)) \to 0,$$

as $k \to \infty$, and similarly for $\mathcal{W}^u$. Consequently, if $g : M \to \mathbf{R}$ is a continuous measurement on M, then

$$\lim_{k\to\infty} |g(f^k(p)) - g(f^k(q))| = 0, \, \forall q \in W^s(p),$$

and

$$\lim_{k\to\infty} |g(f^{-k}(p)) - g(f^{-k}(q'))| = 0, \, \forall q' \in W^u(p).$$

Also, the foliations $\mathcal{W}^u$ and $\mathcal{W}^s$ are absolutely continuous, by [16]. If a foliation $\mathcal{W}$ is absolutely continuous, it makes sense to speak, at least locally, of "almost every leaf of $\mathcal{W}$." A measurable set A is *essentially $\mathcal{W}$-saturated* if it essentially consists of essentially whole $\mathcal{W}$-leaves.

If f is *not* ergodic, then a standard argument, originally due to Hopf (cf. [5], p. 297), shows that there is a measurable set A, with $m(A) > 0$ and $m(M \setminus A) > 0$, such that A is essentially $\mathcal{W}^u$-saturated *and* essentially $\mathcal{W}^s$-saturated. In what follows, we show how this gives a contradiction.

For such an f, let a be a density point of A, and let b be a density point of $M \setminus A$. In appropriately-chosen rectilinear coordinates, a small cube of side-length w centered at a intersects A in a set of very high density in the cube. Similarly for a small cube around b. The next step is the crucial one in [5]. Divide each cube into long, thin prisms of cross-sectional width w and height $w^{2/3}$. At least one of these prisms, say P, will have a high concentration of A. Similarly, there is prism P' near b with a high concentration of $M \setminus A$. In Section 4, we show that, for f sufficiently C^2-close to φ_1, nested inside the prisms P and P' are dynamical-prisms called *juliennes* J and J', in which A and $M \setminus A$ are also highly concentrated.

Now by Lemma 2.1, if f is sufficiently close to φ_1 in the C^1 sense, then there is a $\mathcal{W}^{u,s}$-path τ consisting of three arcs of bounded length that joins the centers of J and J'. Again, if f is C^2-close to φ_1, then the $\mathcal{W}^u$ and $\mathcal{W}^s$ holonomy maps are uniformly α-Hölder-continuous, with α close to 1, and it follows that the holonomy maps don't distort juliennes too much. In Subsection 4.6, we show that if an essentially $\mathcal{W}^u$- ($\mathcal{W}^s$ -) saturated set is highly concentrated in J then its image under the $\mathcal{W}^u$- ($\mathcal{W}^s$ -) holonomy is highly concentrated in another julienne. Applying this argument to A along successive legs of the path τ, we arrive at the conclusion A is highly concentrated in J', a contradiction.

In Section 4.8 we prove Theorem 4.1.

4.1 Holonomy Revisited

Let $\mathcal{V}$ be the neighborhood of φ_1 given in Lemma 2.1 and for $f \in \mathcal{V}$, let N be a generous bound on the lengths of the three-legged $\mathcal{W}^{u,s}(f)$ paths given by that lemma. For $p \in M$ and $R \leq N$ let $\gamma : [0,1] \to W^u(p)$ be a leafwise geodesic with $\gamma(0) = p$ of length $l(\gamma) = R$. Let h^u_R denote the holonomy

$$h^u_R := h^u_\gamma : \mathcal{H}^{cs}_{loc}(p) \to \mathcal{H}^{cs}_{loc}(\gamma(1)).$$

Similarly, for f close to φ_1 in the C^2 sense, let π_R^u denote the $\mathcal{W}^u(f)$- holonomy between $\mathcal{W}_{loc}^{cs}(f)$-transversals of leafwise distance R apart. (Incidentally, the discussion that follows holds equally for the $\mathcal{W}^s$-holonomy, π_R^s). In Chapter 3 we showed that for R bounded, π_R^u, while not necessarily smooth, is uniformly α-Hölder with α near 1. In this section, we investigate further the properties of π_R^u.

4.2 Absolute Continuity

Pre-perturbation, the unstable holonomy h_R^u is smooth because the foliation $\mathcal{H}^u$ is C^1. This implies that the Jacobian of h_R^u is uniformly bounded above and below. In this subsection we show that for $f \doteq \varphi_1$ in the C^2 sense, and for R bounded, the Jacobian $\mathrm{Jac}(\pi_R^u)$ exists, is continuous and uniformly bounded above and below (away from 0).

Lemma 4.2 (Unstable holonomy Jacobians converge uniformly) *As $f \to \varphi_1$ in the C^2 sense, π_R^u converges uniformly to h_R^u and Jac(π_R^u) converges uniformly to Jac(h_R^u).*

Proof of Lemma 4.2: As $f \to \varphi_1$ in the C^2-sense, the local disk families $\mathcal{W}_{loc}^{cs}(f)$ converge uniformly to $\mathcal{H}_{loc}^{cs}$ (see the proof of Proposition 3.1), hence we have

$$\pi_R^u \quad \rightrightarrows \quad h_R^u.$$

To show $\mathrm{Jac}(\pi_R^u) \quad \rightrightarrows \quad \mathrm{Jac}(h_R^u)$, we first show that $\mathrm{Jac}(\pi_R^u)$ is uniformly bounded in a neighborhood of φ_1 in $\mathrm{Diff}^2(M)$. The convergence then follows from the Lebesgue dominated convergence theorem (compare [5], Lemma 2.2).

