Hölder Regularity of Horocycle Foliations

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1 Introduction

Let $M$ be a $C^\infty$, nonpositively curved manifold. A *horosphere* in $M$ is the projection to $M$ of a limit of metric spheres in the universal cover $\tilde{M}$ (see §2). A *horospherical foliation* $\mathcal{H}$ is a foliation of the unit tangent bundle $T^1M$ whose leaves consist of unit normal vector fields to horospheres.\(^1\)

While regularity of horospherical foliations has been studied extensively for negatively curved manifolds $M$, considerably less is known in the non-positively curved case. The most general result is due to P. Eberlein: if $M$ is complete and nonpositively curved, then horospheres are $C^2$, which implies that the individual leaves of $\mathcal{H}$ are $C^1$. Further, the tangent distribution $TH$ depends continuously the basepoint $v \in T^1M$ (see [9]).

Beyond Eberlein’s theorem, smoothness results have consisted mainly of counterexamples ([2],[5]); in particular, the best one could hope for in the case of a general compact, nonpositively curved $M$ is for $TH$ to be Hölder-continuous. In this paper we prove

**Theorem I':** Let $S$ be a compact, real-analytic, nonpositively curved surface. Then $TH$ is Hölder.

Theorem I' is actually a corollary of a more general result, Theorem I below.

The problem of finding the regularity of horospherical foliations has a long history, which we briefly summarize here.

\(^1\)As we explain in §2, there are two such foliations, $\mathcal{H}^-$ and $\mathcal{H}^+$, called *stable* and *unstable* horospherical foliations, respectively. In this discussion, we use $\mathcal{H}$ to denote either of these.
E. Hopf showed in [7] that if \( M \) is a compact, negatively curved surface, then \( T\mathcal{H} \) is \( C^1 \). Under the assumption that the sectional curvatures of \( M \) are \( 1/4 \)-pinched, Hopf’s result was generalized by M. Hirsch and C. Pugh [10] to any dimension. It follows from the work of D. V. Anosov [1] that \( T\mathcal{H} \) is always Hölder, when \( M \) is compact and negatively curved.

In Anosov’s theorem, the conclusion “Hölder” cannot be improved to “\( C^1 \)”.

Returning to the compact, nonpositive curvature case, Gerber and V. Niţică [5] have examples of real-analytic surfaces showing that \( T\mathcal{H} \) in Theorem I’ can fail to have a Hölder exponent greater than \( 1/2 \). In particular, \( T\mathcal{H} \) can be non-Lipschitz.

A related issue is that of the regularity of \( T\mathcal{H} \) along the leaves of \( \mathcal{H} \); that is, how smooth are the leaves of \( \mathcal{H} \)? For \( M \) compact and negatively curved, Anosov [1] showed that the leaves of \( \mathcal{H} \) are \( C^\infty \). In the case of nonpositive curvature, Eberlein’s “\( C^1 \)” conclusion cannot be improved to “\( C^2 \)”;

**Theorem II**: Let \( S \) be a compact, real-analytic, nonpositively curved surface. Then the leaves of \( \mathcal{H} \) are uniformly \( C^{1+\text{Lipschitz}} \).

Our interest in these questions arose while studying the ergodic properties of the geodesic flow for analytic, nonpositively curved surfaces. We asked whether the time-one map of such a flow remains ergodic under suitable perturbations. Related results for negatively curved manifolds use Hölder continuity of the horospherical foliations in a central way ([6], [13], [12]). We hope that Theorems I and II can be used to establish similar results for
certain nonpositively curved surfaces.

1.1 Statement of Results

Throughout this paper we always assume that manifolds are boundaryless. We follow the usual convention of referring to horospheres as “horocycles” when $M$ has dimension 2.

Theorem I: Let $S$ be a compact surface with a $C^\infty$ metric of nonpositive curvature $K$ satisfying the following conditions:

1) If $\gamma$ is a geodesic that is not closed, then there is no infinite time interval $I$ for which $K(\gamma(t)) = 0$, for all $t \in I$.

2) If $\gamma$ is a closed geodesic, then there exists a $t$ such that $K$ does not vanish to infinite order at $\gamma(t)$.

Then the leaves of the horocycle foliations $H^+$ and $H^-$ are uniformly $C^{1+\text{Lipschitz}}$.

Theorem II: Let $S$ be a compact surface with a $C^\infty$ metric of nonpositive curvature satisfying the conditions of Theorem I. Then the tangent distributions $TH^+$ and $TH^-$ are Hölder-continuous.

Proof of Theorems I' and II' from I and II. If $S$ is real-analytic, then the set of points in $S$ where $K$ vanishes is a real-analytic subvariety in $S$. In particular, $K$ cannot vanish on an infinite time interval on a non-closed geodesic nor can $K$ vanish to all orders at a point, unless it vanishes identically on the surface. In this case, $S$ is a flat torus or Klein bottle and the horocycle foliations are analytic. □

Remark: It is an open question whether Theorems I and II hold without hypotheses 1) and 2). It is also not known whether there exist $C^\infty$ surfaces that fail to satisfy hypothesis 1), except if the curvature vanishes identically. There are Lipschitz metrics with this property [3]. At the end of §3 we give an example to show that the estimates on the curvatures of the horocycles that are used in our proofs do not hold without hypothesis 2).

We also have an easier version of Theorem I, with a weaker conclusion, but which holds without the assumptions 1) and 2).
**Proposition III:** Let $S$ be a compact surface with a $C^4$ metric of nonpositive curvature. Then the leaves of the horocyclic foliations $\mathcal{H}^+$ and $\mathcal{H}^-$ are uniformly $C^{1+1/2}$; that is, $TH^\pm$ is uniformly $1/2$-Hölder along leaves.

### 1.2 Outline of the proofs

To prove these results, we study the dependence on $v \in T^1S$ of solutions to the scalar Riccati equation

$$u'(t) + u(t)^2 + K(\sigma_v(t)) = 0,$$

where $\sigma_v(t)$ is the unit-speed geodesic determined by $v$, and $K : S \to \mathbf{R}$ is the curvature. In §2 we explain how Hölder regularity of $TH$ amounts to Hölder dependence on $v$ of the “unstable” solutions to the Riccati equations.

In §3 we turn to a study of these Riccati equations. The analysis begins by taking the difference of two Riccati solutions $u_0$ and $u_1$ along geodesics determined by $v_0$ and $v_1$, to obtain

$$ (u_1 - u_0)' = -(u_1 + u_0)(u_1 - u_0) + (K_0 - K_1), \quad (1.1) $$

where $K_0$ and $K_1$ are the curvatures of $S$ along these geodesics. To obtain our regularity results, we need $|(u_1 - u_0)(0)|$ to be small relative to the distance between $v_0$ and $v_1$. It is apparent from (1.1) that $|u_1 - u_0|$ decreases rapidly if $u_1 + u_0$ is large relative to $|K_0 - K_1|$. The remainder of the proofs is devoted to estimating the sizes of these terms.

The proof of Proposition III depends only on Lemmas 3.1 and 3.2. For the proofs of Theorems I and II, we need the additional Lemmas 3.3 - 3.5. The proof of the lower bound in Lemma 3.3 is presented in §4.

### 1.3 Acknowledgements

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2 Preliminaries

Let $M$ be a complete $n$-dimensional manifold of nonpositive sectional curvatures and let $\tilde{M}$ be its universal cover. We now define the horospherical foliations discussed in the introduction. For a unit vector $v$ let $\sigma_v$ denote the geodesic in $M$ (or $\tilde{M}$) with $\sigma'_v(0) = v$. Vectors $v, w \in T^1\tilde{M}$ are asymptotic if there exists a constant $C > 0$ such that for all $t > 0$, $\text{dist}(\sigma_v(t), \sigma_w(t)) \leq C$. Nonpositive curvature and simple connectivity imply that for every $v \in T^1\tilde{M}$ and $p \in \tilde{M}$, there is a unique vector $Z_v(p) \in T^1p\tilde{M}$ such that $Z_v(p)$ is asymptotic to $v$. Fixing $v$, this defines a radial vector field $Z_v$ on $\tilde{M}$. The vector $v$ also determines a Busemann function $F_v : \tilde{M} \to \mathbb{R}$ by:

$$F_v(p) = \lim_{t \to \infty} (\text{dist}(p, \sigma_v(t)) - t).$$

It is well-known (see, e.g. [9]) that $F_v$ is $C^1$, $Z_v$ is the gradient of $-F_v$, and each level set $F_v^{-1}(c)$ is the limit of geodesic spheres of radius $t + c$ centered at $\sigma_v(t)$. Moreover, as was shown by Eberlein, Busemann functions are $C^2$, and consequently their level sets are $C^2$ [9].

For $v \in T^1\tilde{M}$, define the stable and unstable horospheres $h^-(v)$ and $h^+(v)$ determined by $v$ to be the level sets $F_v^{-1}(0)$ and $F_v^{-1}(0)$, respectively. The leaves of the stable and unstable horospherical foliations $\mathcal{H}^-$ and $\mathcal{H}^+$ of $T^1\tilde{M}$ are defined by:

$$\mathcal{H}^-(v) = \{Z_v(p) : p \in h^-(v)\}$$

and

$$\mathcal{H}^+(v) = \{-Z_v(p) : p \in h^+(v)\}.$$ 

Since Busemann functions are $C^2$, the leaves of $\mathcal{H}^\pm$ are $C^1$, and the tangent distributions $T\mathcal{H}^\pm$ are defined.

We project the horospheres from $\tilde{M}$ into $M$ to obtain horospheres for vectors in $T^1M$. Similarly, we obtain the horospherical foliation of $T^1M$.

We are interested in the regularity of $T\mathcal{H}^\pm$, which reduces to the regularity of the sectional curvature of the horospheres. These sectional curvatures are determined by solutions to certain Riccati and Jacobi equations. We now restrict to the case where $M$ is a surface, $S$, and these equations can be reduced to scalar ones.

