# A stably Bernoullian diffeomorphism that is not Anosov

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## Introduction

In this paper we construct new examples of diffeomorphisms with robust statistical properties. Our interest in these examples arose from the insight they might give into a collection of natural, optimistic and still unresolved problems in the global theory of dynamical systems.

For a compact $C^\infty$ manifold $M$ with volume element $\mu$, let $\mathrm{Diff}^r(M)$ be the space of $C^r$ diffeomorphisms of $M$ and let $\mathrm{Diff}^r_\mu(M)$ be the elements of $\mathrm{Diff}^r(M)$ that preserve $\mu$. Endow these spaces with the $C^r$ topology, $r \geq 1$.

**Theorem I:** *Let $\mu$ be Lebesgue measure on the torus $\mathbf{T}^3 = \mathbf{R}^3/\mathbf{Z}^3$. There is a $C^1$-open set $\mathcal{U} \subset \mathit{Diff}^2_\mu(\mathbf{T}^3)$ such that for each $g \in \mathcal{U}$,*

1. *$g$ is Bernoulli;*
2. *$g$ has Lyapunov exponents nonzero $\mu$-a.e.;*
3. *$g$ is not homotopic to an Anosov diffeomorphism.*

Previously the only open sets of diffeomorphisms known to satisfy either property 1 or property 2 were the Anosov diffeomorphisms. Recall that a

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diffeomorphism $g : M \to M$ is *Anosov* if there exists a continuous, $Tg$-invariant splitting of the tangent bundle to $M$,

$$TM = E^u \oplus E^s,$$

and constants $C > 0$ and $\lambda > 0$ such that, for all $n \geq 0$,

$$\begin{array}{lcll} v \in E^u & \Rightarrow & \|Tg^{-n}(v)\| \leq Ce^{-\lambda n}\|v\|, & \quad \text{and} \\ v \in E^s & \Rightarrow & \|Tg^{n}(v)\| \leq Ce^{-\lambda n}\|v\|. & \end{array}$$

Anosov diffeomorphisms are examples of *uniformly hyperbolic* dynamical systems; other examples include Axiom A diffeomorphisms and attractors. The ergodic theory of such systems has been studied extensively, beginning with the pioneering work of Anosov, Sinai and Bowen-Ruelle in the 1960's and '70's (see, e.g., [KH]). All Anosov diffeomorphisms of $\mathbf{T}^3$ are of the special form $g = \rho \circ h \circ \rho^{-1}$, where $\rho$ is a homeomorphism and $h$ is a linear automorphism of $\mathbf{T}^3$, none of whose eigenvalues has modulus 1 [Mann].

A real number $\lambda$ is a *Lyapunov exponent* of the diffeomorphism $g : M \to M$ if there exists a nonzero vector $v \in TM$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|Tg^n(v)\| = \lambda. \tag{1}$$

By Osceledets' Theorem [Os], there is a set $L \subseteq M$, which has full measure with respect to any $g$-invariant measure, such that the limit in (1) exists for all $v \in T_xM$ with $x \in L$. For a given $x \in L$, there are finitely many different exponents, and for some vectors, this limit may be zero. It is immediate from the definition that the Lyapunov exponents of an Anosov diffeomorphism are all nonzero. At the other extreme, an automorphism of $\mathbf{T}^3$ with 1 as an eigenvalue has an exponent 0 (everywhere). To prove Theorem I, we start with such an automorphism and perturb so that the 0 exponent becomes positive, almost everywhere. A volume-preserving diffeomorphism whose exponents are nonzero almost everywhere is called *nonuniformly hyperbolic*. This term was introduced by Pesin, who was able to establish many ergodic properties for nonuniformly hyperbolic systems, previously known to hold for uniformly hyperbolic ones.

The diffeomorphisms in Theorem I have some other remarkable properties which we describe below. Before turning to them and to Theorem II, we set our result in context.

One of the achievements of the theory of uniformly hyperbolic dynamical systems were the theorems of Sinai, Ruelle and Bowen on invariant measures on the attractors of a system. These attractors and measures are now called Sinai-Ruelle-Bowen measures and SRB measures (or SRB attractors), for short. They may also be called ergodic attractors.[1]

Given $f \in \mathrm{Diff}^r(M)$ (not necessarily preserving $\mu$), a closed, $f$-invariant set $A \subset M$ and an $f$-invariant ergodic measure $\nu$ on $A$, we define $B(A)$, the *basin* of $A$, to be the set of points $x \in M$ such that $f^n(x) \to A$ and for every continuous function $\phi : M \to \mathbf{R}$

$$\lim_{n\to\infty} \frac{1}{n}(\phi(x) + \cdots + \phi(f^n(x))) \to \int_A \phi(x)\ d\mu.$$

**Definition:** $\nu$ is an *SRB measure* and $A$ is an *SRB (or ergodic) attractor* if the Lebesgue measure of $B(A)$ is positive.

It follows from the definition that a diffeomorphism has at most countably many SRB measures. Sinai, Ruelle and Bowen proved that for $r \geq 2$ and $f$ an Axiom A, no-cycle diffeomorphism (see [Si], [Ru], [Bo]), almost every point in $M$ with respect to Lebesgue measure $\mu$ is in the basin of an SRB measure, and there are only finitely many SRB measures. The first natural question is if the first part of this conclusion is a generic property in $\mathrm{Diff}^r(M)$ for $r \geq 2$. (By *generic* we mean: containing a countable intersection of open-dense sets).

**Question 1:** *For $r \geq 2$, is it true for generic $f$ in $\mathrm{Diff}^r(M)$ that the union of the basins of the SRB attractors of $f$ has full Lebesgue measure in $M$?*

In the volume-preserving case Pesin [Pe] proved that for $f$ a $C^2$, nonuniformly hyperbolic diffeomorphism, $M$ may be written as the disjoint union of at most countably many invariant sets of positive measure on which $f$ is ergodic. Thus for nonuniformly hyperbolic volume preserving diffeomorphisms the answer to Question 1 is "yes." The supports of the ergodic measures are the SRB attractors. Pesin asked if nonuniform hyperbolicity was generic in $\mathrm{Diff}^2_\mu(M)$.

[1] We take some of the conclusions of the theorems of Sinai, Ruelle, and Bowen as a definition and warn the reader that the use of SRB measure or attractor is not uniform in the literature. For a survey of SRB measures (using a different definition) see [You2].

Pesin's question is answered in the negative by Cheng and Sun [CS], and Herman. In [He] (see also [Yoc]), Herman showed that on any manifold $M$ of dimension at least 2, and for sufficiently large $r$, there are open sets of volume preserving $C^r$ diffeomorphisms of $M$ all of which posess positive measure sets of codimension one invariant tori, and on each torus the diffeomorphism is $C^1$ conjugate to a diophantine translation. In this case all of the exponents are 0 on these tori.

Our proof gives some hope that a variant of Pesin's original question might hold for volume-preserving diffeomorphisms: either all exponents are zero ($\mu$-a.e.), or, as with our examples, the system may be perturbed to become stably nonuniformly hyperbolic.

**Question 2a):** *For $r \geq 1$, is it true for generic $f$ in $\mathrm{Diff}^r_\mu(M)$ that for almost every ergodic component of $f$, either all of the Lyapunov exponents of $f$ are 0 or none are 0 ($\mu$-a.e.)?*

A special case of 2a) is 2b).