The continuity of $\mathrm{Jac}(\pi_R^u)$ is proved in [16]. Inspecting the proof of Theorem 2.1 in [16], we see that the value of $\mathrm{Jac}_p(\pi_R^u)$ is given by the limit:

$$\operatorname{unif}\lim \frac{\det(f^{-n}|_{T_y D_p})}{\det(f^{-n}|_{T_y D_q})},$$

which in turn is bounded by:

$$\operatorname{unif}\lim \frac{\det(Tf^{-n}|_{T_y D_p})}{\det(Tf^{-n}|_{T_y D_q})},$$

which, by the chain rule, is bounded by a function of:

$$\sum_{k=0}^{\infty} |\det(T^{cs}_{f^{-k}(p)} f^{-1}) - \det(T^{cs}_{f^{-k}(q)} f^{-1})|.$$

Finally, this expression is bounded by

$$\sum_{k=0}^{\infty} (\lambda^{-\theta})^k \cdot d(p,q),$$

where $\lambda = \inf_x m(T_x f|_{E^u})$ and θ is the Hölder exponent of the distribution E^u. Since these quantities vary uniformly in a neighborhood $\mathcal{U}$ of φ_1 in $\mathrm{Diff}^1(M)$, it follows that the Jacobian $\mathrm{Jac}(\pi^u_R)$ is uniformly bounded in $\mathcal{U}$.□

When S has constant curvature, the horocyclic flow preserves the volume form, and consequently $\mathrm{Jac}(h^u_R)$ is identically equal to 1, regardless of the distance R between transversals. Due to metric rigidity, this is probably never the case when the curvature of S is variable. Nonetheless, its Jacobian is continuous, and thus uniformly bounded if R is bounded; Lemma 4.2 then implies that the value of $\mathrm{Jac}(\pi^u_R)$ is uniformly bounded, and we have:

Lemma 4.3 (Uniform bounds for unstable holonomy Jacobians) *Given N, there is a neighborhood $\mathcal{N}_1$ of φ_1 in $Diff^2(M)$ and a constant $j > 1$ such that for all $f \in \mathcal{N}_1$, if $R \leq N$, then*

$$j^{-1} < Jac(\pi^u_R) \leq j.$$

4.3 Behavior Along $\mathcal{W}^c$ Leaves

Restricted to flow lines $\mathcal{H}^c$, the holonomy h^u_R is an isometry. This is just a restatement of the fact that the Anosov splitting is invariant under $T\varphi_t$ and the flow φ_t is an isometry along center leaves. Post-perturbation, nearly the same is true. We have:

Lemma 4.4 *(cf. [5], Lemma 2.3) The $\mathcal{W}^u(f)$-holonomy map π^u_R preserves the center foliation $\mathcal{W}^c$. If $p' = \pi^u_R(p)$, then π^u_R sends $W^c_{loc}(p)$ to $W^c_{loc}(p')$, is C^1 and is uniformly nearly isometric when restricted to $W^c_{loc}(p)$.*

Proof of Lemma 4.4: The proof carries through exactly as in [5].□

4.4 Juliennes and Rectangles

In this section we recall some of the material in [5], sections 4-5. Much of the material in these sections carries over *mutatis mutandis* to our setting. Here f is a C^2 volume-preserving perturbation of the time-one map φ_1 of the geodesic flow on $M = T_1S$, where S is a closed, negatively-curved surface.

4.5 Definitions

Let $TM = E^u \oplus E^c \oplus E^s$ be the Tf-invariant splitting of TM into unstable, center and stable directions. This splitting is Hölder continuous. For each $p \in M$, choose smooth (C^∞) local coordinates $(x, y, z)_p$ on M so that

$$p = (0,0,0)_p, \quad \text{span }(\frac{\partial}{\partial x})_p = E^u_p, \quad \text{span }(\frac{\partial}{\partial y})_p = E^s_p, \quad \text{span }(\frac{\partial}{\partial z})_p = E^c_p.$$

Let Z_p be the plane $z = 0$.

A *rectangle* centered at $p_0 = (x_0, y_0, 0)_p$ of width $2w$ and height $2h$ is a rectilinear prism in p-coordinates:

$$R(p, p_0, w, h) = [x_0 - w, x_0 + w] \times [y_0 - w, y_0 + w] \times [-h, h].$$

A *square* $S(p, p_0, w) = [x_0 - w, x_0 + w] \times [y_0 - w, y_0 + w]$ is the Z_p- slice of a rectangle. Rectangles have minimal dynamical significance, but they can be used to compose rectilinear cubes and hence are useful in the context of the Lebesgue Density Theorem.

By contrast, juliennes are dynamically-constructed approximations to rectangles. As we shall see, juliennes behave well under holonomy maps. They are constructed as follows. We first alter the coordinate axes in the Z_p plane. The laminations $\mathcal{W}^{cu}, \mathcal{W}^{cs}$ give transverse laminations of Z_p:

$$\mathcal{L}^u = \mathcal{W}^{cu} \cap Z_p, \qquad \mathcal{L}^s = \mathcal{W}^{cs} \cap Z_p.$$

The leaves L^u, L^s of these laminations form Hölder coordinates for Z_p.

Let $L^u(p_0)$ and $L^s(p_0)$ be, respectively, the intersections of $L^u_{loc}(p_0)$ and $L^s_{loc}(p_0)$ with the square $S(p, p_0, w)$. For w small enough, $L^u(p_0)$ and $L^s(p_0)$ are smooth curves transverse to each other which define coordinate axes for the square

$$\Sigma(p, p_0, w) = \bigcup L^u_{loc}(r) \cap L^u_{loc}(s),$$

where r and s range over $L^u(p_0)$ and $L^s(p_0)$ respectively.

The nonlinear square $\Sigma(p, p_0, w)$ forms the base of a *julienne*

$$J(p, p_0, w, h) = \cup_{r \in \Sigma(p,p_0,w)} W^c_h(r),$$

where $W^c_h(r)$ is the arc of $W^c_{loc}(r)$ lying between the planes $z = \pm h$.

The center foliation arcs comprising the vertical direction of a julienne are integral curves of a Hölder-continuous vector field and so we have some control over how vertical strips deviate horizontally. The square $\Sigma(p, p_0, w)$, on the other hand, is nearly tangent to the very non-integrable distribution $E^u \oplus E^s$ (pre-perturbation, this distribution is totally non-integrable), so we can't hope to control the vertical deviation of a horizontal strip if w is too large relative to h. It turns out that if w is sufficiently small relative to h, then juliennes will approximate rectangles.