Let $v \in T^1_p\tilde{S}$ and $w \in T_p\tilde{S}$, and let $J_\pm [J_\pm]$ be the stable [unstable] Jacobi field along $\sigma_v$ with $J_\pm(0) = w$ [$J_\pm(0) = w$]. (The stable Jacobi field is defined $J_- = \lim_{n \to \infty} J_n$, where $J_n$ is the Jacobi field along $\sigma_v$ with $J_n(0) = w$ and
The unstable Jacobi field \( J_+ \) is defined by the same formula, except replacing \( \lim_{n \to \infty} \) by \( \lim_{n \to -\infty} \). If \( Z_v \) is the radial vector field defined above, then \( \nabla_w Z_v = J'_-(0) \), by Proposition 3.1 in [9]. Now assume \( w \) is a unit vector perpendicular to \( \sigma_v \) and let \( E \) be the continuous, unit-length vector field along \( \sigma_v \) that is perpendicular to \( \sigma_v \) and satisfies \( E(0) = w \). Then \( J_-(t) = j_-(t) E(t) \), where \( j_- \) is a real-valued function that satisfies the scalar Jacobi equation:

\[
j''_-(t) = -K(\sigma_v(t)) j_-(t).
\]

Let \( u_- = j'_-/j_- \). Then \( u_- \) satisfies the scalar Riccati equation

\[
u'_-(t) + u_-(t)^2 + K(\sigma_v(t)) = 0.
\]

Since \( j_-(0) = 1 \),

\[
u_-(0) = j'_-(0) = \langle \nabla_w Z_v, w \rangle = k_-(v),
\]

where \(-k_-(v)\) is the geodesic curvature of \( h_-(v) \) at \( v \). The function \( u_- \) is called the stable solution to the Riccati equation along \( \sigma_v \); since \( J_- \) was constructed as \( \lim_{n \to -\infty} J_n \), it follows that \( u_-(t) = \lim_{n \to -\infty} u_n(t) \), where, for \( n > 0 \), \( u_n \) is the solution to the Riccati equation along \( \sigma_v \) with \( u_n(n) = -\infty \). The unstable Riccati solution \( u_+ \) along \( \sigma_v \) is similarly defined in terms of \( J_+ \) and satisfies \( u_+(t) = \lim_{n \to -\infty} u_n \), where, for \( n < 0 \), \( u_n \) is the solution to the Riccati equation along \( \sigma_v \) with \( u_n(n) = +\infty \). A similar argument to the one summarized in equation (2.1) shows that \( u_+(0) = k_+(v) \), where \( k_+(v) \) is the geodesic curvature of \( h^+(v) \) at \( v \). Since \( K \leq 0 \), it follows that \( u_-(t) \leq 0 \) for all \( t \), and \( u_+(t) \geq 0 \) for all \( t \). Moreover, if \( K(\sigma_v(t_0)) < 0 \) for some \( t_0 \), then \( u_-(t) < 0 \), for all \( t < t_0 \), and \( u_+(t) > 0 \), for all \( t > t_0 \). (These inequalities are easy consequences of Lemma 3.1 below.)

A function \( f \) from a metric space \((X_1, d_1)\) to a metric space \((X_2, d_2)\) is Hölder-continuous of exponent \( \alpha \in (0,1] \) if there exists a constant \( C > 0 \) such that for all \( p, q \in X_1 \),

\[
d_2(f(p), f(q)) \leq C(d_1(p, q))^\alpha.
\]

The function \( f \) is Lipschitz if it is Hölder with exponent 1. We say that \( f \) is Hölder (or Lipschitz) at a point \( p \in X_1 \) if there is a constant \( C = C(p) > 0 \) such that inequality (2.2) holds for all \( q \in X_1 \). A family of functions \( \mathcal{F} \) from
X_1 to X_2 is uniformly Hölder (or Lipschitz) if there is a single constant C such that (2.2) holds for all p, q ∈ X_1 and for all f ∈ F.

Throughout this paper all geodesics have unit speed. We will use Fermi coordinates (s, x) along a geodesic γ in ˜S, where s is the time parameter along γ, and x is the signed distance to γ. Then the curves s = constant are unit-speed geodesics perpendicular to γ. We will frequently use φ to denote the angle between a vector v and the curve x = constant; unless stated otherwise, such angles will be signed angles in [−π/2, π/2] chosen so that <(∂/∂x, x = a) = π/2.

3 Proofs of Theorems I and II

This section contains the proofs of Theorems I and II, with the exception of the proof of the lower bound on the curvatures of horocycles in Lemma 3.3. This lower bound is proved in §4.

The following lemma contains facts which are routinely used in the study of Riccati and Jacobi equations. For example, part (iii) is the Comparison Lemma in [2] and it is also a special case of a well-known differential inequality ([7], Chapter III, Corollary 4.2). Part (v) is a special case of the Sturm Comparison Theorem.

Lemma 3.1 Let A and B be real numbers, let K, K_0, K_1 : R → R be continuous functions, and let u, u_0, u_1 satisfy the Riccati equations u' = −u^2 − K, u'_i = −u_i^2 − K_i, i = 0, 1 respectively. Let y = u_1 − u_0. Let J_0, J_1 satisfy the Jacobi equations J''_i = −K_i J_i, i = 0, 1. Then

(i) y' = −(u_0 + u_1)y + K_0 − K_1

(ii) y(B) = \int_A^B (K_0(t)) − K_1(t)) \exp \left[ − \int_t^B (u_0(\tau) + u_1(\tau)) d\tau \right] dt

+ y(A) \exp \left[ − \int_A^B (u_0(\tau) + u_1(\tau)) d\tau \right]

(iii) If A ≤ B, u_0(A) ≤ u_1(A) and K_1(t) ≤ K_0(t) for t ∈ [A, B], then u_0(B) ≤ u_1(B).

(iv) If A ≤ B, K(t) ≤ 0 for t ∈ [A, B], and u(A) ≥ 0, then

\[ u(B) ≥ \frac{u(A)}{(B − A)u(A) + 1} \]
(v) If $A \leq B$, $J_0(A) = J_1(A) = 0$, $0 \leq J'_0(A) \leq J'_1(A)$ and $K_1(t) \leq K_0(t) \leq 0$ for $t \in [A, B]$, then $J_0(B) \leq J_1(B)$.

**Proof.** Property (i) is obtained by subtracting the Riccati equation for $u_0$ from the Riccati equation for $u_1$, and (ii) follows from the formula for the solution of first order linear differential equations. It is clear from (ii) that if $A \leq B$, $y(A) \geq 0$ and $K_0(t) - K_1(t) \geq 0$ for $t \in [A, B]$, then $y(B) \geq 0$. This proves (iii). Now (iv) is a special case of (iii), because if $K_0 \equiv 0$ and $u_0(A) = u(A)$, then $u_0(t) = u(A)/(t - A)u(A) + 1$. For the proof of (v), see §10.2 of [4]. □

The following lemma will be applied to the curvature function $f = K$. In this lemma the complete surface $S$ could easily be replaced by a complete Riemannian manifold.

**Lemma 3.2** If $f$ is a nonpositive function on a complete surface $S$ such that $|(d^2/dt^2)(f(\sigma(t)))|$ exists and is uniformly bounded from above along all geodesics $\sigma$, then there exist constants $L_1, L_2 > 0$ such that for all $p, q \in S$,

$$|f(p) - f(q)| \leq L_1 \epsilon \sqrt{-f(p)} + L_2 \epsilon^2, \quad (3.1)$$

where $\epsilon = \text{dist}(p, q)$.

**Proof.** Let $L = \sup\{|(d^2/dt^2)(f(\sigma(t)))| : \sigma \text{ is a geodesic on } S\}$. Let $p \in S$, and let $\sigma$ be a geodesic on $S$ such that $\sigma(0) = p$ and $\sigma'(0)$ is in a direction of the greatest increase of $f$ at $p$; i.e., $(d/dt)|_{t=0}(f(\sigma(t))) = ||Df_p||$. Let $g : \mathbb{R} \to \mathbb{R}$ satisfy $g(0) = f(\sigma(0)), g'(0) = (d/dt)|_{t=0}(f(\sigma(t)))$ and $g''(t) = -L$ for all $t$.

Then for $t \geq 0$,

$$0 \geq f(t) \geq g(t) = f(p) + ||Df_p||t - \frac{1}{2}Lt^2.$$ 

Setting $t = ||Df_p||/L$, we obtain $||Df_p|| \leq \sqrt{-2Lf(p)}$. Let $L_1 = \sqrt{2L}, L_2 = L/2$. Then the lemma follows from Taylor’s Theorem. □

In the following lemmas, we begin invoking our hypotheses 1) and 2) on the surface $S$. 

8
Lemma 3.3  Suppose $S$ is a complete surface of nonpositive curvature $K$ and $\gamma(s)$ is a closed geodesic on $S$ such that $K$ vanishes to order $m-1$ on $\gamma$, where $m \in \{2, 4, 6, \ldots\}$. (I.e., if $(s, x)$ are the Fermi coordinates along the lift $\tilde{\gamma}$ of $\gamma$ to $\tilde{S}$, then

$$\left. \frac{\partial^k}{\partial x^k} \right|_{x=0} K(s, x) = 0$$

for $k = 0, 1, \ldots, m-1$ and all $s$.) Also assume that there is at least one point, say $\gamma(0)$, such that $K$ does not vanish to order $m$ at $\gamma(0)$. (I.e., $(\partial^m / \partial x^m)K(0, x) \neq 0$.) Then there exists a neighborhood $U$ of $T^1\tilde{\gamma}$ in $T^1\tilde{S}$ and a positive constant $C$ such that for any $v \in U$ with footpoint having second Fermi coordinate $x = a$ the curvatures $k_-(v)$ and $k_+(v)$ of the stable and unstable horocycles satisfy

$$C \max \left( |a|^{m/2}, |\phi|^{m/(m+2)} \right) \leq -k_-(v) \leq C^{-1} \max \left( |a|^{m/2}, |\phi|^{m/(m+2)} \right)$$

and

$$C \max \left( |a|^{m/2}, |\phi|^{m/(m+2)} \right) \leq k_+(v) \leq C^{-1} \max \left( |a|^{m/2}, |\phi|^{m/(m+2)} \right),$$

where $\phi = \angle(v, x = a)$.

The upper bounds on $-k_-(v)$ and $k_+(v)$ are proved in [5], Theorem 3.1. The assumption that there is a point on $\gamma$ where the curvature does not vanish to order $m$ is not needed to obtain these upper bounds. The lower bounds are proved in §4.

Note that the hypothesis of Lemma 3.3 could not hold for odd $m$, because $K$ does not change sign.

Lemma 3.4  If $S$ is a surface satisfying the hypotheses of Theorems I and II, then there is a constant $C > 0$ such that for all $v \in T^1S$,

$$-k_-(v) \geq C \sqrt{-K(p)} \tag{3.2}$$

and

$$k_+(v) \geq C \sqrt{-K(p)}, \tag{3.3}$$

where $p$ is the footpoint of $v$.  

9
Proof. If $\gamma$ is a closed geodesic on $S$ along which $K$ vanishes to order $m - 1$, for $m \in \{2, 4, 6, \ldots\}$, then there is a constant $C_1 > 0$ such that $-K(p) \leq C_1|a|^m$ for $p$ in a neighborhood of $\gamma$ with Fermi coordinates $(s, a)$. If, in addition, there is a point on $\gamma$ at which $K$ does not vanish to order $m$, then the lower bounds on $-k_-(v)$ and $k_+(v)$ in Lemma 3.3 imply that (3.2) and (3.3) are satisfied for $v$ with footprint in some neighborhood of $T^1\gamma$, for some constant $C > 0$.

By hypotheses 1) and 2) of Theorems I and II, there are at most finitely many closed geodesics along which $K$ vanishes. Therefore (3.2) and (3.3) hold for $v$ in a neighborhood $U$ of the union of the unit tangent bundles of such geodesics.