**Question 2b):** *For $r \geq 1$ does the generic ergodic diffeomorphism in $\mathrm{Diff}^r_\mu(M)$ have either no exponent equal to 0 or all exponents equal to 0 ($\mu$-a.e.)?*

Question 2a) has an affirmative answer for 2-dimensional $M$ in the case $r = 1$; Mañé has shown that the generic diffeomorphism in $\mathrm{Diff}^1_\mu(M)$ either has all of its Lyapunov exponents zero or is an Anosov diffeomorphism. Theorem I shows that $C^1$ - open sets of $C^2$, volume-preserving diffeomorphisms with nonzero exponents need not be Anosov in general.

An analogue of Question 2 for $\mathrm{Diff}^r(M)$ is the following.

**Question 3:** *For $r \geq 1$, is it true for the generic $f$ in $\mathrm{Diff}^r(M)$ and any weak limit $\nu$ of the push forwards $f^n_*\mu$ that almost every ergodic component of $\nu$ has some exponents not equal to 0 ($\nu$-a.e.)? All exponents not equal to 0?*

Question 2b) is closest in spirit to Theorem I, which in turn gives some credence to the possibility that 2b) is true. Question 3) might be a way to approach Question 1) along the lines of [Pe], [PS1], [PS4]. There might even be only *finitely* many SRB attractors, for almost every $f$ in the generic $k$-parameter family in $\mathrm{Diff}^r(M)$, as suggested in [PS2], or densely in $\mathrm{Diff}^r(M)$ as suggested in [Pa].

We now turn to another novel aspect of the diffeomorphisms we construct. This property concerns one of the invariant expanding Lyapunov directions; by construction, this line field is tangent to a foliation of $\mathbf{T}^3$.

**Theorem II:** *Let $\mathcal{U}$ be the set of diffeomorphisms in Theorem I. For every $g \in \mathcal{U}$, there is a foliation $\mathcal{W}^c_g$ of $\mathbf{T}^3$ by $C^2$ circles which is preserved by $g$. This foliation has the following properties:*

1. *There is an equivariant fibration $\pi : \mathbf{T}^3 \to \mathbf{T}^2$ such that $\pi g = h\pi$ where $h$ is the linear endomorphism of $\mathbf{T}^2$ given by a hyperbolic matrix $A \in Sl(2, \mathbf{Z})$. The leaves of the foliation $\mathcal{W}^c_g$ are the fibers of the map $\pi$. In particular, the set of periodic leaves is dense in $\mathbf{T}^3$.*

2. *There exists $\lambda > 0$ such that, for $\mu$-almost every $w \in \mathbf{T}^3$, if $v \in T_w\mathbf{T}^3$ is tangent to the leaf of $\mathcal{W}^c_g$ containing $w$, then*
$$\lim_{n\to\infty} \frac{1}{n} \log \|T_w g^n v\| = \lambda.$$

3. *Consequently, there exists a set $S \subseteq \mathbf{T}^3$ of full $\mu$-measure that meets every leaf of $\mathcal{W}^c_g$ in a set of leaf-measure 0.*

Property 3 stands in contrast to the measure-theoretic properties of other dynamically-invariant foliations, specifically the unstable foliations of a hyperbolic (or even nonuniformly hyperbolic) diffeomorphism. Unstable foliations have the property of *absolute continuity:* any set of full measure meets almost every leaf in a set of leafwise full measure. Absolutely continuous foliations satisfy a Fubini-type theorem; this enabled Anosov to prove ergodicity for volume-preserving hyperbolic systems. Foliations exhibiting the behavior in 3 have been referred to as "Fubini's Nightmare," (also "Fubini Foiled"); Katok had previously constructed an example of a dynamically-invariant foliation with this property, which is presented in [Mi]. Katok's example made us aware that diffeomorphisms such as those we construct are indeed possible.

Lai-Sang Young has previously constructed open sets of $C^1$ cocycles that are not uniformly hyperbolic and which have exponents nonzero [You1]. These cocycles, however, are not the natural cocycle associated to the derivative of a map.

We remark here that our construction can be slightly modified to obtain diffeomorphisms of $\mathbf{T}^n$, for any $n \geq 3$, that satisfy the conclusions of Theorems I and II. In this modification, the automorphism of $\mathbf{T}^2$ in the next section is replaced by an automorphism of $\mathbf{T}^{n-1}$ with one-dimensional expanding eigenspace.

We thank Michael Herman for many conversations which clarified questions 1-3 for us. Some of this material was presented and discussed in his seminar during March 1998. The question of whether perturbing a normally-hyperbolic, non-Anosov diffeomorphism could produce nonzero exponents was initially raised by Lai-Sang Young during a conversation about these questions. We thank her for reminding us that examples such as the ones we construct might exist. Numerical experiments conducted by Chai Wah Wu and later by Niels Sondergaard convinced us of their existence and we are indebted to them. We also thank Charles Pugh and Clark Robinson for useful conversations.

# 1 The construction

The ingredients in our construction are: a linear Anosov diffeomorphism $h : \mathbf{T}^2 \to \mathbf{T}^2$, circle-valued $C^3$ functions $\varphi : \mathbf{T}^2 \to \mathbf{T}$ and $\psi : \mathbf{T} \to \mathbf{T}$, and vectors $v_0 \in \mathbf{R}^2$, $w_0 \in \mathbf{Z}^2$. We specify in this section how to select these ingredients; an example of a suitable choice is $h(x, y) = (2x+y, x+y)$, $\varphi(x, y) = \sin(2\pi y)$, $\psi(z) = \sin(2\pi z)$, $v_0 = ((1 + \sqrt{5})/2, 1)$, and $w_0 = (1, 1)$. These define a 2-parameter family $f_{a,b}$ of diffeomorphisms of $\mathbf{T}^3$ by the formula:

$$f_{a,b} = g_a \circ h_b, \tag{2}$$

where

$$h_b(x, y, z) = (h(x, y), z + w_0 \cdot (x, y) + b\varphi(x, y)),$$

and

$$g_a(x, y, z) = ((x, y) + a\psi(z)v_0, z).$$

The set $\mathcal{U}$ in Theorem I will be a neighborhood of $f_{a,b}$, for a suitable choice of parameters.

Let $h : \mathbf{T}^2 \to \mathbf{T}^2$ be a linear Anosov diffeomorphism, i.e. a map of the form $h(p) = Ap$, for some hyperbolic matrix $A \in Sl(2, \mathbf{Z})$. The expanding and contracting eigenspaces for $A$ are both one-dimensional. Choose $v_0 \in \mathbf{R}^2$ to be an expanding eigenvector for $A$.

The diffeomorphisms $h_b$ in (2) are circle extensions of $h$. We describe this construction. If $\theta : \mathbf{T}^2 \to \mathbf{T}$ is any circle-valued function, we may construct the *circle extension* (or *skew product*) *of* $h$ *by* $\theta$, denoted $h_\theta : \mathbf{T}^3 \to \mathbf{T}^3$, by the formula

$$h_\theta(p, z) = (h(p), z + \theta(p)),$$

for $p \in \mathbf{T}^2$, and $z \in \mathbf{T}$. It is easy to see that $h_\theta$ preserves $\mu$. We will work with skew products that are stably ergodic; we shall see that they are exactly characterized by the following condition.

**Definition:** Let $k = \det(A - I)$. Say that $\theta$ *satisfies the cocycle condition* if there are no solutions to the equation

$$k\theta(p) \quad = \quad \Phi(h(p)) - \Phi(p) + c, \tag{3}$$

with $\Phi \in C^0(\mathbf{T}^2, \mathbf{T})$ and $c \in \mathbf{T}$.