More precisely, under appropriate conditions, a julienne will contain a scaled copy of a rectangle and *vice versa*. By a *scaled* julienne we mean: fix a real number $c \in (0, 1)$ and let

$$cR = R(p, p_0, cw, ch), \quad \text{and} \quad cJ = J(p, p_0, cw, ch).$$

The next lemma shows that if w and h are related by: $w = k \cdot h^{3/2}$, where $k > 0$ is a constant bounded above and below from zero, then the associated families of rectangles and juliennes are comparable at small w-scales. Assume that $p_0 \in M$ lies in the coordinate systems at both p_1 and p_2. There are four different objects to compare:

$$R_1 = R(p_1, p_0, w, h) \qquad J_1 = J(p_1, p_0, w, h)$$

$$R_2 = R(p_2, p_0, w, h) \qquad J_2 = J(p_2, p_0, w, h).$$

Lemma 4.5 (Julienne Nesting Lemma) *(cf. [5], Lemma* 3.1*) Fix constants* $c \in (0,1)$ *and* $k_2 > k_1 > 0$. *As* $f \to \varphi_1$ *in the* C^2 *sense and* $w = kh^{3/2} \to 0$ *with* $k \in [k_1, k_2]$ *and while* $h^{-1} \cdot d(p_0, p_1)$ *and* $h^{-1} \cdot d(p_0, p_2)$ *stay bounded, then:*

$$cR_1 \subset J_2 \subset c^{-1}R_1 \quad \textit{and} \quad cR_2 \subset J_1 \subset c^{-1}R_2.$$

Proof of Lemma 4.5: We can choose a neighborhood $\mathcal{N}_0$ of f in $\mathrm{Diff}^2 M$ such that the center line field E^c for f is uniformly (in the sense of Proposition 3.1) α-Hölder continuous with $\alpha > \frac{1}{2}$. Since the adapted local coordinates at p are C^∞, the proof carries through as in [5].□

4.6 Density Estimates

Let A be an essentially $\mathcal{W}^u$-saturated and denote by A^u the *unstable saturate* of A. It consists entirely of whole $\mathcal{W}^u$-leaves – those that meet A in a set of full leaf measure. Then A and A^u differ by a zero-set, so $m(A \cap R) = m(A^u \cap R)$ for any measurable set R. Define the *conditional measure* of a set A relative to R (with $0 < m(R) < \infty$) in the standard way:

$$m(A : R) = \frac{m(A \cap R)}{m(R)}.$$

Because we will be swapping between rectangles and juliennes, we need an estimate of how density decays in the process. Lemma 4.5 allows us to swap the two at small scales with little sacrifice in measure. Since we will slide juliennes along $\mathcal{W}^u$ leaves, preserving the $\mathcal{W}^{cs}$ lamination, it is convenient to concentrate on the density of a set A inside a $\mathcal{W}^{cs}$-leaf. The strategy in [5] is to study first the density of A in a rectilinear approximation of the $\mathcal{W}^{cs}$-leaf, and then, using Lemma 4.5, to make a similar statement about actual $\mathcal{W}^{cs}$-leaves. In this way, the non-absolute continuity of the $\mathcal{W}^{cs}$ can be compensated for by approximating by a smooth foliation at small scales.

The non-linear *center stable slice* of a julienne $J = J(p, p_0, w, h)$ is defined:

$$J^{cs}_x = W^{cs}_{loc}(x) \cap J,$$

for x in the $\mathcal{L}^u$-leaf $L^u_w(p_0)$. The center stable *midslice* is the the leaf $J^{cs} = J^{cs}_{p_0}$.

Lemma 4.6 *(cf. [5], Lemma* 3.3*) Fix* $\rho' \leq 1$ *and* $k_2 > k_1 > 0$. *Let* $d \in (0,1)$ *and* $.1 \leq \rho \leq \rho'$ *be given. If* $k \in [k_1, k_2]$, *and* $w = kh^{3/2}$ *is small enough and if* f C^2-*approximates* φ_1 *well enough then for any essentially* $\mathcal{W}^u$-*saturated set* A,

$$m(A : J) \geq \rho \quad \Rightarrow \quad m(A^u : J^{cs}) \geq d\rho,$$

$$m(A^u : J^{cs}) \geq \rho \quad \Rightarrow \quad m(A : J) \geq d\rho.$$

Proof of Lemma 4.6: See [5]. There is a slight error in the proof there: they say that the mutual densities of R^{cs} and J^{cs} are nearly 1. This is clearly not true, but the Lemma follows easily from the fact that $\mathrm{Jac}(\pi^u_R) \quad \rightrightarrows \quad 1$ as $R \to 0$.□

The next lemma is essential for proving the estimates in Section 4.1.

Lemma 4.7 *(cf. [5], Lemma 3.4) Fix $k_2 > k_1 > 0$. As $w = kh^{3/2} \to 0$ with $k \in [k_1, k_2]$ and $f \to \varphi_1$ in the C^2 sense,*

$$\frac{m(J^{cs})}{4wh} \quad \rightrightarrows \quad 1,$$

where $J = J(p, p, w, h)$ and m denotes the area on the leaf $W^{cs}_{loc}(p)$. Also, if J^{cs}_+ denotes the part of J^{cs} lying to the right of $W^c_h(p)$, then

$$\frac{m(J^{cs}_+)}{2wh} \quad \rightrightarrows \quad 1.$$

Proof of Lemma 4.7: The proof is the same as in [5]; in our case, $\mathrm{Jac}(\pi^u) \doteq 1$ because $\mathrm{Jac}(\pi^u_R) \quad \rightrightarrows \quad 1$ as $R \to 0$.□

4.7 Holonomy and Juliennes

Again let $J^{cs}(p, q, w, h)$ denote the center-stable midslice of the julienne $J(p, q, w, h)$.

Lemma 4.8 (Julienne Holonomy Lemma) *Fix constants $R \geq 0$, $k_2 > k_1 > 0$ and $c \in (0, 1)$. Fix $p \in M$, and let $j_p = Jac_p(h^u_R)$ and $q = \pi^u_R(p)$. As $f \to \varphi_1$ in the C^2 sense and $w = kh^{3/2} \to 0$, with $k \in [k_1, k_2]$,*

$$cJ^{cs}(q, q, j_p \cdot w, h) \subset \pi^u_R(J^{cs}(p, p, w, h)) \subset c^{-1} J^{cs}(q, q, j_p \cdot w, h).$$

Proof of Lemma 4.8:(compare [5], Lemma 4.1) To simplify notation, let J denote the julienne $J(p, p, w, h)$ and let J' denote $J(q, q, j_p \cdot w, h)$. Similarly define J^{cs}, J'^{cs}, J^{cs}_+, and J'^{cs}_+. By Lemma 4.4, the image of J^{cs} under π^u_R is a union of center manifolds $W^c_{loc}(q)$, each with length approximately equal to 1. This alone does not ensure that the image of J^{cs} does not look very ragged along the top. For example, with this information alone, the image of J^{cs} could resemble the image in Figure 11. Nearly half of the density of julienne in that figure has disintegrated.