Now for $v \in T^1S, k_-(v)$ vanishes only if $K(\sigma_v(t)) = 0$ for all $t \geq 0$, and $k_+(v)$ vanishes only if $K(\sigma_v(t)) = 0$ for all $t \leq 0$. Thus by hypothesis 1), $k_-(v)$ and $k_+(v)$ each vanish only for $v$ in the unit tangent bundle of a closed geodesic along which $K$ vanishes. Then (3.2) and (3.3) extend to the complement of $U$ in $T^1S$ for some $C > 0$ by the continuity of $k_-, k_+$, and $K$. □

**Lemma 3.5** Let $S$ be a surface satisfying the hypotheses of Theorems I and II. Let $\gamma_0$ and $\gamma_1$ be geodesics on $S$ or $\tilde{S}$, let $K_i(t) = K(\gamma_i(t))$, for $i = 0, 1$ and $t \in \mathbb{R}$, and let $A$ and $B$ be real numbers with $A \leq B$. Let $u_i$ be a solution to the Riccati equation $u'_i = -u_i^2 - K_i, i = 0, 1$, where $u_0$ is the unstable solution and $u_1(A) \geq 0$. Let $y = u_1 - u_0$. Then there exist positive constants $C_1$ and $C_2$, which depend only on $S$, such that

$$|y(B)| \leq C_1 \epsilon + C_2(B - A) \epsilon^2 + |y(A)| \exp\left[-\int_A^B (u_0(\tau) + u_1(\tau)) \, d\tau\right],$$

where $\epsilon = \max\{\text{dist}(\gamma_0(t), \gamma_1(t)) : A \leq t \leq B\}$.

**Proof.** By part (ii) of Lemma 3.1, we have

$$|y(B)| \leq \int_A^B |K_0(t) - K_1(t)| \exp\left[-\int_t^B (u_0(\tau) + u_1(\tau)) \, d\tau\right] \, dt + |y(A)| \exp\left[-\int_A^B (u_0(\tau) + u_1(\tau)) \, d\tau\right]$$

10
Moreover, if we apply Lemma 3.2 to \( f = K \), we obtain constants \( L_1, L_2 > 0 \) such that
\[
|K_0(t) - K_1(t)| \leq L_1 \epsilon \sqrt{-K_0(t)} + L_2 \epsilon^2,
\]
where \( \epsilon \) is as in the statement of the present lemma. Also, \( u_1(\tau) \geq 0 \) for \( \tau \geq A \). Thus
\[
|y(B)| \leq L_1 \epsilon \int_A^B \sqrt{-K_0(t)} \exp \left[ - \int_t^B u_0(\tau) \, d\tau \right] dt + L_2 (B - A) \epsilon^2 + |y(A)| \exp \left[ - \int_A^B (u_0(\tau) + u_1(\tau)) \, d\tau \right].
\]
(3.4)

Now, by Lemma 3.4, we obtain
\[
u_0(\tau) \geq C_3 \sqrt{-K_0(\tau)}
\]
for some constant \( C_3 > 0 \), for all \( \tau \in [A, B] \). Therefore the first term on the right hand side of inequality (3.4) is bounded above by
\[
\frac{L_1 \epsilon}{C_3} \int_A^B g(t) \exp \left[ - \int_t^B g(\tau) \, d\tau \right] dt,
\]
(3.5)
where \( g(\tau) = C_3 \sqrt{-K_0(\tau)} \). The expression in (3.5) is equal to
\[
\frac{L_1 \epsilon}{C_3} \left( 1 - \exp \left[ - \int_A^B C_3 \sqrt{-K_0(\tau)} \, d\tau \right] \right),
\]
which is less than or equal to \( L_1 \epsilon / C_3 \).
This proves the lemma with \( C_1 = L_1 / C_3 \) and \( C_2 = L_2 \). \( \square \)

**Proof of Theorem I.** Let \( \gamma_0 \) and \( \gamma_1 \) be two geodesics on \( \tilde{S} \) such that \( \gamma_0'(0) \) and \( \gamma_1'(0) \) are on the same unstable horocycle; i.e.,
\[
\lim_{t \to -\infty} \text{dist}_{\tilde{S}}(\gamma_0(t), \gamma_1(t)) = 0.
\]

To prove that the leaves of the unstable horocycle foliation of \( T^1 \tilde{S} \) (or \( T^1 S \)) are uniformly \( C^{1+\text{Lipschitz}} \), it suffices to show that there exists a constant \( C > 0 \) (depending only on \( S \)) such that
\[
|k_+^{(1)}(\gamma_1(0)) - k_+^{(1)}(\gamma_0(0))| \leq C \epsilon,
\]
11
where $\epsilon = \text{dist}_{\tilde{S}}(\gamma_0(0), \gamma_1(0))$. Since $K \leq 0$, $\text{dist}_{\tilde{S}}(\gamma_0(t), \gamma_1(t)) \leq \epsilon$, for $t \leq 0$. Let $u_i$ be unstable Riccati solutions along $\gamma_i$, $i = 0, 1$. Let $y = u_1 - u_0$. Then $y(0) = k_+(\gamma'_1(0)) - k_+(\gamma'_0(0))$. Now apply Lemma 3.5 with $A = -1/\epsilon$ and $B = 0$. We obtain

$$|y(0)| \leq C_1 \epsilon + C_2 \epsilon + |y(A)| \exp \left[ - \int_A^0 (u_0(\tau) + u_1(\tau)) \, d\tau \right].$$

If $u_0(A) = u_1(A)$, then $y(A) = 0$ and we have the desired estimate. Thus we may assume $u_0(A) \neq u_1(A)$, say $u_1(A) > u_0(A)$. Then

$$|y(A)| \exp \left[ - \int_A^0 (u_0(\tau) + u_1(\tau)) \, d\tau \right] \leq u_1(A) \exp \left[ - \int_A^0 u_1(\tau) \, d\tau \right] \leq u_1(A) \exp \left[ - \int_0^{-A} u_1(\tau + A) \, d\tau \right] \leq u_1(A) \exp \left[ - \int_0^{-A} u_1(A) \, d\tau \right] \leq u_1(A) \left( \frac{1}{A} \right) \leq -\frac{1}{A} = \epsilon.$$

This completes the proof. \( \square \)

The following proof is similar to the proof of Theorem I, but is independent of Lemmas 3.3 - 3.5.

**Proof of Proposition III.** Let $S$ be a compact surface with a $C^4$ metric of nonpositive curvature. Then $K$ is $C^2$ and Lemma 3.2 applies to $f = K$. We use the notation in the proof of Theorem I, but let $A = -1/\sqrt{\epsilon}$ and $B = 0$. By Lemmas 3.1(ii) and 3.2, we have

$$|y(0)| \leq \int_A^0 |K_0(t) - K_1(t)| \, dt + |y(A)| \exp \left[ - \int_A^0 (u_0(t) + u_1(t)) \, dt \right] \leq \epsilon L_1 \int_A^0 \sqrt{-K_0(t)} \, dt + L_2 \epsilon^{3/2} + |y(A)| \exp \left[ - \int_A^0 (u_0(t) + u_1(t)) \, dt \right].$$

By the argument at the end of the proof of Theorem I, the third term in the preceding expression is bounded above by $-1/A = \sqrt{\epsilon}$. Therefore, if $\epsilon \leq 1$, we obtain

$$|y(0)| \leq \sqrt{\epsilon} L_1 \left( \max_{p \in M} \sqrt{-K(p)} \right) + L_2 \epsilon^{3/2} + \sqrt{\epsilon} \leq C \sqrt{\epsilon}$$

12
for some $C > 0$. This proves that the leaves of $\mathcal{H}^+$ are uniformly $C^{1+1/2}$. □

We now turn to the proof of Theorem II. We need an additional lemma.

**Lemma 3.6** If $S$ is a surface satisfying the hypotheses of Theorems I and II, then there is a constant $C > 0$ such that for all $v \in T^1S$,

$$Ck_+(v) \leq -k_-(v) \leq C^{-1}k_+(v).$$

**Proof.** This lemma follows from Lemma 3.3 in the same way that Lemma 3.4 follows from Lemma 3.3. □

**Proof of Theorem II.** We must show that there are constants $\alpha, C > 0$ such that

$$|k_+(v_1) - k_+(v_0)| \leq C(\text{dist}(v_0, v_1))^{\alpha}, \text{ for all } v_0, v_1 \in T^1S. \quad (3.6)$$

Let $\Gamma \subset T^1S$ be the set of unit vectors which are tangent to closed geodesics along which $K$ vanishes.

**Step 1.** We first show that it suffices to prove (3.6) in the case $v_0$ and $v_1$ have the same footpoint. I.e., we will show that (3.6) will follow for some $C > 0$ if there is a constant $\bar{C} > 0$ such that for all $p \in S$ and $v_0, v_1 \in T^1_pS$,

$$|k_+(v_1) - k_+(v_0)| \leq \bar{C}\theta^\alpha, \text{ where } \theta = \langle v_0, v_1 \rangle. \quad (3.7)$$

We now define a “horocyclic coordinate system,” $(S, X, \Phi) = (S, X, \Phi)_{v_0}$ about a vector $v_0 \in T^1S$ with footpoint $p_0$. Let $W = W_{v_0} = -Z_{-v_0}$ where $Z_{-v_0}$ is the radial vector field consisting of vectors asymptotic to $-v_0$. (See §2.) A vector $v$ in a neighborhood of $v_0$ has coordinates $(S, X, \Phi)$ if $v$ makes signed angle $\Phi$ with $W_{v_0}$ and the footpoint $p$ of $v$ can be obtained as follows. Let $h^+(v_0)$ be the unstable horocycle determined by $v_0$, as defined in §2. Start at $p_0$ and go along $h^+(v_0)$ for a signed distance $X$, to a point $\tilde{p}$. Then apply the geodesic flow to $W(\tilde{p})$ for time $S$ to arrive at the vector $W(p)$. (See Figure 3.1.) Note that the curves $X = \text{constant}, \Phi = 0$ are geodesics and the curves $S = \text{constant}, \Phi = 0$ are unstable horocycles. Also, along any geodesic $X = \text{constant}, \partial/\partial S$ is a Jacobi field corresponding to the variation of geodesics defined by $S \to (X = S)$. Therefore the $(S, X)$ coordinate system is $C^2$ in a neighborhood of $p_0$ in $S$ and the $(S, X, \Phi)$ coordinate system is $C^1$ in a neighborhood of $v_0$ in $T^1S$. Moreover, there is an $\eta > 0$ such that
the $(\mathcal{S}, X, \Phi)_{v_0}$ coordinate system is defined on an open ball $\mathcal{B}(v_0, \eta)$ about $v_0$, for each $v_0$ in $T^1S$. We may also assume that $\eta$ is chosen sufficiently small so that the images of the maps $(\mathcal{S}, X, \Phi)_{v_0} : \mathcal{B}(v_0, \eta) \to \mathbb{R}^3$ lie in $(-1, 1) \times (-1, 1) \times (-1, 1)$. The $C^1$ norms of these maps are uniformly bounded by some positive constant $C_3$.