The cocycle condition has a simpler formulation when $\theta$ is homotopic to a constant map. Indeed, it follows from the proof of Proposition 11.1 in [BW] that for such a $\theta$, equation (3) in the definition may equivalently be replaced by

$$\theta(p) \quad = \quad \Phi(h(p)) - \Phi(p) + c, \tag{4}$$

where $\Phi$ ranges over maps homotopic to a constant. The following proposition is a special case of Corollary B3 in [BW].

**Proposition 1.1** *If $\theta$ is $C^2$ and satisfies the cocycle condition, then $h_\theta$ is stably ergodic (in fact, stably a $K$-system). That is, there is a $C^1$-open neighborhood $\mathcal{E}$ of $h_\theta$ in $\mathit{Diff}^2_\mu(\mathbf{T}^3)$ such that all $g \in \mathcal{E}$ are ergodic ($K$-systems).*

*Conversely, if $h_\theta$ is stably ergodic, then $\theta$ satisfies the cocycle condition.*

The cocycle condition is easily seen to hold for an open and dense set of $\theta \in C^r(\mathbf{T}^2, \mathbf{T})$, for any $r \geq 2$. An example of a function satisfying this condition (for *any* linear Anosov $h$) is $\theta(x, y) = \sin(2\pi x)$, which, in addition, is homotopic to a constant map.

Choose the function $\varphi$ to be $C^3$, homotopic to a constant map and satisfying the cocycle condition. Choose $w_0 \in \mathbf{Z}^2$ to be any nonzero vector, and for $b \in \mathbf{R}$, let

$$\varphi_b(x, y) = w_0 \cdot (x, y) + b\varphi(x, y).$$

Observe that $h_b = h_{\varphi_b}$.

**Lemma 1.2** *If $\varphi$ is homotopic to a constant map and satisfies the cocycle condition, then $\varphi_b$ satisfies the cocycle condition, for all $b \neq 0$. For these $b$, $h_b$ is stably ergodic and stably $K$.*

**Proof:** Suppose that $\varphi_b$ does not satisfy the cocycle condition, for some $b \neq 0$. Then there exist $\Phi \in C^0(\mathbf{T}^2, \mathbf{T})$ and $c \in \mathbf{T}$ such that for all $(x, y) \in \mathbf{T}^2$,

$$k\varphi_b(x, y) = kw_0 \cdot (x, y) + kb\varphi(x, y) = \Phi \circ h(x, y) - \Phi(x, y) + c, \tag{5}$$

where $k$ is the determinant of $A - I$.

Since $kw_0 \in (A - I)(\mathbf{Z}^2)$, there exist integers $r$ and $s$ such that

$$kw_0 \cdot (x, y) = \Psi \circ h(x, y) - \Psi(x, y), \tag{6}$$

with $\Psi(x, y) = rx + sy$. Combining (6) with (5), we obtain

$$\Psi \circ h - \Psi + kb\varphi = \Phi \circ h - \Phi + c.$$

Setting $\overline{\Phi} = \Phi - \Psi$, gives:

$$kb\varphi = \overline{\Phi} \circ h - \overline{\Phi} + c.$$

Thus $b\varphi$ fails to satisfy the cocycle condition. But $b\varphi$ is homotopic to a constant map. By the remarks following the definition of the cocycle condition, there exist $\hat{\Phi} \in C^0(\mathbf{T}^2, \mathbf{T})$, homotopic to a constant map, and $c' \in \mathbf{T}$ such that

$$b\varphi = \hat{\Phi} \circ h - \hat{\Phi} + c'.$$

But then

$$\varphi = b^{-1}\hat{\Phi} \circ h - b^{-1}\hat{\Phi} + b^{-1}c',$$

and so $\varphi$ fails to satisfy the cocycle condition. This contradicts the hypothesis of the lemma.

Thus for all $b \neq 0$, $h_{\varphi_b}$ satisfies the cocycle condition, and by Proposition 1.1, the skew product $h_b = h_{\varphi_b}$ is stably ergodic and stably $K$.$\Box$

Finally, choose $\psi : \mathbf{T} \to \mathbf{T}$ to be any $C^3$, nonconstant function that is homotopic to a constant map. We will also use $\psi$ to denote the function on $\mathbf{T}^3$:

$$\psi(x, y, z) = \psi(z)$$

and use $\psi'$ for the function

$$\psi'(x, y, z) = \psi'(z).$$

To summarize, we choose $h$ to be linear Anosov, $\varphi$ to be $C^3$, homotopic to a constant and satisfying the cocycle condition, $\psi$ to be $C^3$ and homotopic to a constant, $v_0$ to be an expanding eigenvector for $h$, and $w_0$ to be any nonzero vector in $\mathbf{Z}^2$. Let $f_{a,b} : \mathbf{T}^3 \to \mathbf{T}^3$ be defined by (2). Observe that $f_{a,b}$ is volume-preserving, since $\det T f_{a,b} = 1$ everywhere.

Theorems I and II follow from

**Proposition 1.3** *For all $b > 0$ sufficiently small, there exists a positive number $a(b)$ such that, for $|a| \in (0, a(b))$, there is a neighborhood $\mathcal{U}$ of $f_{a,b}$ in $Diff^2_\mu(\mathbf{T}^3)$ in which the conclusions of Theorems I and II hold.*

# 2 Proofs

In this section we prove Proposition 1.3. We first need to establish a few facts about partially hyperbolic diffeomorphisms.

## 2.1 Partial Hyperbolicity

A diffeomorphism $g : M \to M$ is *partially hyperbolic* if there exists a $Tg$-invariant continuous splitting of the tangent bundle to $M$ into three subbundles

$$TM = E^u_g \oplus E^c_g \oplus E^s_g,$$

with at least one of $E^u_g$ or $E^s_g$ nontrivial, and constants $\lambda^s_g < \lambda^c_g < \mu^c_g < \mu^u_g$, with $\lambda^s_g < 1 < \mu^u_g$ such that, for all $p \in M$:

$$v \in E^u_g(p) \quad \Rightarrow \quad \|Dg(p)v\| \geq \mu^u_g \|v\|,$$

$$\begin{aligned} v \in E^c_g(p) \quad &\Rightarrow \quad \mu^c_g\|v\| \geq \|Dg(p)v\| \geq \lambda^c_g\|v\|, \qquad \text{and} \\ v \in E^s_g(p) \quad &\Rightarrow \quad \lambda^s_g\|v\| \geq \|Dg(p)v\|, \end{aligned}$$

with respect to some Riemannian metric on $M$. The bundles $E^u_g$, $E^c_g$, and $E^s_g$ are called the unstable, center, and stable bundles of $g$, respectively. Anosov diffeomorphisms are partially hyperbolic diffeomorphisms for which $E^c$ is trivial. The property of partial hyperbolicity is open in the $C^1$ topology on $C^r$ diffeomorphisms of $M$ (see [BP] or [HPS] for a discussion of partially hyperbolic diffeomorphisms).