Fortunately, we have the additional information from Proposition 3.1 that for $f \doteq \varphi_1$ the holonomy π^u_R is uniformly α-Hölder continuous, with $\alpha > \frac{2}{3}$, as is its inverse. This implies that the picture in Figure 11 cannot occur, for if the width of the top edge of J^{cs} is w, then the width of its image is $\leq H \cdot w^\alpha$, and

$$Hw^\alpha << (wk^{-1})^{\frac{2}{3}} = h$$

at small w-scales (since k is bounded). Set $e = \sqrt{c}$. For w small enough the image of the top boundary of J^{cs} lies between the planes $z = eh$ and $z = e^{-1}h$ in the $(x, y, z)_{p'}$-coordinate system. Similarly, the image of the bottom boundary of J^{cs} lies between the planes $z = -eh$ and $z = -e^{-1}h$. See Figure 12. Now π^u_R is absolutely continuous by Lemma 4.2 and its Radon-Nikodym derivative $\mathrm{Jac}(\pi^u_R)$ is nearly constant and equal to j_p. Let $b = \sqrt{e}$. For w small enough, it follows that we have:

$$b \cdot j_p \leq \mathrm{Jac}_q(\pi^u_R) \leq b^{-1} \cdot j_p,$$

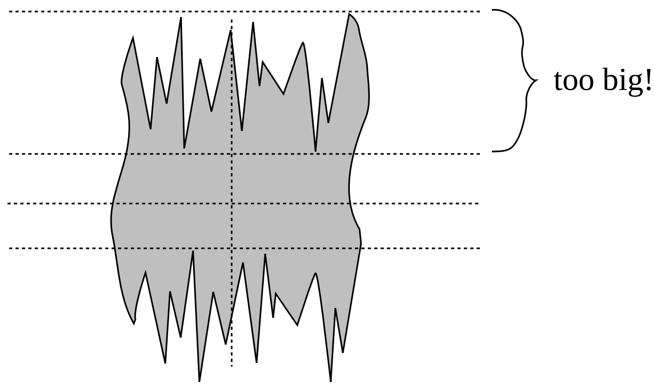

Figure 11: Center lines are preserved, but the top and bottom are ragged.

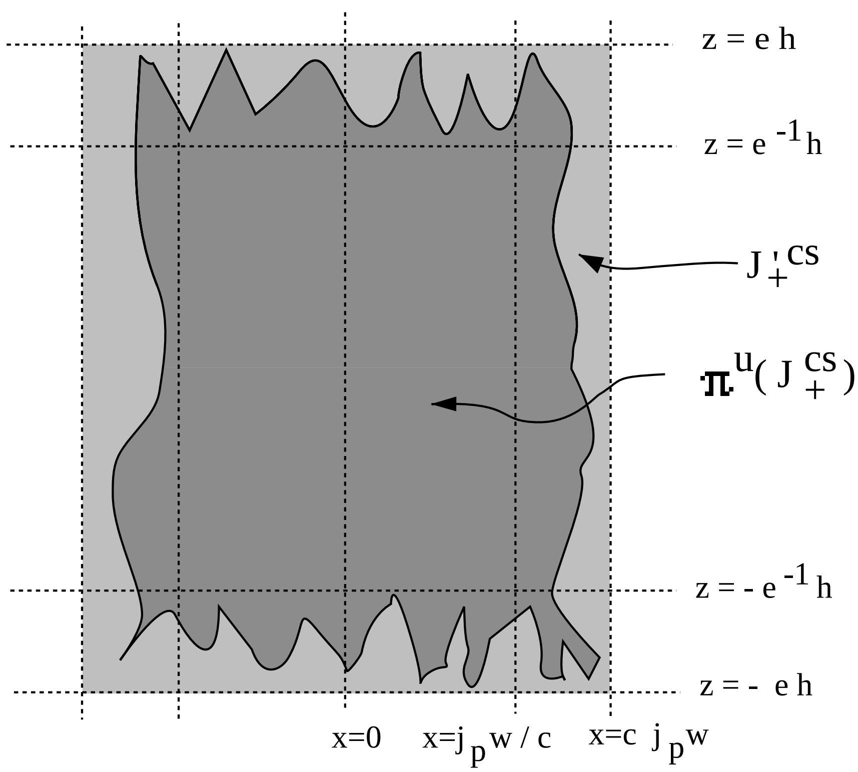

Figure 12: The image of J^{cs} in the p'-coordinate system.

for all $q \in J^{cs}$. Also, π^u_R is orientation-preserving, so the right half-julienne J^{cs}_+ is mapped to the right half of $W^{cs}_{loc}(p')$.

By Lemma 4.4, π^u_R carries the rightmost edge of J^{cs}_+ to another center manifold $W^c_{loc}(r)$, that lies either wholly to the left or wholly to the right of the rightmost edge $W^c_{loc}(q')$ of J'^{cs}. Suppose that r lay to the left of the point cq', so that $\pi^u_R(J^{cs}_+)$ lay entirely in the positive region bounded by $W^c_{loc}(cq')$ and the planes $z = \pm e^{-1}h$. By Lemma 4.7, the area of this region is well-approximated by :

$$c \cdot (j_p w) \cdot (2e^{-1}h) = 2ewhj_p.$$

But this region contains $\pi^u(J^{cs}_+)$, which has area well-approximated by:

$$2wh \cdot (j_p \cdot b),$$

which gives a contradiction, since $b > e$.