Suppose $\text{dist}(v_0, v_1) < \eta$ and $v_0, v_1 \in T^1S$. Let $(\mathcal{S}, X, \Phi) = (\mathcal{S}, X, \Phi)_{v_0}$. Assume that (3.7) holds. Note that $k_+$ is uniformly Lipschitz along unstable horocycles (by Theorem I) and $k_+$ is uniformly $C^1$ (in fact, $C^\infty$) along the orbits of the geodesic flow. Thus there exist positive constants $C_4, C_5, C_6$ such that

\[
|k_+(v_1) - k_+(v_0)| \leq C_4|X| + C_5|\mathcal{S}| + \tilde{C}|\Phi|^\alpha \\
\leq 3 \max(C_4, C_5, \tilde{C}) \|(X, \mathcal{S}, \Phi)\|^\alpha \\
\leq 3 \max(C_4, C_5, \tilde{C})C_6^\alpha \left(\text{dist}(v_0, v_1)\right)^\alpha. \tag{3.8}
\]

By increasing $C_6$, if necessary, we may assume that $|k_+(v_1) - k_+(v_0)|$ is less than or equal to the expression in (3.8) even if we drop the assumption that $\text{dist}(v_0, v_1) < \eta$. This completes the reduction of (3.6) to (3.7), and we proceed with the proof of (3.7).

**Step 2. Application of Lemma 3.5.**

Let $p \in \mathcal{S}$, let $v_0, v_1 \in T^1p\mathcal{S}$ and let $\theta = \angle(v_0, v_1)$. For $i = 0, 1$, let $\gamma_i$ be a geodesic on $\mathcal{S}$ with $\gamma_i(0) = v_i$. For $0 \leq r \leq 1$, let $\gamma_r$ be a continuous variation of geodesics with $\gamma(r) = \gamma_0(0) = \gamma_1(0) = p$ and $\angle(\gamma_0(0), \gamma_r(0)) = r\theta$. Let

\[
T = \max\{T_0 : \text{length of curve } r \to \gamma_r(t), \ 0 \leq r \leq 1, \text{ is less than or equal to } \sqrt{\theta}, \text{ for } -T_0 \leq t \leq 0\}. \tag{3.9}
\]

Let $K_r(t) = K(\gamma_r(t))$ and let $J_r$ be the perpendicular Jacobi field along $\gamma_r$ defined by $J_r(t) = (d/dr)(\gamma_r(t))$. Let $J_r(t) = j_r(t)E_r(t)$, where $E_r$'s are unit normal fields along $\gamma_r$'s oriented so that $j_r(t) > 0$ for $t < 0$. Then $j_r(0) = 0$ and $j_r(0) = -\theta$. By comparing with the $K \equiv 0$ case and applying Lemma 3.1 (v) (with $t$ replaced by $-t$), we have $\tilde{j}_r(-T) \geq \theta T$. From the definition of $T$ we have

\[
\sqrt{\theta} = \int_0^1 j_r(-T)dr \geq \theta T. \tag{3.10}
\]
Therefore $T \leq 1/\sqrt{\theta}$. (See Figure 3.2.) Similarly, by comparing with the $K \equiv K_{\min}$ case, where $K_{\min} < 0$ is the minimum value of the curvature function on $S$, we obtain $j_r(-T) \leq (\theta/\sqrt{|K_{\min}|})\sinh(\sqrt{|K_{\min}|T})$. Thus there exists $\theta_0 > 0$ such that if $\theta < \theta_0$, then $T > 1$. Since it is clear that there is a $\tilde{C}$ such that (3.7) holds for $\theta \geq \theta_0$, we will henceforth assume that $T > 1$. (This will be used in Step 3.)

Let $u_i, i = 0, 1$, be the unstable solution of $u_i' = -u_i^2 - K_i$. Let $y = u_1 - u_0$ and apply Lemma 3.5 with $A = -T, B = 0$ and $\epsilon = \sqrt{\theta}$. We obtain constants $C_7, C_8, C_9 > 0$ such that

$$|k_+ + (v_1) - k_+ (v_0)| = |y(0)|$$

$$\leq C_7 \sqrt{\theta} + C_8 T \theta + |y(-T)| \exp \left[ - \int_{-T}^{0} (u_0(t) + u_1(t)) \, dt \right]$$

$$\leq (C_7 + C_8) \sqrt{\theta} + C_9 (\exp \left[ - \int_{-T}^{0} u_0(t) \, dt \right]) (\exp \left[ - \int_{-T}^{0} u_1(t) \, dt \right]).$$

(3.11)

Thus we must estimate $\exp \left[ - \int_{-T}^{0} u_0(t) \, dt \right]$ and $\exp \left[ - \int_{-T}^{0} u_1(t) \, dt \right]$ from above. We first estimate a related integral.

**Step 3.** Let $w_r = j'_r / j_r$. Then $w_r' = -w_r^2 - K_r$, $w_r(0) = -\infty$, and $w_r(t) < 0$ for $t < 0$. In this step, we will show that there is a constant $C_{10} > 0$ such that

$$\exp \left[ \int_{-T}^{1} w_0(t) \, dt \right] \leq C_{10} \sqrt{\theta}. \quad (3.12)$$

We have

$$\frac{j_r(-1)}{j_r(-T)} = \exp \left[ \int_{-T}^{-1} \frac{j'_r}{j_r} \, dt \right] = \exp \left[ \int_{-T}^{-1} w_r \, dt \right].$$

Thus

$$j_r(-T) = j_r(-1) \exp \left[ - \int_{-T}^{-1} w_r \, dt \right]. \quad (3.13)$$

Since $j_r(0) = 0$ and $j'_r(0) = -\theta$, we see that $j_r(-1) \leq C_{11}\theta$, for some $C_{11} > 0$ (by comparing with the case $K \equiv K_{\min}$ and applying Lemma 3.1 (v)). Combining this fact with (3.10) and (3.13), we obtain

$$\theta^{-1/2} \leq C_{11} \int_{0}^{1} \exp \left[ - \int_{-T}^{-1} w_r \, dt \right] \, dr. \quad (3.14)$$
If the average of the quantities \( \exp \left[ -\int_{-T}^{-1} w_r \, dt \right] \) for \( 0 \leq r \leq 1 \) in (3.14) could be replaced by \( \exp \left[ -\int_{-T}^{0} w_0 \, dt \right] \) the desired inequality (3.12) would follow. To make this type of replacement we now show that there is a constant \( C_{12} > 0 \) such that

\[
\exp \left[ \int_{-T}^{-1} |w_r - w_0| \, dt \right] \leq C_{12}, \quad \text{for all } r, \ 0 \leq r \leq 1. \tag{3.15}
\]

Fix \( r, 0 \leq r \leq 1 \). Let \( \tilde{y} = w_r - w_0 \), and let \( t \) and \( t_0 \) satisfy \(-T \leq t \leq -1 \) and \(-1 < t_0 < 0\). By Lemma 3.1(ii) with \( A = t_0, B = t \), we have

\[
\tilde{y}(t) = -\int_t^{t_0} (K_0(s) - K_r(s)) \exp \left[ \int_t^{s} w_0(\tau) + w_r(\tau) \, d\tau \right] ds
\]

\[
+ \tilde{y}(t_0) \exp \left[ \int_t^{t_0} w_0(\tau) + w_r(\tau) \, d\tau \right].
\]

Since \( w_0(\tau), w_r(\tau) \leq 0 \) for \( \tau < 0 \), this implies

\[
|\tilde{y}(t)| \leq \int_t^{t_0} |K_0(s) - K_r(s)| \exp \left[ \int_t^{s} w_0(\tau) \, d\tau \right] ds + |\tilde{y}(t_0)|. \tag{3.16}
\]

If \( u_- \) denotes the stable Riccati solution along \( \gamma_0 \), then for \( \tau < 0 \),

\[
|w_0(\tau)| \geq |u_-(\tau)| = k_-(\gamma_0(\tau)) \geq C_{13} \sqrt{-K(\gamma_0(\tau))},
\]

for some \( C_{13} > 0 \), by Lemma 3.4. Then the same argument as in the proof of Lemma 3.5 shows that the first term on the right side of (3.16) is bounded by \( C_{14} \sqrt{\theta} \) for some constant \( C_{14} > 0 \), where \( C_{14} \) does not depend on \( t_0 \). Thus

\[
|\tilde{y}(t)| \leq C_{14} \sqrt{\theta} + |\tilde{y}(t_0)|.
\]

By comparing with constant curvature 0 and constant curvature \( K_{\text{min}} \) and applying Lemma 3.1(iii),

\[
\sqrt{|K_{\text{min}}|} \coth \tau \sqrt{|K_{\text{min}}|} \leq w_r(\tau) \leq \frac{1}{\tau} \text{ for } \tau < 0.
\]

A couple of applications of L’Hôpital’s rule shows that

\[
\lim_{\tau \to 0} \sqrt{|K_{\text{min}}|} \coth \tau \sqrt{|K_{\text{min}}|} - \frac{1}{\tau} = 0,
\]
and it follows that \( \lim_{t_0 \to 0} |\tilde{y}(t_0)| = \lim_{t_0 \to 0} |w_r(t_0) - w_0(t_0)| = 0. \)

Therefore \( |\tilde{y}(t)| \leq C_1 \sqrt{\theta} \) for \(-T \leq t \leq -1\). Since \( T \leq 1/\sqrt{\theta} \), this proves (3.15) and with \( C_{12} = C_{14} \). By rewriting (3.14) and applying (3.15) we obtain

\[
\dot{\theta}^{-1/2} \leq C_{11} \int_{-T}^{1} \exp \left[ - \int_{-T}^{1} w_0 \, dt \right] \exp \left[ \int_{-T}^{1} w_0 - w_r \, dt \right] \, dr
\leq C_{11} C_{12} \exp \left[ - \int_{-T}^{1} w_0 \, dt \right].
\]

This proves (3.12).

**Step 4.** We now prove that there are positive constants \( \hat{C} \) and \( \beta \) such that

\[
\exp \left[ - \int_{-T}^{1} u_0 \, dt \right] \leq \left( \frac{\hat{C}}{k_+(v_0)} \right)^\beta \left( \exp \left[ \int_{-T}^{1} w_0 \, dt \right] \right)^\beta,
\]

provided \( k_+(v_0) \neq 0 \).