The unstable and stable bundles of a $C^r$ partially hyperbolic diffeomorphism are always integrable and tangent to foliations $\mathcal{W}^u_g$ and $\mathcal{W}^s_g$, whose leaves are $C^r$ [HPS]. A partially-hyperbolic diffeomorphism is *dynamically coherent* if, in addition, the bundles $E^c_g \oplus E^u_g$ and $E^c_g \oplus E^s_g$, and $E^c_g$ are integrable and tangent to foliations $\mathcal{W}^{uc}_g$, $\mathcal{W}^{sc}_g$, and $\mathcal{W}^c_g$, whose leaves are at least $C^1$. If, in addition, $E^c_g$ is $C^1$, then $g$ is contained in an open set of dynamically coherent diffeomorphisms in $\mathrm{Diff}^1(M)$([PS3], Theorem 2.3). Dynamical coherence that is $C^1$-stable we shall call *stable dynamical coherence*.

Partial hyperbolicity and dynamical coherence for compact group extensions of Anosov diffeomorphisms is well-known and is proved in [BP], Theorem 2.2. We summarize the conclusions of this theorem for circle extensions in the next lemma. Let $f = f_{0,b} = h_b = h_{\varphi_b}$, and let $\pi_0 : \mathbf{T}^3 \to \mathbf{T}^2$ be the projection $\pi_0(x, y, z) = (x, y)$. Observe that $f$ covers the linear automorphism $h : \mathbf{T}^2 \to \mathbf{T}^2$, with $h \circ \pi_0 = \pi_0 \circ f$. The fibers of $\pi_0$ are circles, which foliate $\mathbf{T}^3$ trivially. The diffeomorphism $f$ preserves this foliation, permuting the leaves isometrically.

**Lemma 2.1** *For all $b$, the skew product $f = h_{\varphi_b}$ is partially hyperbolic and dynamically coherent. The center distribution $E^c_f$ is tangent to the fibers of $\pi_0$. For $i = u$ or $s$, each leaf of $\mathcal{W}^{ic}_f$ is the product of a leaf of $\mathcal{W}^i_h$ with a circle fiber. Each $\mathcal{W}^i_f$ leaf is the graph of a $C^2$ function from a leaf of $\mathcal{W}^i_h$ into the circle $\mathbf{T}$. For any $z_0 \in \mathbf{T}$, the translation $(p, z) \mapsto (p, z + z_0)$ carries $\mathcal{W}^i_f$ leaves to $\mathcal{W}^i_f$ leaves.*

*The Riemannian metric may be chosen so that:*

$$\lambda^s_f = m^{-1}, \qquad \lambda^c_f = \mu^c_f = 1, \qquad \textit{and} \qquad \mu^u_f = m, \tag{7}$$

*where $m$ is the expanding eigenvalue of $h$.*

The next proposition follows from Lemma 2.1 and the theory of normally-hyperbolic diffeomorphisms in [HPS].

**Proposition 2.2** *For all $b$, for $a$ sufficiently small, $f_{a,b}$ is partially hyperbolic and stably dynamically coherent. The unstable, center, and stable bundles are 1-dimensional. The center-unstable bundle $E^u_{a,b} \oplus E^c_{a,b}$ of $f_{a,b}$ does not depend on $a$ or $b$; for $w \in \mathbf{T}^3$,*

$$E^u_{a,b} \oplus E^c_{a,b}(w) = E^{uc}(w) = \{(rv_0, s) \,|\, r, s \in \mathbf{R}\}.$$

*There is a $C^1$-open neighborhood $\mathcal{U}_0$ of $f_{a,b}$ in $\mathit{Diff}^2(\mathbf{T}^3)$, such that, for $g \in \mathcal{U}_0$, the center bundle $E^c_g$ is tangent to a foliation $\mathcal{W}^c_g$ that satisfies conclusion 1 of Theorem II.*

*If, in addition, $b \neq 0$, then for $a$ sufficiently small, $f_{a,b}$ is stably ergodic and stably $K$.*

**Proof:** Since stable ergodicity and partial hyperbolicity are $C^1$-open, it suffices to establish these properties for the skew products $f_{0,b}$. Fix $b$ and let $f = f_{0,b}$. By Lemma 2.1, $f$ is partially hyperbolic. By Lemma 1.2, $f$ is stably ergodic (and stably $K$) if $b \neq 0$.

It also follows from Lemma 2.1 that, for $w \in \mathbf{T}^3$,

$$E^u_f \oplus E^c_f(w) = E^{uc} = \{(rv_0, s) \,|\, r, s \in \mathbf{R}\},$$

and

$$E^s_f \oplus E^c_f(w) = \{(rv_0', s) \,|\, r, s \in \mathbf{R}\},$$

where $v_0'$ spans the contracting eigenspace for $h$.

Suppose now that $a$ is small and nonzero. Then $f_{a,b}$ is partially hyperbolic with splitting $T\mathbf{T}^3 = E^u_{a,b} \oplus E^c_{a,b} \oplus E^s_{a,b}$. For $a$ small, $E^u_{a,b} \oplus E^c_{a,b}$ is very close to $E^{uc}$. A direct calculation shows that $Tf_{a,b}$ preserves $E^{uc}$. Since $E^u_{a,b} \oplus E^c_{a,b}$ is the fixed point of a contraction operator, it is unique. It follows that $E^u_{a,b} \oplus E^c_{a,b} = E^{uc}$.

The remaining conclusions of Proposition 2.2 follow from Corollary 8.3 in [HPS]. We recall the outline of the argument here and refer the reader to [HPS], p. 116ff., for the details, including the definitions of normal hyperbolicity and plaque expansiveness.

We verify that for $g \in \mathrm{Diff}^2(\mathbf{T}^3)$ sufficiently $C^1$-close to $f = f_{0,b}$, the center foliation $\mathcal{W}^c_g$ has the properties listed in part 1 of Theorem II. Along the way, we will show that $f$ is stably dynamically coherent. Dynamical coherence of $f$ and (7) imply that $f$ is 2-normally hyperbolic at the $C^\infty$ foliations $\mathcal{W}_f^{uc}$, $\mathcal{W}_f^{sc}$, and $\mathcal{W}_f^c$. Smoothness of these foliations implies that they are plaque expansive. (Alternately, the fact that $f$ covers a hyperbolic diffeomorphism implies plaque expansiveness.) Normally hyperbolic, plaque expansive diffeomorphisms are structurally stable, in the following sense.

**Theorem 2.3** *([HPS], Theorem 7.1) If $f : M \to M$ is $C^r$, $r$-normally hyperbolic and plaque expansive at the foliation $\mathcal{F}$, then for any $C^r$-diffeomorphism $g : M \to M$ sufficiently $C^1$-close to $f$, there exists a foliation $\mathcal{F}'$ of $M$ and a homeomorphism $H : M \to M$ satisfying*

1. *$H$ sends leaves of $\mathcal{F}'$ to leaves of $\mathcal{F}$,*
2. *$fH(L) = Hg(L)$, for every leaf $L$ of $\mathcal{F}$, and*
3. *$g$ is $r$-normally-hyperbolic and plaque expansive at $\mathcal{F}'$*

Applying Theorem 2.3 to $f = f_{0,b}$, we obtain that for $g \in \mathrm{Diff}^2(\mathbf{T}^3)$ sufficiently $C^1$-close to $f$, there exist foliations $\mathcal{W}_g^{uc}$, $\mathcal{W}_g^{sc}$, and $\mathcal{W}_g^c$, with $C^2$ leaves, homeomorphic to the foliations $\mathcal{W}_f^{uc}$, $\mathcal{W}_f^{sc}$, and $\mathcal{W}_f^c$, respectively. They are tangent to the distributions $E_g^c \oplus E_g^u$, $E_g^c \oplus E_g^s$, and $E_g^c$, respectively. The leaves of $\mathcal{W}_g^c$ are obtained by interesecting the leaves of $\mathcal{W}_g^{uc}$ and $\mathcal{W}_g^{sc}$. Thus $f$ is stably dynamically coherent.