Suppose instead that r lies to the right of the point $c^{-1}q'$, so that $\pi^u_R(J^{cs})$ wholly contained the the positive region bounded by W^c_{loc} and the planes $z = \pm eh$. The area of this region is well-approximated by

$$c^{-1} \cdot (j_p w) \cdot (2eh) = 2e^{-1}whj_p.$$

But the area of $\pi^u(J^{cs}_+)$ is well-approximated by:

$$2wh \cdot (j_p \cdot b^{-1}),$$

again giving a contradiction. The same argument applied to the left side of the julienne implies the result.□

Lemma 4.9 (Julienne Holonomy Density Lemma) *Fix $\rho' < 1$. Let $c, \rho \in (0,1)$ be constants such that $.1 \le c\rho < \rho \le \rho'$ and let f, φ_1, R,k_1,k_2, j_p, p, q, w and h have the same meanings as in Lemma 4.8. If $k \in [k_1, k_2]$ and $w = kh^{3/2}$, is small enough, if f C^2 approximates φ_1 well enough, and if A is an essentially $\mathcal{W}^u$-saturated set which has density $\ge \rho$ in $J(p,p,w,h)$, then A has density $\ge c\rho$ in $J(q,q,j_p \cdot w, h)$.*

Proof of Lemma 4.9:(compare [5], Lemma 4.2) Let $c, \rho > .1$ be given and pick d so that $1 > d > c^{1/4}$ and $1 + 10(1 - d^{-1}) > \sqrt{c}$. Note that since $\rho \ge .1$, this implies that $1 - d^{-1}(1 - d\rho) > \sqrt{c}\rho$. Let J, J' *etc.* be as in the proof of Lemma 4.8. Since A is essentially $\mathcal{W}^u$-saturated, Lemma 4.6 implies that if f is close to φ_1 in the C^2 sense and w is small enough, then we will have:

$$m(A^u : J^{cs}) \ge d\rho,$$

or, in other words:

$$m(A^u \cap J^{cs}) \ge d\rho \cdot m(J^{cs}).$$

Denote by $A^{u\prime}$ the complement $M \setminus A^u$ and note that $A^{u\prime}$ is the unstable saturate of $M \setminus A$. We have that:

$$m(A^{u\prime} \cap J^{cs}) \leq (1 - d\rho)m(J^{cs}).$$

Since $A^{u\prime}$ is $\mathcal{W}^u$-saturated, π_R^u sends $A^{u\prime} \cap J^{cs}$ to $A^{u\prime} \cap \pi_R^u(J^{cs})$. By Lemma 4.2, $\mathrm{Jac}(\pi_R^u)$ is nearly constant and equal to j_p, and so:

$$d^{1/2} j_p \leq \mathrm{Jac}_q(\pi_R^u) \leq d^{-1/2} j_p,$$

for all $q \in J^{cs}$. This in turn implies that:

$$m(A^{u\prime} \cap \pi_R^u(J^{cs})) \leq d^{-1/2} j_p (1 - d\rho) m(J^{cs}),$$

or, in other words:

$$m(A^{u\prime} : \pi_R^u(J^{cs})) \leq \frac{j_p(1 - d\rho)m(J^{cs})}{\sqrt{d} m(\pi_R^u(J^{cs}))}.$$

Again, since $\mathrm{Jac}(\pi_R^u)$ is nearly constant, we have

$$\frac{j_p \cdot m(J^{cs})}{m(\pi_R^u(J^{cs}))} \leq d^{-1/2},$$

and so it follows that:

$$m(A^u : \pi_R^u(J^{cs})) \geq 1 - d^{-1}(1 - d\rho) \geq \sqrt{c}\rho.$$

By Lemma 4.9, $\pi_R^u(J^{cs})$ and $J^{\prime cs}$ are highly concentrated in each other, and so

$$m(A^u : J^{\prime cs}) \geq d\sqrt{c}\rho.$$

Finally, Lemma 4.6 implies that:

$$m(A^u : J^{\prime cs}) \geq d^2\sqrt{c}\rho = c\rho.$$

Implicitly we are using here the fact that j_p is uniformly bounded above and below, so that these constants can be chosen uniformly at the same time for J and J'. The lemma is proved.□

The lemmas of this section hold equally for the π_R^s-holonomy maps.

4.8 Proof of Main Theorem

In this section we put together the results of the previous sections to prove the Main Theorem. The proof follows the lines of the proof of the main theorem in [5].

Suppose f is not ergodic. Then, as discussed in the beginning of this chapter, there is an essentially $\mathcal{W}^{u,s}(f)$-saturated set A such that $m(A) > 0$ and $m(M \setminus A) > 0$. Let a be a density point of A and let b be a density point of $M \setminus A$.

Lemma 2.1 implies that if f is in a neighborhood $\mathcal{V}$ of φ_1 in $\mathrm{Diff}^2(M)$, then there is a three-legged $\mathcal{W}^{u,s}$-path of length $\leq N$ connecting any two points of T_1S. Lemma 4.3 implies that there exists a constant $e \in (0,1)$ such that for $f \in \mathcal{V}$, if τ is any three-legged $\mathcal{W}^{u,s}(f)$-path of length $\leq N$, and if k represents the Jacobian of the concatenated π^u/π^s holonomy maps along τ, then

$$e \leq k \leq e^{-1}.$$

Pick constants $\rho_1, d, \rho_2 \in (0,1)$, depending on e, as follows. First pick $\rho_1 > \max\{.1, 1-e^2\}$. Now pick $d < 1$ so that

$$d > \frac{1-e^2}{\rho_1}.$$

Now pick $\rho_2 < 1$ such that

$$\rho_2 > \max\{1 - e^2 d\rho_1, e^{-2}(1-d\rho_1)\}.$$

Choose h small enough so that the density of A in a rectilinear cube C_a about a of side-length $2h$ is $\geq \sqrt{\rho_1}$. Also choose h so that the density of $M \setminus A$ in a rectilinear cube C_b about b of side-length $2h$ is $\geq \sqrt{\rho_2}$. Divide the cube C_a centered at a into rectilinear boxes of width $2w$ and height $2h$, where $w = h^{3/2}$. Similarly, divide the cube C_b centered at b into rectangles of width $2w$ and height $2h$. The density of A in one of these boxes, say $R(a, a_0, w, h)$ is $\geq \sqrt{\rho_1}$, and the density of $M \setminus A$ in some box $R(b, b_0, w, h)$ is $\geq \sqrt{\rho_2}$. Lemma 4.5 then implies that A is highly concentrated in the julienne $J_{a_0} := J(a_0, a_0, w, h)$ and $M \setminus A$ is highly concentrated in $J_{b_0} := J(b_0, b_0, w, h)$. We may assume, then, that we initially chose h small enough, so that $m(A : J_{a_0}) \geq \rho_1$ and $m(A : J_{b_0}) \geq \rho_2$.