Suppose \( k_+(v_0) \neq 0 \) (i.e., \( v_0 \notin \Gamma \)). Since \( u_0 \) and \( u_- \) are, respectively, the unstable and stable Riccati solutions along \( \gamma_0 \), it follows from Lemma 3.6 (with \( \beta = \hat{C} \)) that

\[
-u_0(t) \leq \beta u_-(t) \text{ for all } t \in \mathbb{R}.
\]

Thus

\[
\exp \left[ - \int_{-T}^{1} u_0 \, dt \right] \leq \left( \exp \left[ \int_{-T}^{1} u_- \, dt \right] \right)^\beta.
\]

We now find an upper bound for the ratio

\[
\frac{\exp \left[ \int_{-T}^{1} u_- \, dt \right]}{\exp \left[ \int_{-T}^{1} w_0 \, dt \right]} = \exp \left[ \int_{-T}^{1} z \, dt \right],
\]

where \( z(t) = u_-(t) - w_0(t) \), which is positive for \( t < 0 \). By Lemma 3.1(ii), with equality of the two curvature functions and \( A = t, B = -1 \), we have

\[
\exp \left[ \int_{-T}^{1} z \, dt \right] = \exp \left[ z(-1) \int_{-T}^{1} \exp \left[ \int_{t}^{1} u_-(\tau) + w_0(\tau) \, d\tau \right] \, dt \right]
\]

(3.20)
Fix $\tau \leq -1$. Given the values $\tilde{A} = u_-(1)$ and $\tilde{B} = w_0(-1)$, $\tilde{B} < \tilde{A} < 0$, the largest possible value for $u_-(\tau) + w_0(\tau)$ would occur if $K_0(t)$ were identically 0 for $t \leq -1$ (by Lemma 3.1(iii)). It then follows from (3.20) that we can obtain an upper bound for

$$\exp \left[ \int_{-\tau}^{-1} z \, dt \right] = \exp \left[ \int_{-\tau}^{-1} u_+ - w_0 \, dt \right]$$

by replacing $u_-$ and $w_0$ in (3.21) by Riccati solutions for the curvature 0 case, with the values $\tilde{A}$ and $\tilde{B}$, respectively at $t = -1$. Thus

$$\exp \left[ \int_{-\tau}^{-1} z \, dt \right] \leq \exp \left[ \int_{-\tau}^{-1} \frac{\tilde{A}}{(t+1)A + 1} - \frac{\tilde{B}}{(t+1)B + 1} \, dt \right]$$

$$= \left| \frac{\tilde{B} + \frac{1}{(t+1)}}{A + \frac{1}{(t+1)}} \right| \leq \left| \frac{\tilde{B}}{A} \right| = \left| \frac{w_0(-1)}{u_-(1)} \right|$$

By the analog of Lemma 3.1(iv) for negative Riccati solutions,

$$u_-(1) \leq \frac{u_-(0)}{-u_-(0) + 1}. \quad (3.23)$$

Also, since $K$ is bounded from below, both $|u_-(0)|$ and $|w_0(-1)|$ are bounded from above by a constant. Therefore, by (3.22) and (3.23), there exists a constant $C_{15} > 0$ such that

$$\exp \left[ \int_{-\tau}^{-1} z \, dt \right] \leq \frac{C_{15}}{|u_-(0)|} = \frac{C_{15}}{|k_-(v_0)|}. \quad (3.24)$$

If we let $\bar{C} = C_{15} \sup \{(k_+(v))/|k_-(v)|) : v \notin \Gamma\}$, which is finite by Lemma 3.6, then (3.17) follows from (3.18), (3.19) and (3.24).

**Step 5. Completion of the proof.** Combining the results of steps 3 and 4, we obtain

$$\exp \left[ -\int_{-\tau}^{-1} u_0 \, dt \right] \leq \left( \frac{C_{10} C}{k_+(v_0)} \right)^{\beta} \theta^{\beta/2}. \quad (3.25)$$

The same argument shows that this inequality also holds with $u_0$ and $v_0$ replaced by $u_1$ and $v_1$, respectively. These inequalities, together with (3.11), imply that

$$|k_+(v_1) - k_+(v_0)| \leq (C_7 + C_8)\sqrt{\theta} + C_9 \left( C_{10} \bar{C} \right)^{\beta} \min((k_+(v_0))^{-\beta}, (k_+(v_1))^{-\beta}) \theta^{\beta/2}$$
If \( k_+(v_0) \) and \( k_+(v_1) \) are both less than or equal to \( \theta^{1/4} \), then \( |k_+(v_1) - k_+(v_0)| \leq 2\theta^{1/4} \). If at least one of \( k_+(v_0) \) and \( k_+(v_1) \) is greater than \( \theta^{1/4} \), then

\[
|k_+(v_1) - k_+(v_0)| \leq (C_7 + C_8)\sqrt{\theta} + C_9\left(C_{10} \sim C\right)^{\beta} \theta^{3/4}
\]

In both cases, (3.7) holds for \( \alpha = \min(1/4, \beta/4) \) and some positive constant \( \bar{C} \).

Although we do not have counterexamples to Theorems I and II if hypothesis 2) is omitted, we now give an example to show that the crucial Lemmas 3.4 and 3.6 fail to hold without hypothesis 2). This example satisfies hypothesis 1).

**Example.** Let \( S \) be a compact surface containing a closed right circular cylinder \( C \) with negative curvature on \( S \setminus C \). Let \( \gamma \) be a closed geodesic along the boundary of \( C \) and let \( S \) be constructed so that for some \( \epsilon > 0 \), the \( \epsilon \) neighborhood of \( \gamma \) in \( S \) is a surface of revolution, and in Fermi coordinates \((s, x)\) along \( \gamma \), we have

\[
K(s, x) = \begin{cases} 
-e^{-1/x}, & \text{for } 0 < x < \epsilon, -\infty < s < \infty \\
0, & \text{for } -\epsilon < x \leq 0, -\infty < s < \infty.
\end{cases}
\]

It follows from a minor modification of Theorem 2.3 in [5] that there is a constant \( C_1 > 0 \) such that if \( v_\phi \) is a vector with footpoint on \( \gamma \) which makes an angle \( \phi \) with \( \gamma \) and which has a positive component in the \( \partial/\partial x \) direction, then

\[
-k_-(v_\phi) > C_1\phi|\ln \phi|.
\]

Let \( \sigma = \sigma_{v_\phi} \) and let \( T_1 = T_1(\phi) \) be chosen so that \( \{\sigma(t) : -T_1 \leq t \leq 0\} \) is a component of the intersection of \( \sigma \) and \( C \). Then there is a constant \( C_2 > 0 \) such that \( T_1 > C_2/\phi \). Let \( u_+ \) be the unstable Riccati solution along \( \sigma \). Since \( K(\sigma(t)) = 0 \) for \( -T_1 \leq t \leq 0 \),

\[
k_+(v_\phi) = u_+(0) = \frac{u_+(-T_1)}{T_1u_+(-T_1) + 1} \leq \frac{1}{T_1} < C_3\phi,
\]

where \( C_3 = 1/C_2 \). By (3.25) and (3.26) we see that the second inequality in the conclusion of Lemma 3.6 does not hold for any constant \( C \).
We now show that the same example also fails to satisfy the conclusion of Lemma 3.4. Let $0 < x_0 < \epsilon$, let $0 < \phi < \pi / 2$, and let $v_\phi$ and $\sigma = \sigma_{v_\phi}$ be as above. Let $T_2 = \sup \{ T > 0 : \text{dist}(\sigma(t), \gamma) \leq x_0 \text{ for } 0 \leq t \leq T \}$. By comparison with the curvature 0 case (using a particular case of Lemma 4.2 from §4), we have $T_2 \leq x_0 / \sin \phi \leq 2x_0 / \phi$. Since $u_+ = -u_+^2 - K$, we obtain

$$u_+(T_2) \leq u_+(0) + \int_0^{T_2} -K(\sigma(t)) \, dt$$

$$\leq u_+(0) + T_2 e^{-1/x_0}$$

$$\leq C_3 \phi + \frac{2x_0 e^{-1/x_0}}{\phi}.$$  

Let $\phi = \sqrt{x_0 e^{-1/(2x_0)}}$, let $w = \sigma'(T_2)$, and let $p$ be the footpoint of $w$. Then

$$k_+(w) = u_+(T_2) \leq \sqrt{x_0}(C_3 + 2)e^{-1/2x_0}, \quad (3.27)$$

while $\sqrt{K(p)} = e^{-1/(2x_0)}$. Since the coefficient $\sqrt{x_0}(C_3 + 2)$ in (3.27) can be made arbitrarily small, the second inequality in the conclusion of Lemma 3.4 does not hold for any constant $C > 0$.

### 4 Lower Bounds on Curvatures of Horocycles

In this section we establish the lower bound given in Lemma 3.3 on the curvatures, $k_+(v)$, of the unstable horocycles. We consider the curvatures of these horocycles at vectors $v$ that are close to $T^1 \gamma$, where $\gamma$ is a closed geodesic along which the curvature $K$ of $S$ vanishes identically. As in hypothesis 2) of Theorems I and II, we will assume that there is a point $q$ on $\gamma$ such that $K$ does not vanish to infinite order at $q$. Assume that $q$ is chosen so that the order to which $K$ vanishes is minimized (over points of $\gamma$) at $q$. The geodesic $\sigma_v$ determined by $v$ wraps around $S$ many times very close to $\gamma$. (See Figure 4.1.) In those time intervals when $\sigma_v$ passes close to $q$, $K$ is bounded from above by a negative function of the distance from $\sigma_v$ to $\gamma$. On the complements of these intervals we only assume that $K$ is nonpositive. The following lemma gives an upper bound for solutions to the Riccati equation which will apply in this situation.

**Lemma 4.1 (Estimate for intervals of alternating curvatures.)** Let $A, B, K_1$ be positive constants. Let $n$ be a positive integer and let $I_0, I'_0, I_1, I'_1, \ldots, I_n, I'_n$
be closed intervals (arranged in the natural order from left to right) that partition \([-T, 0]\), where \(T = \sum_{i=0}^{n} (|I_i| + |I_i'|)\). Assume:

1) All intervals \(I_i, I_i'\) are non-empty, except possibly \(I_0\) and \(I_n\).

2) \(|I_i| \geq A\) for \(i = 1, \ldots, n - 1\), and \(|I_n| \geq A\) if \(|I_n'| > 0\).

3) In the case \(n = 1\), \(|I_i| \geq A\) for \(i = 0\) or \(1\).

4) \(|I_i'| \leq B\) for \(i = 0, \ldots, n\).

Let \(K_0\) be a constant such that \(0 < K_0 < K_1\) and let \(u\) be a solution to the Riccati equation \(u' = -u^2 - K\), where \(K \leq -K_0\) on \(I_i\), and \(K \leq 0\) on \(I_i'\), for \(i = 0, \ldots, n\).

Then there exists a positive constant \(C\) which depends only on \(A, B\) and \(K_1\) such that:

i) If \(T \geq K_0^{-1/2}\) and \(u(-T) \geq 0\), then \(u(0) \geq C \sqrt{K_0}\) and

ii) If \(u(-T) \geq \sqrt{K_0}\), then \(u(0) \geq C \sqrt{K_0}\).

(The assertion ii) is still true if we delete the assumption 3).)

**Proof.** Let \(y_1(t) = 1/t\) for \(t > 0\) and let \(y_2(t) = \sqrt{K_0} \tanh (\sqrt{K_0} t)\) for \(t \geq 0\). Any solution to the Riccati equation with \(K \equiv 0\) and positive initial value follows a horizontal translate of the graph of \(y_1\). Similarly, any solution to the Riccati equation with \(K \equiv -K_0\) and nonnegative initial value less than \(\sqrt{K_0}\) follows a horizontal translate of the graph of \(y_2\). (If the initial value were greater than or equal to \(\sqrt{K_0}\), the solution would remain greater than or equal to \(\sqrt{K_0}\) for all future time.)

Note that \(y_1'(t) < 0\) and \(|y_1'(t)|\) is decreasing for \(t > 0\), while \(y_2'(t) > 0\) and \(y_2'(t)\) is decreasing for \(t > 0\). We now compare \(|y_1'|\) with \(y_2\). We have \(|y_1'| = t^{-2} = y_1^2\) and \(y_2' = K_0(1 - \tanh^2(\sqrt{K_0} t)) = K_0 - y_2^2\). Let \(C_1\) be a constant satisfying \(0 < C_1 < 1\) that will be specified below. If \(y_1(t) \leq C_1 \sqrt{K_0}\) for some \(t > 0\), then \(|y_1'(t)| \leq C_1^2 K_0\); and if \(y_2(t) \leq C_1 \sqrt{K_0}\) for some \(t > 0\), then \(y_2'(t) \geq K_0(1 - C_1^2)\).