We also obtain a a homeomorphism $H : \mathbf{T}^3 \to \mathbf{T}^3$ satisfying the conclusions of Theorem 2.3, with $\mathcal{F} = \mathcal{W}_f^c$ and $\mathcal{F}' = \mathcal{W}_g^c$. Define $\pi : \mathbf{T}^3 \to \mathbf{T}^2$ by $\pi(w) = \pi_0 \circ H(w)$. Then the fibers of $\pi$ are the leaves of $\mathcal{W}_g^c$, and

$$\pi \circ g = \pi_0 \circ H \circ g = \pi_0 \circ f \circ H = h \circ \pi_0 \circ H = h \circ \pi,$$

completing the proof.□

## 2.2 Smoothness of $E^u_{a,b}$ inside the family $f_{a,b}$

In order to determine the behavior of the Lyapunov exponents of $f_{a,b}$, we will need to differentiate the unstable line field $E^u_{a,b}$ with respect to $a$. We will use:

**Lemma 2.4** *There exists a neighborhood $\mathcal{A}$ of $(0,0)$ in $\mathbf{R}^2$, such that, for each $w \in \mathbf{T}^3$, the function*

$$(a,b) \mapsto E^u_{a,b}(w)$$

*is $C^2$ on $\mathcal{A}$. The first two derivatives of this function depend uniformly on $w$.*

**Remark:** It is not in general true that in the space of partially-hyperbolic diffeomorphisms, the map $g \mapsto E^u_g(w)$, for a fixed $w$, depends smoothly on $g$. To see that the dependence is not smooth in general, let $f : \mathbf{T}^4 \to \mathbf{T}^4$ be the real-analytic Anosov diffeomorphism constructed by Anosov in [An] whose unstable distribution $E^u_f$ is not $C^1$. For $v \in \mathbf{T}^4$, let $\tau_v$ be translation in $\mathbf{T}^4$ by $v$. Let $f_v = \tau_v \circ f \circ \tau_{-v}$. The unstable distribution for $f_v$ is $E^u_{f_v} = \tau_{v*} E^u_f$, which is the unstable distribution for $f$ translated by $\tau_v$. Then $f_v$ is clearly a $C^\omega$ family of diffeomorphisms, but $v \mapsto E^u_{f_v}(w)$ cannot be $C^1$ for all $w \in \mathbf{T}^4$.

Nonetheless, there is a form of "smooth dependence" for the unstable bundle $E^u_{g_0}$ of a $C^r$ Anosov diffeomorphism $g_0$. Namely, if $g$ is $C^r$ and sufficiently $C^1$-close to $g_0$, and $h_g = g \circ h_g \circ g_0^{-1}$ is the homeomorphism conjugating $g$ to $g_0$ (see, e.g., [KH] §18.2), then the map $g \mapsto E^u_g(h_g(w))$ is $C^{r-2}$ in a neighborhood of $g_0$, for fixed $w$. This is proved, for example, in [Ru2]. For partially-hyperbolic diffeomorphisms that are not structurally stable, it is less clear how to formulate an analogous statement.

In our situation, because the diffeomorphisms $f_{a,b}$ have a common center-unstable foliation, which is $C^\infty$, one can show that the unstable line field $E^u_{a,b}$ varies in a $C^2$ fashion along the leaves of this foliation. The difficulties we have just outlined in the general case do not arise. In fact, the same argument that gives smooth dependence of $E^u_{a,b}$ along center-unstable leaves also gives smooth dependence on the parameters $a, b$.

**Proof of Lemma 2.4:** It does not affect the smoothness of $e^u$ if we scale the functions $\varphi$ and $\psi$ by a positive constant. Thus we may assume that

$$(\|\psi\|_0 \|v_0\| + \|\varphi\|_0 + 1)^2 < m, \tag{8}$$

where $m > 1$ is the unstable eigenvalue of $h$.

By Proposition 2.2, there is a neighborhood $\mathcal{A}_0$ of $\omega_0 = (0,0)$ in $\mathbf{R}^2$ such that for all $\omega \in \mathcal{A}_0$, $f_\omega$ is partially hyperbolic, stably dynamically coherent, and, for all $w \in \mathbf{T}^3$,

$$E^u_\omega \oplus E^c_\omega(w) = E^{uc}(w) = \{(rv_0, s) \,|\, r, s \in \mathbf{R}\}.$$

The splitting $E^u_{\omega_0} \oplus E^c_{\omega_0} \oplus E^s_{\omega_0}$ is obtained from the eigenspace decomposition of the linear map $f_{\omega_0}$. The center bundle $E^c_{\omega_0}$ is spanned by $\partial/\partial z$. Assume that $\mathcal{A}_0$ is chosen so that $E^u_\omega$ is transverse to $E^c_{\omega_0}$.

Let $\mathcal{W}^{uc}$ be the $C^\infty$ foliation tangent to $E^{uc}$. The leaves of $\mathcal{W}^{uc}$ are smoothly permuted by $f_\omega$. Let $X$ be the disjoint union of the leaves of $\mathcal{W}^{uc}$. Because the foliation structure of $\mathcal{W}^{uc}$ is preserved by all of the $f_\omega$, there is a well-defined, $C^3$ map $\mathcal{F} : \mathcal{A}_0 \times X \to \mathcal{A}_0 \times X$ given by:

$$\mathcal{F}(\omega, w) = (\omega, f_\omega(w)).$$

On $\mathcal{A}_0 \times X$, put the metric:

$$d((\omega_1, w_1), (\omega_2, w_2)) = \max\{d_{\mathcal{A}_0}(\omega_1, \omega_2), d_X(w_1, w_2)\},$$

where $d_X$ is the induced Riemannian metric on $X$ and $d_{\mathcal{A}_0}((a_1, b_1), (a_2, b_2)) = \max\{|a_1 - a_2|, |b_1 - b_2|\}$. With respect to this metric, there exists a constant $\rho$ such that

$$d(\mathcal{F}(\omega_1, w_1), \mathcal{F}(\omega_2, w_2)) \geq \rho d((\omega_1, w_1), (\omega_2, w_2)),$$

for all $\omega_1, \omega_2 \in \mathcal{A}_0$ and $w_1, w_2 \in X$. The constant $\rho$ is the inverse of the Lipschitz norm of $\mathcal{F}^{-1}$. A straightforward estimate shows that by shrinking the size of the neighborhood $\mathcal{A}_0$, we may bring $\rho$ arbitrarily close to:

$$(\|\psi\|_0 \|v_0\| + \|\varphi\|_0 + 1)^{-1}.$$

Let $\mathcal{B}$ be the trivial bundle over $\mathcal{A}_0 \times X$ whose fiber $\mathcal{L}_w = \mathcal{L}$ over $w$ is the set of all linear maps $L : E^u_{\omega_0}(w) \to E^c_{\omega_0}(w)$. Since $E^u_{\omega_0}$ and $E^c_{\omega_0}$ are 1-dimensional, so is $\mathcal{L}$. We think of $\mathcal{B}$ as the product $\mathcal{A}_0 \times X \times \mathbf{R}$.