Let $c = (d)^{\frac{1}{3}}$, where d was chosen above, and assume we have chosen h small enough so that Lemma 4.9 holds for this value of c, with $e < k < e^{-1}$, $\rho' = \sqrt{\rho_1}$, and with $R = N$. By Lemma 2.1, there is a three-legged $\mathcal{W}^{u,s}$-path connecting a_0 to b_0 of length $\leq N$. Let $a_0 = q_0 \to q_1 \to q_2 \to q_3 = b_0$ be the $\mathcal{W}^{u,s}$-path from a_0 to b_0 and let $j_{q_0}, j_{q_1}, j_{q_2}$ be the Jacobians of the corresponding holonomy maps, with concatenated Jacobian $k_0 := j_{q_0} j_{q_1} j_{q_2}$. Lemma 4.9 then implies that the density of A in $J(q_1, q_1, j_{q_0} w, h)$ is at least $c\rho_1$. Continuing along the $\mathcal{W}^{u,s}$-path, applying Lemma 4.9 two more times, we arrive at the conclusion that the density of A in $J(b_0, b_0, k_0 w, h)$ is at least $c^3 \cdot \rho_1 = d\rho_1$. We show now that this gives a contradiction. There are two cases to consider. Suppose first that $k_0 < 1$, so that the julienne $J(b_0, b_0, k_0 w, h)$ is contained in the julienne $J(b_0, b_0, w, h)$ (see Figure 13). Lemma 4.5 then implies that the density of A in $J(b_0, b_0, w, h)$ is well-approximated by $(k_0)^2 \cdot d\rho_1$, which in turn is well-approximated (below), by $e^2 d\rho_1$. Since the density of $M \setminus A$ in $J(b_0, b_0, w, h)$ is at least ρ_2, and $\rho_2 > 1 - e^2 d\rho_1$, this gives a contradiction. Suppose on the other hand that $k_0 > 1$, so that the julienne $J(b_0, b_0, w, h)$ is contained in the julienne $J(b_0, b_0, k_0 w, h)$ (see Figure 14). The density of $M \setminus A$ in $J(b_0, b_0, k_0 w, h)$ is bounded above by $1 - d\rho_1$, which, again by Lemma 4.5 implies that the density of $M \setminus A$ in $J(b_0, b_0, w, h)$ is less than $e^{-2}(1 - d\rho_1)$. This gives a contradiction, since $\rho_2 > e^{-2}(1 - d\rho_1)$. The theorem is proved. □

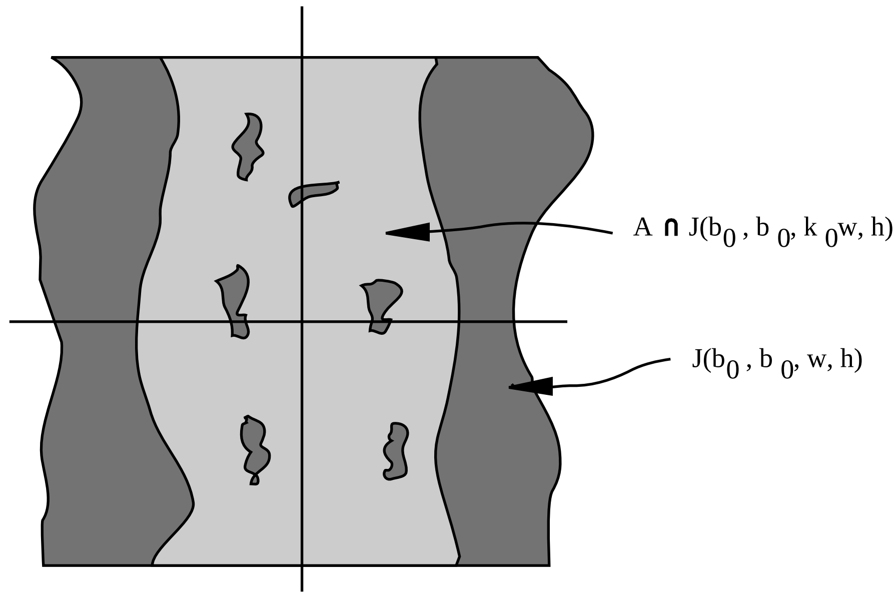

Figure 13: $k_0 < 1$: The density of A in $J(b_0, b_0, w, h)$ is not too small.

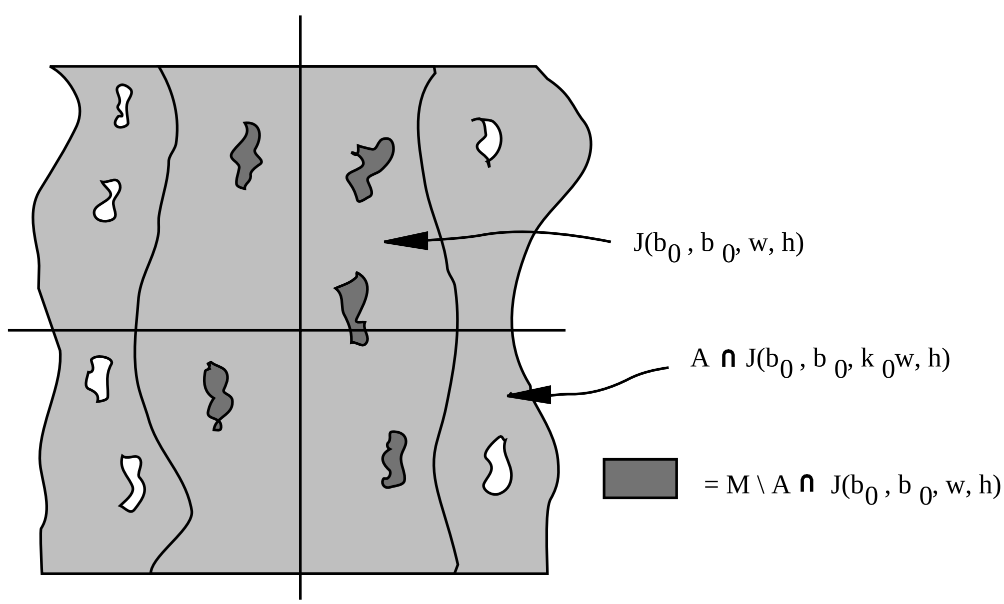

Figure 14: $k_0 > 1$: The density of $M \setminus A$ in $J(b_0, b_0, w, h)$ is not too large.