Let \(A = \sum_{i=0}^{n} |I_i|\) and \(B = \sum_{i=0}^{n} |I_i'|\). It follows from assumptions 1) - 4) that there is a positive constant \(C_2\), depending only on \(A\) and \(B\), such that \(A > C_2 T\). Let \(C_1 = \min(\sqrt{C_2/2}, \sqrt{A/(A + B)})\).
Let $u$ be a solution to the Riccati equation $u' = -u^2 - K$, where $K \leq -K_0$ on $I_i$, and $K \leq 0$ on $I'_i$, for $i = 0, \ldots, n$. Let $w$ be the piecewise smooth solution to the Riccati equation $w' = -w^2 - \tilde{K}$ with $w(-T) = u(-T)$, $\tilde{K} \equiv -K_0$ on int $(I_i)$ and $\tilde{K} \equiv 0$ on int $(I'_i)$, for $i = 0, \ldots, n$. By the Comparison Lemma 3.1(iii), if $u(-T) \geq 0$, then $u(0) \geq w(0)$.

**Proof of i).** Assume $T \geq K_0^{-1/2}$ and $u(-T) \geq 0$. 

**Case 1.** Suppose that $w(t) \geq C_1 \sqrt{K_0}$ for all $t \in [-T, 0]$. By the above estimates on $y'_j$ and $y'_2$, we have

\[
w(0) \geq K_0[A(1 - C^2_1) - BC^2_1] + w(-T) \\
\quad \geq K_0T[C_2(1 - C^2_1) - (1 - C_2)C^2_1] \\
\quad \geq \sqrt{K_0}(C_2 - C^2_1) \\
\quad \geq (C_2/2)\sqrt{K_0}.
\]

**Case 2.** Suppose $w(t) \geq C_1 \sqrt{K_0}$ for some $t \in [-T, 0]$. Let $t_0$ be the largest $t \in [-T, 0]$ such that this inequality holds. Then there is an $i \in \{0, 1, \ldots, n\}$ such that $t_0 \in I'_i = [a'_i, b'_i]$. Moreover, by Lemma 3.1,

\[
w(b'_i) \geq \frac{w(t_0)}{(b'_i - t_0)w(t_0) + 1} \geq \frac{C_1 \sqrt{K_0}}{BC_1 \sqrt{K_1} + 1}.
\]

If $i = n$, then $b'_i = 0$ and $w(0) = w(b'_i)$. If $i < n$, then we consider $j$, $i < j \leq n$, such that $|I_j| \geq A$. Let $I_j = [a_j, b_j]$ and $I'_j = [a'_j, b'_j]$, where $b_j = a'_j$. By the same type of estimate as in Case 1,

\[
w(b'_j) \geq K_0[A(1 - C^2_1) - BC^2_1] + w(a_j) \\
\quad \geq w(a_j),
\]

by the choice of $C_1$. Repeated application of this inequality gives $w(0) = w(b'_n) \geq w(b'_i) \geq w(b'_j) \geq w(b'_k)$, if $|I_k| \geq A$, and it gives $w(b'_{n-1}) \geq w(b'_i)$, if $|I_n| < A$. But in the latter case, $|I'_n| = 0$ and $w(0) = w(b_n) \geq w(b'_{n-1}) \geq w(b'_k)$.

Thus, if we let $C_3 = \min(C_2/2, C_1/[BC_1 \sqrt{K_1} + 1])$ and combine the results of Cases 1 and 2, we have

\[u(0) \geq w(0) \geq C_3 \sqrt{K_0}.
\]

**Proof of ii).** Assume $u(-T) \geq \sqrt{K_0}$. Since $C_1 < 1$, $w(-T) = u(-T) \geq C_1 \sqrt{K_0}$. Then by the argument in Case 2 of the proof of i), $u(0) \geq w(0) \geq C_4 \sqrt{K_0}$, where $C_4 = C_1/(BC_1 \sqrt{K_1} + 1)$. \qed
The next lemma is due to Keith Burns. This lemma will enable us to estimate the lengths of time intervals that geodesics $\sigma_v$ in $S$ (or in $\tilde{S}$) spend in certain regions, by making the analogous estimates for a surface of constant negative curvature. Note that there is no assumption on the sign of the curvatures.

**Lemma 4.2** Let $S_0$ and $S_1$ be complete surfaces with curvatures $K_0$ and $K_1$, respectively. For $i \in \{0,1\}$, let $\gamma_i$ be a unit-speed geodesic in $S_i$, and let $(s_i,x_i)$ be Fermi coordinates along $\gamma_i$. Let $I_0$, $I_1$, and $J$ be intervals in $\mathbb{R}$, with $0 \in J$, such that, for $i \in \{0,1\}$, the map

$$p \mapsto (s_i(p),x_i(p))$$

is a diffeomorphism from a neighborhood $N_i$ of $\gamma_i(I_i)$ onto $I_i \times J$.

Assume the following condition on the curvatures $K_0$ and $K_1$:

1) For all $p_0 \in N_0$ and $p_1 \in N_1$,

$$x_0(p_0) = x_1(p_1) \Rightarrow K_0(p_0) \geq K_1(p_1),$$

For $i \in \{0,1\}$, let $\sigma_i : [0,T_i] \rightarrow N_i$ be a unit-speed geodesic segment. For $t \in [0,T_i]$, let $\phi_i(t)$ be the signed angle between $\sigma_i'(t)$ and the curve $x_i = x_i(\sigma_i(t))$, chosen to lie in the interval $[-\pi/2, \pi/2]$, and consistent with $\angle(\partial/\partial s, \partial/\partial x) = \pi/2$. Let $d_i(t) = x_i(\sigma_i(t))$ be the signed distance from $\sigma_i(t)$ to $\gamma_i$. Suppose further that

2) For $i \in \{0,1\}$, and for $t \in (0,T_i)$, $d_i(t) \geq 0$, and $\phi_i(t) \neq 0$,

3) $d_0(0) = d_1(0)$,

4) $\phi_0(0) \leq \phi_1(0)$.

Then, for all $t_0 \in [0,T_0]$ and $t_1 \in [0,T_1]$ satisfying $d_0(t_0) = d_1(t_1)$, we have

i) $\phi_0(t_0) \leq \phi_1(t_1)$, and

ii) If $\phi_0(t_0) > 0$, then $t_0 \geq t_1$, and if $\phi_0(t_0) < 0$, then $t_0 \leq t_1$. 

23
Proof. It is a straightforward exercise to verify that the conclusions of the lemma hold if either \( \phi_0 \) or \( \phi_1 \) takes one of the values \( \pm \pi/2 \), so we may assume that \( \phi_i[0,T_i] \subseteq (-\pi/2,\pi/2) \), for \( i \in \{0,1\} \).

If \( \phi_0 \) and \( \phi_1 \) have opposite sign on the intervals \((0,T_0)\) and \((0,T_1)\), respectively, then \( d_0 \) and \( d_1 \) are equal only at \( t = 0 \), and the lemma is trivially true. Suppose that \( \phi_0 \) and \( \phi_1 \) have the same sign on \((0,T_0)\) and \((0,T_1)\), respectively, and that \( \phi_0(0) = \phi_1(0) = 0 \). Let \( \sigma_{1,r} \) be a continuous variation of geodesics in \( S_1 \) with \( \sigma_{1,r}(0) = \sigma_1(0) \) and \( \angle(\sigma'_{1,r}(0),\sigma'_1(0)) = r \). Suppose that the conclusions of the lemma hold for the pair of geodesics \( \sigma_0 \) and \( \sigma_{1,r} \) for all \( r \neq 0 \) sufficiently small. Since the geodesics \( \sigma_{1,r} \) converge uniformly on compact time intervals to \( \sigma_1 \), as \( |r| \to 0 \), it follows that the conclusions of the lemma will also hold for the pair \( \sigma_0 \) and \( \sigma_1 \).

It suffices, then, to prove the lemma in the case \( \phi_1(0) \neq 0 \). By considering a variation of geodesics in \( S_0 \) that contains \( \sigma_0 \), we see that it suffices to prove the lemma in the case \( \phi_i(0) \neq 0 \). Assume that \( \phi_i[0,T_i] \subseteq (0,\pi/2) \), for \( i \in \{0,1\}; \) the case where the \( \phi_i \) take values in \((-\pi/2,0)\) is treated similarly.

For \( i \in \{0,1\} \), let \( S_i \) and \( \mathcal{X}_i \) be the unit-speed vector fields in the directions of \( \partial/\partial s_i \) and \( \partial/\partial x_i \), respectively. For \( p_i \in \mathcal{N}_i \), let \( A_i(p_i) \) be the geodesic curvature at the point \( p_i \) of the curve \( x_i = x_i(p_i) \) in \( \mathcal{N}_i \), so that

\[
\nabla_{S_i} \mathcal{X}_i = A_i S_i.
\]

Observe that if \( \beta_i : J \to \mathcal{N}_i \) is the geodesic segment tangent to \( \partial/\partial x_i \) with \( x_i(\beta_i(0)) = 0 \), then for \( x \geq 0 \), the function \( w_i(x) = A_i(\beta_i(x)) \) is the solution to the Riccati equation

\[
w'_i(x) = -w_i(x)^2 - K_i(\beta_i(x)),
\]

with initial condition \( w_i(0) = 0 \). Lemma 3.1, part (iii), combined with assumption 1, gives that

\[
A_0(\beta_0(x)) \leq A_1(\beta_1(x)),
\]

for all nonnegative \( x \in J \). Thus, for \( p_0 \in \mathcal{N}_0, p_1 \in \mathcal{N}_1 \),

\[
x_0(p_0) = x_1(p_1) \geq 0 \Rightarrow A_0(p_0) \leq A_1(p_1). \tag{4.1}
\]

We now calculate \( d'_i(t) \) and \( \phi'_i(t) \), for \( t \in [0,T_i) \). In the following calculation, we drop the subscript \( i \) for ease of reading. Notice that

\[
\sigma'(t) = \cos \phi(t) \mathcal{S}(\sigma(t)) + \sin \phi(t) \mathcal{X}(\sigma(t)),
\]