With respect to the $C^\infty$ splitting $TX = E^{uc} = E^u_{\omega_0} \oplus E^c_{\omega_0}$, the map $Tf_\omega|_{E^{uc}}$ can be written:

$$Tf_\omega = \begin{pmatrix} A_\omega & B_\omega \\ C_\omega & K_\omega \end{pmatrix},$$

where $A_\omega : E^u_{\omega_0} \to E^u_{\omega_0}$, $B_\omega : E^c_{\omega_0} \to E^u_{\omega_0}$, $C_\omega : E^u_{\omega_0} \to E^c_{\omega_0}$, and $K_\omega : E^c_{\omega_0} \to E^c_{\omega_0}$. These maps depend in a $C^2$ fashion on $\omega$ and on the basepoint $w \in \mathbf{T}^3$. When $\omega = \omega_0$, we have $B = C = 0$, $K = 1$, and $A = m$.

Define a bundle map $\mathcal{F}_\sharp : \mathcal{B} \to \mathcal{B}$, covering $\mathcal{F}$, by:

$$\mathcal{F}_\sharp(\omega, w, L) = (\omega, f_\omega(w), (C_\omega(w) + K_\omega(w)L)(A_\omega(w) + B_\omega(w)L)^{-1}).$$

Then $\mathcal{F}_\sharp$ is $C^2$, contracts fibers of $\mathcal{B}$ at the weakest by a factor $\sigma \doteq m^{-1}$, and has strongest base contraction by the factor $\rho \doteq (\|\psi\|_0\|v_0\| + \|\varphi\| + 1)^{-1}$. These estimates depend uniformly on the size of the neighborhood $\mathcal{A}_0$. Thus, by inequality (8), there is a neighborhood $\mathcal{A} \subseteq \mathcal{A}_0$ of $\omega_0$, in which

$$\sigma\rho^{-2} < 1.$$

By the $C^r$ Section Theorem of [HPS], the unique $\mathcal{F}_\sharp$-invariant section $s : \mathcal{A} \times X \to \mathcal{L}$ is $C^2$. But the graph of $s(\omega, w) : E^u_{\omega_0}(w) \to E^s_{\omega_0}(w)$ is precisely $E^u_\omega(w)$. We conclude that $E^u_\omega(w)$ is a $C^2$ function of $\omega \in \mathcal{A}$ and of $w \in X$. Since $\mathcal{W}^{uc}$ is a $C^\infty$ foliation, these estimates are uniform over $w \in \mathbf{T}^3$. $\square$

### 2.3 Some continuity properties of Lyapunov exponents

Let $g \in \mathrm{Diff}^1_\mu(M)$ be any partially-hyperbolic diffeomorphism of a 3-manifold $M$ whose unstable and stable bundles are 1-dimensional. Suppose that $V^u_g$, $V^c_g$ and $V^s_g$ are continuous nonvanishing vector fields everywhere tangent to the line fields $E^u_g$, $E^c_g$, and $E^s_g$. Then $V^u_g(w)$, $V^c_g(w)$ and $V^s_g(w)$ are a basis for $T_wM$. With respect to this basis, the derivative of $g$ at $w$ is expressed

$$Tg(w) = \begin{pmatrix} r^u_g(w) & 0 & 0 \\ 0 & r^c_g(w) & 0 \\ 0 & 0 & r^s_g(w) \end{pmatrix},$$

where $r^u_g$, $r^c_g$, and $r^s_g$ are the continuous functions:

$$\begin{aligned} r^u_g(w) &= \|Tg(w)V^u_g(w)\| \cdot \|V^u_g(g(w))\|^{-1}, & (9)\\ r^c_g(w) &= \|Tg(w)V^c_g(w)\| \cdot \|V^c_g(g(w))\|^{-1}, & (10)\\ r^s_g(w) &= \|Tg(w)V^s_g(w)\| \cdot \|V^s_g(g(w))\|^{-1}. & (11)\end{aligned}$$

Compactness implies that the functions $\log(r_g^u), \log(r_g^c)$, and $\log(r_g^s)$ are integrable on $M$, and so we may define real-valued functions $e^u$, $e^c$ and $e^s$ of the diffeomorphism $g$ by the formulas:

$$\begin{aligned} e^u(g) &= \int_M \log r_g^u(w)\, d\mu(w) && (12)\\ e^c(g) &= \int_M \log r_g^c(w)\, d\mu(w) && (13)\\ e^s(g) &= \int_M \log r_g^s(w)\, d\mu(w) && (14) \end{aligned}$$

**Lemma 2.5** *The functions $e^u$, $e^c$, and $e^s$ do not depend on the choice of vector fields $V^u$, $V^c$ and $V^s$ and are continuous in a neighborhood of $g$ in $\mathrm{Diff}^1_\mu(M)$.*

*If $g$ is ergodic with respect to $\mu$, then the Lyapunov exponents of $g$ are $e^u(g)$, $e^c(g)$ and $e^s(g)$, $\mu$-almost everywhere.*

**Proof of Lemma 2.5:** The exponents of $g$ are the logarithms of the eigenvalues of the limit:

$$\lim_{n\to\infty} \left((T_w g^n)^t\, T_w g^n\right)^{\frac{1}{2n}}$$

$$= \lim_{n\to\infty} \begin{pmatrix} \left(r_g^u(w)\cdots r_g^u(g^n(w))\right)^{\frac{1}{n}} & 0 & 0 \\ 0 & \left(r_g^c(w)\cdots r_g^c(g^n(w))\right)^{\frac{1}{n}} & 0 \\ 0 & 0 & \left(r_g^s(w)\cdots r_g^s(g^n(w))\right)^{\frac{1}{n}} \end{pmatrix}.$$

By the Ergodic Theorem, these limits converge almost everywhere. Thus the largest Lyapunov exponent at $w$ is

$$e^u(g,w) = \lim_{n\to\infty} \frac{1}{n}(\log(r_g^u(w)) + \cdots + \log(r_g^u(g^n(w)))). \qquad (15)$$

The function $e^u(g,\cdot)$ is measurable, $g$-invariant, and $\int_M e^u(g,w)d\mu(w) = \int_M \log(r_g^u(w))d\mu(w) = e^u(g)$. Hence $e^u(g)$ is the average largest Lyapunov exponent with respect to $\mu$. Since the Lyapunov exponents of $g$ with respect to $\mu$ are $\mu$-a.e. unique, the function $e^u(g)$ does not depend on $V_g^u$. Since $g$ is partially-hyperbolic, the function $g \mapsto E_g^u$ is continuous. The vector field $V_g^u$ can be chosen to depend continuously on $g$. Since $e^u(g)$ depends continuously on $V_g^u$, it follows that $e^u$ is continuous.

Suppose that $g$ is ergodic. Then $e^u(g,w) = e^u(g)$, $\mu$-a.e., and $e^u$ is the largest Lyapunov exponent of $g$, $\mu$-a.e.

The argument is the same for $e^c$ and $e^s$.$\square$

### 2.4 Behavior of exponents inside the family $f_{a,b}$

Let $\mathcal{A}$ be a neighborhood of the origin in $\mathbf{R}^2$ such that $f_\omega$ satisfies the conclusions of Proposition 2.2 and Lemma 2.4, for all $\omega \in \mathcal{A}$. In this subsection, we choose vector fields $V^i(\omega) = V^i(f_\omega)$ for $i = u, c, s$ and define functions $e^i(\omega) = e^i(f_\omega)$ according to equations (12)-(14) in the previous subsection. By Lemma 2.5, these functions depend continuously on the parameter $\omega$ and give the Lyapunov exponents of ergodic $f_\omega$. Because the functions $e^i(\omega)$ are independent of the choice of $V^i(\omega)$, we may choose these vector fields so that the computation of the $e^i$ is as simple as possible. Our goal is to find a parameter value $\omega$ for which $f_\omega$ is ergodic and $e^u(\omega)$, $e^c(\omega)$, and $e^s(\omega)$ are all nonzero.