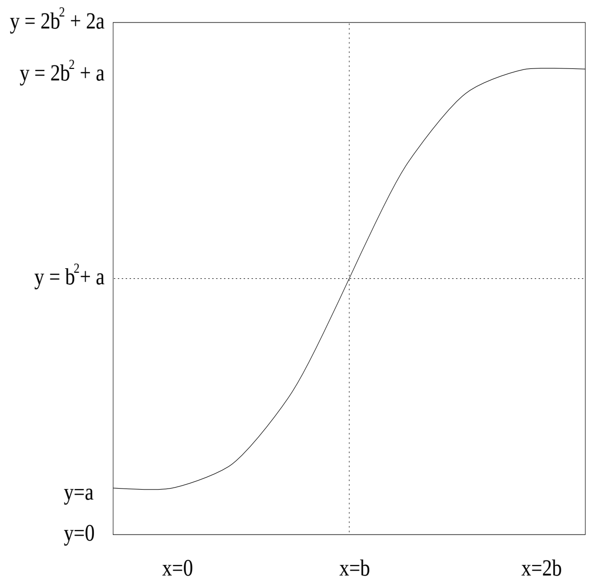

Figure 15: The graph $y = f_c(x)$.

5 Appendix

In this section we construct a uniquely integrable Hölder continuous vector field whose integral foliation is not Hölder continuous.

Given real parameters $c = (a, b)$, with $1 > b > a > 0$, define $f_c : \mathbf{R} \to \mathbf{R}$ by:

$$f_c(x) = \begin{cases} a & \text{if } x < 0 \\ x^2 + a & \text{if } 0 \leq x < b \\ 2b^2 + a - (x - 2b)^2 & \text{if } b \leq x < 2b \\ 2b^2 + a & \text{if } x \geq 2b. \end{cases}$$

The graph of f_c is shown in Figure 15. We use f_c to construct a unit-length vector field X_c, defined in the strip $S_c = 0 \leq y \leq 2b^2 + 2a$ in $\mathbf{R}^2$. Ultimately, we will stack these strips vertically, varying the parameter c, to obtain a vector field on $\mathbf{R}^2$. The slope of the vector field X_c is the vertical linear interpolation between 0 on the lines $x = 0$ and $x = 2b^2 + 2a$, and the unit tangent vector field to the graph of f_c. In other words,

$$X_c(x, y) = \frac{1}{\sqrt{1 + s_c(x, y)^2}} \cdot (\frac{\partial}{\partial x} + s_c(x, y) \cdot \frac{\partial}{\partial y}),$$

where:

$$s_c(x, y) = \begin{cases} f_c'(x)y/f_c(x) & \text{if } y \leq f_c(x) \\ (2b^2 + 2a - yf_c'(x)/(2b^2 + 2a - f_c(x))) & \text{if } f_c(x) < y \leq 2b^2 + 2a. \end{cases}$$

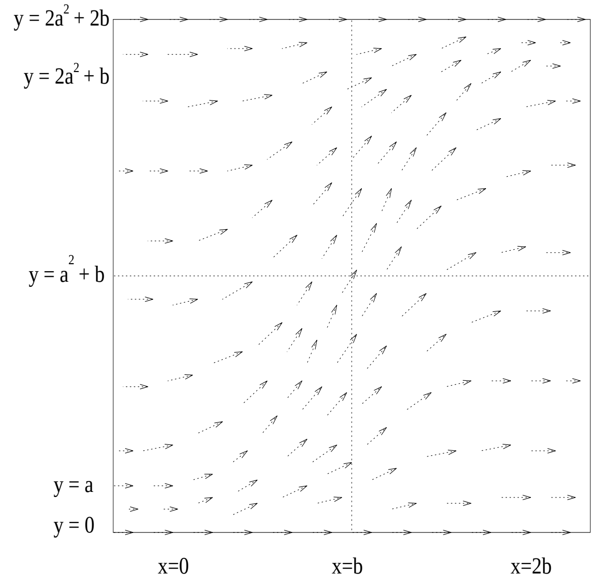

Figure 16: The vector field X_c.

Figure 16 shows X_c. Note that X_c is C^1 and hence uniquely integrable in S_c.

Lemma 5.1 *Given $c = (a, b)$ as above, the function $s_c(x, y)$ is Hölder continuous in the strip S_c, with exponent $\frac{1}{2}$ and constant M, where M is independant of c.*

Proof of Lemma 5.1: The strip S_c is naturally divided into four regions I-IV, as shown in in Figure 17. We first show that s_c is Hölder in a given (closed) region. By symmetry, it suffices to prove the claim for regions I and II.

Suppose (x, y) and (x, y') are contained in the same region, either I or II (that is, $0 \le y, y' \le f_c(x)$). Note that $|f_c'(x)| \le 2|f_c(x)|^{\frac{1}{2}}$. It follows that

$$\begin{aligned} |s_c(x,y) - s_c(x,y')| &= |(y-y')\frac{f_c'(x)}{f_c(x)}| \\ &\le |y-y'| \cdot 2(f_c(x))^{-\frac{1}{2}} \\ &\le 2|y-y'|^{\frac{1}{2}} \end{aligned}$$

Now suppose that (x, y) and (x', y) are both contained in region I of S_c. An application of the Mean Value Theorem shows that

$$|s_c(x,y) - s_c(x',y)| \le 2|x - x'|$$

Similarly, for (x, y) and (x', y) in region II,

$$|s_c(x,y) - s_c(x',y)| \le 6|x - x'|.$$

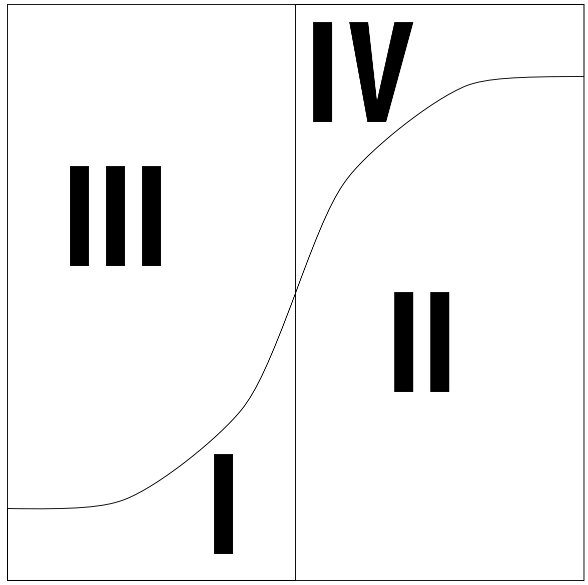

Figure 17: Regions I-IV in the strip S_c.