24
and so
\[
\phi' \cos \phi = \frac{d}{dt} \sin \phi \\
= \frac{d}{dt} \langle \mathcal{X}(\sigma), \sigma' \rangle \\
= \langle \nabla_{\sigma'} \mathcal{X}(\sigma), \sigma' \rangle \\
= \langle \nabla_{\cos \phi} S(\sigma) + \sin \phi \mathcal{X}(\sigma), \sigma' \rangle \\
= \cos \phi \langle \nabla S(\sigma) \mathcal{X}(\sigma), \sigma' \rangle + \sin \phi \langle \nabla \mathcal{X}(\sigma), \sigma' \rangle \\
= \cos \phi \langle \nabla S(\sigma) \mathcal{X}(\sigma), \sigma' \rangle \\
= \cos \phi \langle A(\sigma) S(\sigma), \sigma' \rangle \\
= \cos \phi A(\sigma) \cos \phi,
\]
where the equations are evaluated at \( t \). Replacing the subscript \( i \), we divide both sides of this equation by \( \cos \phi_i(t) \), for \( t \in [0, T_i] \), to obtain:
\[
\phi_i'(t) = A_i(\sigma_i(t)) \cos \phi_i(t). \tag{4.2}
\]
Also,
\[
d_i'(t) = \langle \sigma_i'(t), \mathcal{X}_i(\sigma_i(t)) \rangle \\
= \sin \phi_i(t). \tag{4.3}
\]
Since \( \phi_i(t) \in (0, \pi/2) \) for \( t \in [0, T_i] \), it follows from equation (4.3) that \( d_i' \) is monotone increasing on \( [0, T_i] \). Let \( a = d_0(0) = d_1(0) \) and \( b = \min\{d_0(T_0), d_1(T_1)\} \). For \( x \in [a, b] \), let \( t_i(x) = d_i^{-1}(x) \), and let \( \Phi_i(x) = \phi_i(t_i(x)) \). By equations (4.2) and (4.3) it follows that \( \Phi_i \) and \( t_i \) are differentiable on \( [a, b] \), and for \( x \in [a, b] \),
\[
t_i'(x) = (d_i'(d_i^{-1}(x)))^{-1} \\
= \csc \Phi_i(x), \tag{4.4}
\]
and
\[
\Phi_i'(x) = \phi_i'(t_i(x)) t_i'(x) \\
= A_i(p_i(x)) \cos \Phi_i(x)(\csc \Phi_i(x)) \\
= A_i(p_i(x)) \cot \Phi_i(x), \tag{4.5}
\]
where \( p_i(x) = \sigma_i(t_i(x)) \). (We needed that \( \phi_i'(0) \neq 0 \) to get differentiability of these functions at \( a \).) Clearly \( d_0(p_0(x)) = x = d_1(p_1(x)) \) and \( \cot \Phi_i(x) \) is
positive, for $x \in [a,b]$ and $i \in \{0,1\}$. Combining (4.1) with (4.5), we thereby obtain:

$$\Phi_0(x) = \Phi_1(x) \Rightarrow A_0(p_0(x)) \cot \Phi_0(x) \leq A_1(p_1(x)) \cot \Phi_1(x),$$

for all $x \in [a,b]$. By assumption 4) of the lemma, $\Phi_0(a) \leq \Phi_1(a)$, and so a standard differential inequality (Corollary 4.2 in [7], p. 27) implies that $\Phi_0(x) \leq \Phi_1(x)$, for all $x \in [a,b]$. In other words, for all $t_0, t_1$ with $d_0(t_0) = d_1(t_1)$,

$$\phi_0(t_0) \leq \phi_1(t_1),$$

proving inequality i) of the lemma.

To prove inequality ii) (in the present case where $\phi_0$ is positive), observe that, by inequality i), and equation (4.4),

$$t'_0(x) = \csc \Phi_0(x) \geq \csc \Phi_1(x) = t'_1(x),$$

for all $x \in [a,b]$. Since $t_0(a) = t_1(a) = 0$, it follows that $t_0(x) \geq t_1(x)$, for all $x \in [a,b]$. The case where $\phi_i(0) < 0$ is dealt with similarly. □

The following lemma, together with its analog for stable horocycles, provides the last step in the proof of Lemma 3.3, thereby completing the proofs of Theorems I and II.

**Lemma 4.3** Under the hypothesis of Lemma 3.3, there exists a neighborhood $\mathcal{U}$ of $T^1\tilde{\gamma}$ in $T^1\tilde{S}$ such that for any $v \in \mathcal{U}$ with footpoint having second Fermi coordinate $x = a$, the curvature $k_+(v)$ of the unstable horocycle satisfies

$$k_+(v) \geq C \max(|a|^{m/2}, |\phi|^{m/(m+2)}),$$

where $\phi = \angle(v, x = a)$.

**Proof.** Let $s_0$ be the length of $\gamma$, and let $P : \mathbb{R} \to [0, s_0]$ be the covering map with $P(0) = 0$. For a set $A \subseteq [0, s_0]$, let $\tilde{A}$ denote $P^{-1}(A)$. Since $K$ vanishes to order $m - 1$ on $\tilde{\gamma}$, but does not vanish to order $m$ at $\tilde{\gamma}(0)$, there exist positive constants $C_1$, $C_2$ and $\epsilon$ and an interval $L = [0, s_1]$ for some $s_1 \in (0, s_0)$ such that

$$-C_1x^m \leq K(s, x), \text{ for } |x| < \epsilon, \text{ for all } s,$$
and

\[-C_1 x^m \leq K(s, x) \leq -C_2 x^m, \text{ for } |x| < \epsilon, \text{ for } s \in \hat{L}.\]

Let \( L' \) be the closure of \([0, s_0] \setminus L\).

In this proof we modify our previous convention and let \( \sigma_v \) denote the maximal geodesic segment (possibly of infinite length) in \( \hat{S} \) with initial tangent vector \( v \) (with footpoint in the region where \( |x| < \epsilon \)) which remains in the region where \( |x| < \epsilon \).

The first part of our proof is concerned with the choice of a neighborhood \( \mathcal{U} \) such that for \( v \in \mathcal{U} \), \( \sigma_v \) will assume all \( s \) values in one component of \( \tilde{L} \), while taking \( x \) values in the interval \([\delta, 2\delta]\) for some suitably chosen \( \delta > 0 \). We can accomplish this by controlling the angles \( \sigma_v \) makes with the curves \( x = c \). The central estimates needed to control these angles are summarized in (4.8) and (4.9). These inequalities will also be used later, in Case 1 and Case 2b. Once the neighborhood \( \mathcal{U} \) is chosen, the argument divides into two cases, according to whether the angle \( v \) makes with \( x = a \) is less than or equal to a constant multiple of \( a^{(m+2)/2} \) (Case 1) or greater than or equal to a constant multiple of \( a^{(m+2)/2} \) (Case 2). In Case 1, we rely on \( \sigma_v \) spending a long time in the region where the \( x \) coordinate has magnitude on the order of \( a \). In Case 2, we use the facts that \( \sigma_v \) attains all \( s \) values in a component of \( \tilde{L} \) during the time \( \sigma_v \) is in the region where \( |x| > \delta \) and afterwards \( \sigma_v \) doesn’t spend too much time in the region where the curvature is very close to 0.

Let \( \ell = \max\{||{(\partial/\partial s)}_p|| : |x(p)| \leq \epsilon\} \). If \( 0 < \delta < \min(\epsilon/2, \ell s_0 \pi) \), then a geodesic segment \( \sigma \) in the region of \( \tilde{S} \) where \( \delta \leq x \leq 2\delta \) that starts at \( x = \delta \), ends at \( x = 2\delta \), and satisfies

\[ \angle(\sigma', x = c) \leq \delta/(4\ell s_0), \text{ for } \delta \leq c \leq 2\delta, \]  

must assume all \( s \) values in some interval of length at least \( 2s_0 \). (Reason: If we let \( \nu(c) = \angle(\sigma', x = c) \), then \( 0 < \nu(c) < \pi/4 \) and \( ||{(ds/dx)}_{x=c}\| = (\cot \nu(c))/||{\partial/\partial s}|| \geq (2||{\partial/\partial s}||\sin \nu(c))^{-1} \geq (2\ell\nu(c))^{-1} \geq 2s_0/\delta. \) In particular, the interval of \( s \) values so obtained would include at least one component of \( \hat{L} \). We now show that there is a neighborhood \( \mathcal{U} \) of \( T^1\gamma \) such that if \( \delta > 0 \) is sufficiently small and \( v \in \mathcal{U} \), then any geodesic segment contained in \( \sigma_v \) which goes from \( x = \delta \) to \( x = 2\delta \) satisfies (4.6). By the Gauss-Bonnet Theorem, the angle such a geodesic segment makes with \( x = c \) is monotone
increasing in \( c \) for \( \delta \leq c \leq 2\delta \). Thus it suffices to check that the angle with \( x = 2\delta \) is less than or equal to \( \delta/(4\ell s_0) \).

Fix \( 0 < b < \epsilon \) and let \( K_1 = K_1(b) = -C_1 b^m \). We will apply Lemma 4.2 in the case \( S_0 = \tilde{S} \) and \( S_1 \) is a simply connected surface of constant curvature \( K_1 \). Let \( \gamma_1 \) be a geodesic in \( S_1 \) and let \((s_1, x_1)\) be Fermi coordinates about \( \gamma_1 \).

Suppose \( \sigma [\sigma_1] \) is a geodesic segment on \( \tilde{S} \) which lies in the region where \( 0 \leq x < \epsilon \) \([0 \leq x_1 < \epsilon]\) and which makes angle \( \theta \), \( 0 < \theta < \pi/2 \), with \( \tilde{\gamma} \) \([\gamma_1]\). Let \( \theta_0 [\theta_1] \) be the angle \( \sigma [\sigma_1] \) makes with \( x = b \) \([x_1 = b]\). By Lemma 4.2, we have \( \theta_0 \leq \theta_1 \). A formula from hyperbolic trigonometry gives

\[
\cos \theta = \cosh \left( b\sqrt{-K_1(b)} \right) \cos \theta_1. \tag{4.7}
\]

We may assume that \( \epsilon \) was chosen sufficiently small so that there exists an \( \eta > 0 \) such that (4.7) implies that \( \theta_1 < \pi/4 \), whenever \( 0 < \theta < \eta \) (for any choice of \( b \) with \( 0 < b < \epsilon \)).

We will use the estimates \( \cos z \geq 1 - z^2/2 \) for \( 0 \leq z \leq \pi/2 \) and \( \cos z \leq 1 - z^2/2 \) for \( 0 \leq z \leq \pi/4 \). Also, there is a constant \( C_3 > 0 \) such that if \( 0 \leq z \leq \epsilon^{(m+2)/2}\sqrt{C_1} \), then \( \cosh z \leq 1 + C_3 z^2 \). (Note that \( \epsilon^{(m+2)/2}\sqrt{C_1} > b\sqrt{-K_1(b)} \), if \( 0 < b < \epsilon \).) Then, for \( 0 < \theta < \eta \), (4.7) implies

\[
1 - \frac{\theta^2}{2} \leq \left( 1 + C_3 b^2 |K_1| \right) \left( 1 - \frac{\theta^2}{4} \right).
\]

Thus

\[
\frac{\theta^2}{4} \leq \frac{\theta^2}{2} + C_3 b^2 |K_1|,
\]

and we obtain

\[
\theta_0 \leq \theta_1 \leq \sqrt{2\theta} + 2\sqrt{C_1 C_3 b^{(m+2)/2}}. \tag{4.8}
\]

It also follows from Lemma 4.2 that if \( \tilde{\sigma} \) is a geodesic segment in \( \tilde{S} \) which is tangent to \( x = c \) and goes from \( x = c \) to \( x = b \), for some \( 0 < c < b \), and makes angle \( \theta_0 \) with \( x = b \), then

\[
\theta_0 \leq 2\sqrt{C_1 C_3 b^{(m+2)/2}}. \tag{4.9}
\]

This can be seen by comparing \( \tilde{\sigma} \) with geodesic segments in \( S_1 \) making angle \( \theta \) with \( \gamma_1 \) and letting \( \theta \) go to 0.
Fix $\delta$ such that $0 < \delta < \min(\epsilon/2, \ell s_0 \pi)$ and $2\sqrt{C_3 C_4 (2\delta)^{(m+2)/2}} < \delta/(8\ell s_0)$. This is possible, since $m$ is at least 2. If $\theta < \min(\eta, \delta/(8\sqrt{\ell}))$ and $b = 2\delta$, then $\theta_0 \leq \theta_1 \leq \delta/(4\ell s_0)$. Choose a neighborhood $U$ of $T^1\gamma$ in $T^1\tilde{S}$ such that the footpoints of vectors in $U$ are in the region where $-\delta < x < \delta$ and for each $v \in U$, $\sigma_v$ intersects $\gamma$ with angle of absolute value less than $\min(\eta, \delta/(8\sqrt{\ell}))$ or does not intersect $\gamma$ at all. If $\sigma$ is a geodesic segment contained in $\sigma_v$, for $v \in U$, that goes from $x = \delta$ to $x = 2\delta$, then (4.6) is satisfied, and, by the above discussion $\sigma$ takes all $s$ values in at least one component of $\tilde{L}$. This property will be used in Case 2a below. For Case 1 we will only need the fact that for $v \in U$, $\sigma_v$ makes an angle of absolute value at most $\pi/4$ with any curve $x = c$, where $|c| < \epsilon$.