The first observation, which will imply that $e^s(\omega)$ is the constant function on $\mathcal{A}$, is that the two-dimensional center-unstable Jacobian is preserved by $f_\omega$.

**Lemma 2.6** *Let $m$ be the eigenvalue for $v_0$. There exists a $C^\infty$ 2-form $\alpha$ such that $\alpha$ is nondegenerate on $E^{uc}$, and for all $\omega \in \mathcal{A}$,*

$$f_\omega^*\alpha = m\alpha + \beta_\omega,$$

*where $\beta_\omega$ vanishes on $E^{uc}$:*

$$\beta_\omega(v_1, v_2) = 0, \qquad \forall v_1, v_2 \in E^{uc}.$$

**Proof of Lemma 2.6:** Write $v_0 = (q_1, q_2)$ and let $\alpha = v_0 \cdot (dx, dy) \wedge dz = q_1 dx \wedge dz + q_2 dy \wedge dz$. Clearly $\alpha$ is nondegenerate on $E^{uc}$. Let $(a,b) = \omega$. One calculates that $g_a^*(dx \wedge dz) = dx \wedge dz$ and $g_a^*(dy \wedge dz) = dy \wedge dz$, and so $g_a^*\alpha = \alpha$. Next, note that

$$h_b^*\alpha = m\alpha + c_\omega(x,y) dx \wedge dy,$$

where $c_\omega(x,y) = m \det(v_0 \ \nabla\varphi_b(x,y))$. The form $\beta_\omega = c(x,y) dx \wedge dy$ vanishes on $E^{uc}$, since $dx \wedge dy$ does. Finally,

$$f_\omega^*\alpha = h_b^* g_a^* \alpha = h_b^*\alpha = m\alpha + \beta_\omega.$$

$\square$

We now choose $V^u_\omega$, $V^c_\omega$, and $V^s_\omega$. For $\omega \in \mathcal{A}$, the line $E^u_\omega(w)$ sits inside the plane $E^{uc}(w)$, transverse to the line spanned by $(0,0,1)$. There is a unique vector in $E^u_\omega(w)$ of the form $(v_0, r)$. Let $V^u_\omega(w)$ be this vector; this also defines a continuous function $u_\omega : \mathbf{T}^3 \to \mathbf{R}$, by:

$$V^u_\omega(w) = (v_0, u_\omega(w)).$$

By Lemma 2.4, $u_\omega(w)$ is a $C^2$ function of $\omega \in \mathcal{A}$. The first two derivatives depend continuously on $w \in \mathbf{T}^3$.

Next, for $w \in \mathbf{T}^3$, pick $V^c_\omega(w)$ tangent to $E^c_\omega(w)$ such that

$$\alpha(V^u_\omega(w), V^c_\omega(w)) = 1.$$

Finally, pick $V^s_\omega(w)$ tangent to $E^s_\omega(w)$ such that

$$dx \wedge dy \wedge dz(V^u_\omega(w), V^c_\omega(w), V^s_\omega(w)) = 1.$$

Let $e^u(\omega) = e^u(f_\omega)$, $e^c(\omega) = e^c(f_\omega)$, and $e^s(\omega) = e^s(f_\omega)$ be defined by equations (12)-(14).

**Lemma 2.7** *The functions $e^s$ and $e^u + e^c$ are constant:*

$$e^s(\omega) = -\log(m), \qquad e^u(\omega) + e^c(\omega) = \log(m).$$

*for $\omega \in \mathcal{A}$.*

**Proof of Lemma 2.7:** For $i = u, c, s$, let $r^i_\omega = r^i_{f_\omega}$, be the functions defined by equations (9)-(11), using the vector fields $V^i_\omega$.

The lemma follows if we establish that the following equations hold for all $w \in \mathbf{T}^3$:

$$r^u_\omega(w) r^c_\omega(w) = m, \tag{16}$$
$$r^u_\omega(w) r^c_\omega(w) r^s_\omega(w) = 1. \tag{17}$$

Using Lemma 2.6, we see that

$$\begin{aligned}
\alpha(Tf_\omega(w)V^u_\omega(w), Tf_\omega(w)V^c_\omega(w)) &= f^*_\omega\alpha(V^u_\omega(w), V^c_\omega(w)) \\
&= m\alpha(V^u_\omega(w), V^c_\omega(w)) + \beta_\omega(V^u_\omega(w), V^c_\omega(w)) \\
&= m\alpha(V^u_\omega(w), V^c_\omega(w)) \\
&= m,
\end{aligned}$$

while on the other hand,

$$\begin{aligned}\alpha(Tf_\omega(w)V^u_\omega(w), Tf_\omega(w)V^c_\omega(w)) &= \alpha(r^u_\omega(w)V^u_\omega(f_\omega(w)), r^c_\omega(w)V^c_\omega(f_\omega(w)))\\ &= r^u_\omega(w)r^c_\omega(w)\alpha(V^u_\omega(f_\omega(w)), V^c_\omega(f_\omega(w)))\\ &= r^u_\omega(w)r^c_\omega(w),\end{aligned}$$

which proves (16). Since $f_\omega$ preserves volume,

$$\begin{aligned}1 &= dx\wedge dy\wedge dz(Tf_\omega(w)V^u_\omega(w), Tf_\omega(w)V^c_\omega(w), Tf_\omega(w)V^s_\omega(w))\\ &= r^u_\omega(w)r^c_\omega(w)r^s_\omega(w),\end{aligned}$$

proving (17).□

**Lemma 2.8** *For $\omega = (a,b) \in \mathcal{A}$,*

$$e^u(\omega) = \log(m) - \int_{\mathbf{T}^3} \log(1 - a\psi'(w)u_\omega(w))\,dw.$$

**Proof of Lemma 2.8:** From the definition of $f_\omega$, we have:

$$\begin{aligned}Tf_\omega(w)\begin{pmatrix} v_0 \\ u_\omega(w)\end{pmatrix} &= \begin{pmatrix} [m + a\psi'(f_\omega(w))\,[u_\omega(w) + \nabla\varphi_b(w)\cdot v_0]]\,v_0 \\ u_\omega(w) + \nabla\varphi_b(w)\cdot v_0\end{pmatrix}\\ &= \begin{pmatrix} [m + a\psi'(f_\omega(w))r^u_\omega(w)u_\omega(f_\omega(w))]\,v_0 \\ r^u_\omega(w)u_\omega(f_\omega(w))\end{pmatrix}.\end{aligned}$$

It follows that

$$r^u_\omega(w) = m + a\psi'(f_\omega(w))u_\omega(f_\omega(w))r^u_\omega(w),$$

and so $r^u_\omega(w) = m/(1 - a\psi'(f_\omega(w))u_\omega(f_\omega(w)))$. To obtain the formula, integrate $\log(r^u_\omega(w))$. □

We now study how the function $e^u(a,b)$ varies when $a$ varies, keeping $b$ fixed.