Hence s_c is 1/2 - Hölder continuous with constant 2 along vertical lines in a given region, and Lipschitz with constant 6 along horizontals. For sufficiently small values of a, b , the 1/2 - Hölder constant of s_c along horizontals is also 6. Finally, suppose that (x, y) and (x, y') are in different regions. Then

$$\begin{aligned}
|s_c(x,y) - s_c(x,y')| &\leq |s_c(x,y) - s_c(x, f_c(x))| + |s_c(x, f_c(x)) - s_c(x,y')| \\
&\leq 6(|y - f_c(x)|^{\frac{1}{2}} + |f_c(x) - y'|^{\frac{1}{2}}) \\
&\leq 6\sqrt{2}|y - y'|^{\frac{1}{2}}
\end{aligned}$$

Similarly, if $(x, y), (x', y)$ lie in different regions of S_c, then

$$|s_c(x,y) - s_c(x',y)| \leq 6\sqrt{2}|x - x'|^{\frac{1}{2}}.$$

Finally, for arbitrary $(x, y), (x', y')$ in S_c, we have

$$\begin{aligned}
|s_c(x,y) - s_c(x',y')| &\leq |s_c(x,y) - s_c(x,y')| + |s_c(x,y') - s_c(x',y')| \\
&\leq 6\sqrt{2}(|y - y'|^{\frac{1}{2}} + |x - x'|^{\frac{1}{2}}) \\
&\leq 6 \cdot (2)^{\frac{3}{4}} d((x,y),(x',y'))^{\frac{1}{2}}
\end{aligned}$$

Set $M = 6 \cdot (2)^{\frac{3}{4}}$ and the proof is complete. □

Hence the vector field X_c is Hölder-continuous, with Hölder constant M. Note that by modifying our construction of X_c, we could have replaced the Hölder exponent $\frac{1}{2}$ by any fixed real number between 0 and 1.

We now proceed to the construction of the example.

Proposition 5.2 *There is a vector field X on $\mathbf{R}^2$ that is $1/2$-Hölder continuous, uniquely integrable, whose integral foliation fails to be α-Hölder continuous, for any $\alpha > 0$.*

Proof of Proposition 5.2:

For $n \geq 1$, let $c(n) = (2^{-n}, 2^{-\frac{\sqrt{n}}{2}})$. Let

$$h_n = \sum_{k=n+1}^{\infty} 2(2^{-k} + 2^{-\sqrt{k}}),$$

$$h = \sum_{k=1}^{\infty} 2(2^{-k} + 2^{-\sqrt{k}}),$$

To simplify notation, let $X_n(x, y) = X_{c(n)}(x, y)$, $s_n = s_{c(n)}$, $f_n = f_{c(n)}$, and let $S_n = S_{c(n)}$. Define X as follows:

$$X(x, y) = \begin{cases} X_n(x, y - h_n) & \text{if } h_n \leq y < h_{n+1}, \text{ for some } n \geq 1 \\ \frac{\partial}{\partial x} & \text{otherwise.} \end{cases}$$

Note that the vertical width of the strip S_n is $2(2^{-n} + 2^{-\sqrt{n}})$ so this defines a continuous vector field on $\mathbf{R}^2$. Clearly X is uniquely integrable outside the closed region $0 \leq y \leq h$. It is uniquely integrable in the region $0 < y \leq h$ because it is uniquely integrable in every strip $S_n + (0, h_n)$. Since the lines $y = h_n$ are unique integral curves of X, the line $y = 0$ is also a unique integral curve. So X is uniquely integrable. Let $\mathcal{F}$ be the foliation of $\mathbf{R}^2$ tangent to X.

By Lemma 5.1, X is $1/2$-Hölder-continuous in every strip S_n, with Hölder constant M (which is independent of n). To check that X is Hölder-continuous on all of $\mathbf{R}^2$, it remains to estimate, for $(x, y) \in S_n$ and $(x', y') \in S_{n'}$ with $n' > n$, the quantity:

$$|s_n(x, y) - s_{n'}(x', y')|.$$

Assume without loss of generality that $s_n(x, y) \geq s_{n'}(x', y')$. Then, since $s_n \geq 0$, it follows that:

$$\begin{aligned} |s_n(x, y) - s_{n'}(x', y')| &\leq |s_n(x, y)| \\ &= |s_n(x, y) - s_n(x', h_n)| \\ &\leq M \cdot d((x, y), (x', h_n))^{\frac{1}{2}} \\ &\leq M \cdot d((x, y), (x', y'))^{\frac{1}{2}}. \end{aligned}$$

(A similar argument works when $s_n(x, y) < s_{n'}(x', y')$).

To see that $\mathcal{F}$ fails to be Hölder-continuous at the point $(0, 0)$, first observe that the curves $\{(x, h_n + f_n(x)) \mid x \in \mathbf{R}\}$ are integral curves of X, as are the lines $y = h_n$. For $\delta > 0$, there exists an N such that for $n \geq N$,

$$\begin{aligned} \sup_{x \in [-\delta, \delta]} |(h_n + f_n(x)) - h_n| &= \sup_{x \in \mathbf{R}} f_n(x) \\ &= 2 \cdot 2^{-\sqrt{n}} + 2^{-n}, \end{aligned}$$

which is $O(2^{-\sqrt{n}})$. On the other hand, for all $\delta > 0$ and $n > 0$,

$$\begin{aligned} \inf_{x\in[-\delta,\delta]} |(h_n + f_n(x)) - h_n| &= \inf_{x\in\mathbf{R}} f_n(x) \\ &= 2^{-n}, \end{aligned}$$

which is $O(2^{-n})$. This implies that for every neighborhood U of the origin $(0,0)$ there exists $N > 0$ such that for every $n \geq N$, there exist vertical transversals τ_1 and τ_2 and points $z, z', w, w' \in U$ such that $z, z' \in \tau_1$, $w = \tau_2 \cap \mathcal{F}(z)$, $w' = \tau_2 \cap \mathcal{F}(z')$, and

$$d(w, w') \geq d(z, z')^{\frac{1}{\sqrt{n}}}.$$

Clearly no $\mathcal{F}$-holonomy map originating at the origin is α-Hölder-continuous, for any $\alpha > 0$. $\square$

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