For the rest of the proof fix a choice of $v \in U$, let $x = a$ be the second Fermi coordinate of the footpoint of $v$, and let $\phi = \angle(v, x = a)$. We may assume that $a \geq 0$.

**Case 1.** Suppose $|\phi| \leq C_0 a^{(m+2)/2}$ for some $C_0 > 0$. (In Case 1, $C_0$ may be any positive constant.) If $a = 0$, then $\phi = 0$ and $k_+(v) = 0$; so assume $a > 0$.

**Case 1a.** Suppose $\phi \leq 0$. Then $\sigma_v$ intersects $\tilde{\gamma}$ with angle $\theta$ satisfying $|\theta| \leq |\phi|$ or does not intersect $\tilde{\gamma}$ at all. (See Figure 4.2.) In either case, it follows from (4.8) that

$$|\angle(\sigma_v, x = 2a)| \leq \sqrt{2}|\phi| + 2\sqrt{C_3 C_4 (2a)^{(m+2)/2}}$$

$$\leq C_4 a^{(m+2)/2}$$

for some constant $C_4 > 0$. Therefore $\sigma_v$ is in the region $a \leq x \leq 2a$ during a time interval $[-T, 0]$ where $T \geq (2a - a)/(\sin(C_4 a^{(m+2)/2})) \geq C_4^{-1} a^{-m/2}$. 

**Case 1b.** Suppose $\phi > 0$ and $\sigma_v$ intersects $x = a/2$. (See Figure 4.3.) Then the angle that $\sigma_v$ makes with $x = a/2$ is less than or equal to $\phi$, and it follows from Case 1a, applied at $x = a/2$ instead of $x = a$, that there is a constant $C_5 > 0$ such that $\sigma_v$ is in the region $a/2 \leq x \leq a$ during a time interval $[-T, 0]$ with $T \geq C_5 a^{-m/2}$.

**Case 1c.** Suppose $\phi > 0$ and $\sigma_v$ does not intersect $x = a/2$. (See Figure 4.4.) Then $\sigma_v$ is tangent to $x = a_1$ for some $a_1$ satisfying $a/2 \leq a_1 \leq a$. It follows from Case 1a applied at $x = a_1$ with angle 0 that $\sigma_v$ is in the region $a_1 \leq x \leq 2a_1$ during a time interval $[-T, 0]$ with $T \geq C_6 a^{-m/2}$ for some constant $C_6 > 0$.

**Conclusion of Case 1.** In all three subcases there is a constant $C_7 > 0$ and an interval $[-T, 0]$, where $T \geq C_7 a^{-m/2}$, such that $\sigma_v(t)$ is in the region.
where \(a/2 \leq x \leq 2a\) for \(t \in [-T, 0]\).

Let \(I', I_0, I_1, I_2, \ldots, I_n, I_n\) be a partition of \([-T, 0]\) such that for \(t \in I_i \in [t \in I_i']\) the \(s\) coordinate of \(\sigma_v\) is in \(L \subset [L']\). Then for \(t \in I_i, K(\sigma_v(t)) = -C_2(a/2)^m\). Since \(|\langle \sigma_v, x = constant \rangle| \leq \pi/4\) for \(v \in \mathcal{U}\), there exist positive constants \(A, B\) such that the hypothesis of Lemma 4.1 holds with \(K_0\) a constant multiple of \(a^m\). From part i) of this lemma we conclude that there is a constant \(C_8 > 0\) such that \(K_+(v) \geq C_8a^{m/2}\). Since \(\phi \leq C_0|a|^{(m+2)/2}\), it follows that there exists \(C > 0\) such that \(k_+(v) \geq C \max(|a|^{m/2}, |\phi|^{m/(m+2)})\).

**Case 2.** \(|\phi| \geq C_0a^{(m+2)/2}\). (Here \(C_0\) is a sufficiently large positive constant, as described in Case 2b. This constant depends only on \(C_1\) and \(C_3\).) For starters require that \(C_0 > 1\). Since \(\phi\) cannot be 0 except in the trivial case, when \(a\) is also 0, we will assume \(\phi \neq 0\).

**Case 2a.** Suppose \(\phi < 0\). Let \(T_0 > 0\) be such that \(\sigma_v\) crosses \(x = \delta\) at time \(-T_0\). By the choice of \(\mathcal{U}\), there exist \(T_1, T_2 > 0\) such that \(T_0 \leq T_1 < T_2\) and during the time interval \([-T_2, -T_1]\) the \(s\) values taken by \(\sigma_v\) lie in a component of \(L\) and cover this component. (See Figure 4.5.) Assume that \(T_1\) and \(T_2\) are chosen as small as possible, while satisfying these requirements. The interval of \(s\) values assumed by \(\sigma_v\) in the time interval \([-T_1, -T_0]\) has length less than \(s_0\). Since \(|\langle \sigma_v, x = c \rangle| \leq \pi/4\) for \(\delta \leq c \leq 2\delta\), this implies that \(|T_1 - T_0| \leq \ell'/\sqrt{2}\). Also, we have \(|T_2 - T_1| \geq |L|\). (This inequality depends on the fact that \(||\partial/\partial s|| \geq 1\), which follows from the nonpositive curvature assumption.) If \(t \in [-T_2, -T_1]\), then \(K(\sigma_v(t)) = -C_2\delta^m\). It follows that there is a positive constant \(C_9\) (depending on \(\delta\), but not on \(\phi\)) such that the unstable Riccati solution along \(\sigma_v\) is at least \(C_9\) at \(t = -T_0\). (This could be viewed as a simple case of Lemma 4.1.) By reducing the size of the neighborhood \(\mathcal{U}\), if necessary, we may assume that \(\phi\) satisfies \(\sqrt{C_2}|\phi|^{m/(m+2)} < C_0\). Then the hypothesis of part (ii) of the Lemma 4.1 holds for the time that \(\sigma_v\) is in the region \(|\phi|^{2/(m+2)} \leq x \leq \delta\) with \(K_0 = C_2|\phi|^{2m/(m+2)}\). From this lemma we conclude that the value of the unstable Riccati solution along \(\sigma_v\) is at least \(C_{10}|\phi|^{m/(m+2)}\), for some \(C_{10} > 0\), at the time \(\sigma_v\) crosses \(x = |\phi|^{2/(m+2)}\). Since \(\sigma_v\) makes angle of absolute value at least \(|\phi|\) with \(x = c\) for \(a \leq c \leq |\phi|^{2/(m+2)}\), the length of time \(\sigma_v\) is in the region where \(a \leq x \leq |\phi|^{2/(m+2)}\) is less than or equal to

\[
\frac{|\phi|^{2/(m+2)} - a}{\sin|\phi|} \leq \frac{|\phi|^{2/(m+2)}(1 - C_0^{-2/(m+2)})}{|\phi|/2} \leq C_{11}|\phi|^{-m/(m+2)}
\]
for some $C_{11} > 0$. Then, by Lemma 3.1, $k_+(v)$, the value of the unstable Riccati solution along $\sigma_v$ at time 0, satisfies

$$k_+(v) \geq \frac{C_{10}|\phi|^{m/(m+2)}}{C_{11}|\phi|^{-m/(m+2)}C_{10}|\phi|^{m/(m+2)} + 1} \geq C_{12}|\phi|^{m/(m+2)}$$

for some $C_{12} > 0$.

**Case 2b.** Suppose $\phi > 0$. It follows from (4.8) and (4.9) that if $\phi \geq \sqrt{2C_1C_3a^{(m+2)/2}}$, then $\sigma_v$ crosses $\tilde{\gamma}$ with angle $\theta$ satisfying

$$\theta \geq \frac{1}{\sqrt{2}} \left[ \phi - 2\sqrt{C_1C_3a^{(m+2)/2}} \right]$$

$$\geq \frac{\phi}{\sqrt{2}} \left[ 1 - \frac{2\sqrt{C_1C_3}}{C_0} \right]$$

Choose $C_0$ sufficiently large so that $C_0 > 1$ and $2\sqrt{C_1C_3}/C_0 < 1 - 1/\sqrt{2}$. Then, if $\phi \geq C_0a^{(m+2)/2}$, $\sigma_v$ crosses $\tilde{\gamma}$ with angle $\theta$ satisfying $\phi \geq \theta \geq \phi/2$. By Case 2a, the unstable Riccati solution $u$ along $\sigma_v$ satisfies $u(-T_3) \geq C_{12}^{m/(m+2)} \geq C_{12}^{-m/(m+2)}\phi^{m/(m+2)}$, where $T_3 \geq 0$ is such that $\sigma_v$ crosses $\tilde{\gamma}$ at time $-T_3$. Also, we have $T_3 \leq a/\sin \theta \leq 2a/\theta \leq 4a/\phi$. Hence, by Lemma 3.1(iii),

$$k_+(v) = u(0) \geq \frac{C_{12}^{-m/(m+2)}\phi}{(4a/\phi)(C_{12}^{m/(m+2)}\phi) + 1} \geq C_{13}\phi,$$

for some $C_{13} > 0$.

**Conclusion of Case 2.** In both subcases $k_+(v) \geq C_{14}|\phi|$ for some $C_{14} > 0$. Since $|\phi| \geq C_0|a|^{(m+2)/2}$, this implies that there is a constant $C > 0$ such that $k_+(v) \geq C\max(|a|^{m/2}, |\phi|^{m/(m+2)})$, which completes the proof. □

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Figure 3.1
Figure 3.2
Figure 4.1
Figure 4.2. Case 1a.
Figure 4.3. Case 1b.
Figure 4.4. Case 1c.

\[ \tilde{\gamma} \quad x = \frac{a}{2}, \; x = a, \; x = 2a \]
Figure 4.5. Case 2a.