**Lemma 2.9** *$e^u$ is $C^2$ on $\mathcal{A}$, and*

$$\frac{\partial}{\partial a}e^u(a,b) = \int_{\mathbf{T}^3} \frac{\psi'(w)u_{a,b}(w) + a\psi'(w)\frac{\partial u_{a,b}(w)}{\partial a}}{1 - a\psi'(w)u_{a,b}(w)}dw. \tag{18}$$

**Proof:** By Lemma 2.4, the function $\omega \mapsto u_\omega(w)$ is $C^2$ on $\mathcal{A}$, uniformly in $w$. Then by the formula in Lemma 2.8, $e^u$ is $C^2$ as well. Differentiating this formula with respect to $a$ gives (18). □

Setting $a = 0$ in (18), we obtain:

$$\frac{\partial}{\partial a} e^u(0, b) \;=\; \int_{\mathbf{T}^3} \psi'(w) u_{0,b}(w)\, dw \tag{19}$$

By Lemma 2.1, the distribution $E^u_{0,b}$ for the skew product $f_{0,b}$ is invariant under translations of the form $(x, y, z) \mapsto (x, y, z + z_0)$. This implies that the function $u_{0,b}(x, y, z)$ depends only on $x$ and $y$. On the other hand, $\psi'(x, y, z) = \psi'(z)$ depends only on $z$. The integral in (19) is therefore equal to

$$\int_{\mathbf{T}} \psi'(z)\, dz \int_{\mathbf{T}^3} u_{0,b}(w)\, dw \;=\; 0,$$

since $\psi$ is homotopic to a constant map. We have shown:

**Proposition 2.10** *For* $(a, b) \in \mathcal{A}$,

$$\frac{\partial}{\partial a} e^u(0, b) = 0$$

The behavior of the exponents of $f_{a,b}$ near $(0, 0)$ is thus determined by the second derivative of $e^u$ with respect to $a$. An exact computation of this second derivative is difficult in general. For our purposes, it suffices to compute this derivative at $(0, 0)$.

**Proposition 2.11**

$$\frac{\partial^2}{\partial a^2} e^u(0, 0) \;=\; -u_0^2 \int_0^1 \psi'(z)^2\, dz < 0,$$

*where* $u_0 = (w_0 \cdot v_0)/(m - 1)$.

We prove Proposition 2.11 in the final subsection.

A picture of some unstable Lyapunov exponents computed for $b = 0$ and small values of $a$ is shown in Figure 2.4, where $h(x, y) = (2x + y, x + y)$, $\psi(z) = \cos(2\pi z)$, and $v_0 = (\lambda, 1)$, $\lambda = (1 + \sqrt{5})/2$. A random starting point was chosen for each value of $a$ and $100,000$ iterations were applied to the vector $(\lambda, 1, \lambda)$. Using Propositions 2.10 and 2.11, we computed the second order Taylor expansion with respect to $a$ at $(0, 0)$ to be $e^u(a, 0) \sim \log(m) - \pi^2 m a^2$, where $m = (3 + \sqrt{5})/2$. This approximation in fact fits the data quite well.

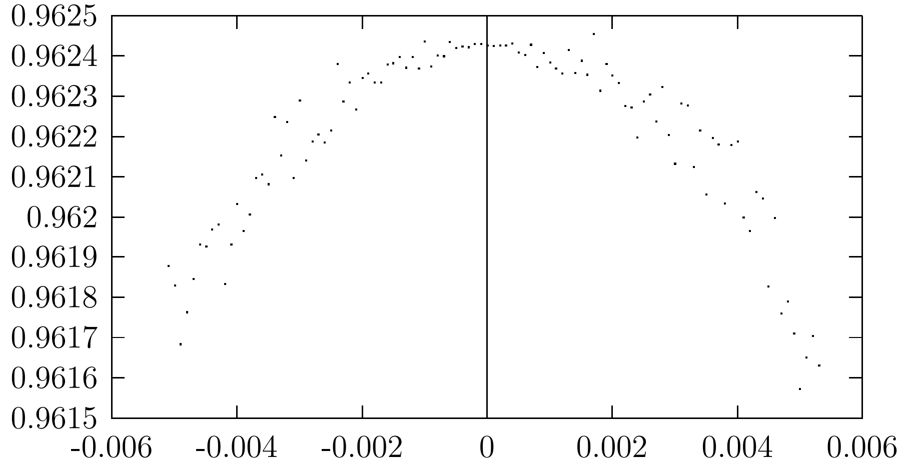


Figure 1: $e^u(a,b)$ for the family $f_{a,b}$, with $b=0$.

## 2.5 Proof of Proposition 1.3

If $|b|$ is sufficiently small, then by Lemma 2.9 and Proposition 2.11, we have

$$\frac{\partial^2}{\partial a^2} e^u(0,b) \quad < \quad 0. \tag{20}$$

By Proposition 2.10,

$$\frac{\partial}{\partial a} e^u(0,b) = 0. \tag{21}$$

Choose $b$ different from 0 and satisfying (20). It follows from (20), (21), and Lemma 2.9 that for $|a|$ sufficiently small and positive, $e^u(a,b)$ is positive and strictly greater than $\log(m)$. But now by Lemma 2.7, $e^c(a,b) = \log(m) - e^u(a,b) > 0$, and $e^s(a,b) = -\log(m)$. By Lemma 2.5, the Lyapunov exponents of $f_{a,b}$ are almost everywhere equal to $e^u(a,b)$, $e^c(a,b)$, and $e^s(a,b)$. Thus for this $b$, and for $|a|$ small and positive, the Lyapunov exponents of $f_{a,b}$ are nonzero almost everywhere.

Let $f = f_{a,b}$, where $a$ and $b$ are chosen so that $f_{a,b}$ is stably a $K$-system and partially hyperbolic, the foliation $\mathcal{W}^c_{a,b}$ exists and has the properties outlined in Proposition 2.2, and the exponents of $f_{a,b}$ are nonzero, $\mu - a.e.$. Let $\mathcal{U}_0$

be the neighborhood of $f_{a,b}$ in $\mathrm{Diff}^2_\mu(\mathbf{T}^3)$ given by Proposition 2.2. Every $g \in \mathcal{U}_0$ is a $K$-system, is partially hyperbolic, and has a center foliation $\mathcal{W}^c_g$ satisfying conclusion 1 of Theorem II. By Lemma 2.5 the exponents of $g \in \mathcal{U}_0$ depend continuously on $g$. Since $f_{a,b}$ is nonuniformly hyperbolic, there exists a $C^1$-open neighborhood $\mathcal{U} \subseteq \mathcal{U}_0$ of $f_{a,b}$ in $\mathrm{Diff}^2_\mu(\mathbf{T}^3)$ so that every $g \in \mathcal{U}$ is nonuniformly hyperbolic. By a theorem of Pesin [Pe], a nonuniformly hyperbolic $K$-system is isomorphic to a Bernoulli shift. Therefore every $g \in \mathcal{U}$ is isomorphic to a Bernoulli shift.

It also follows from Lemma 2.5 that for $g \in \mathcal{U}$, $E^c_g$ is a Lyapunov direction $\mu$-a.e. for $g$, with positive exponent, since $g$ is nonuniformly hyperbolic. This implies conclusion 2 of Theorem II.

Conclusion 3 of Theorem II follows from 2. Let $S$ be the set of $w \in \mathbf{T}^3$ such that $E^c_g(w)$ is a positive Lyapunov direction. Then $S$ has full $\mu$-measure in $\mathbf{T}^3$. Suppose there were a circle $\mathcal{C} \in \mathcal{W}^c_g$ with $\mu_{\mathcal{C}}(\mathcal{C} \cap S) > 0$, where $\mu_C$ is Riemannian measure on $\mathcal{C}$. Then the length of $g^n(\mathcal{C})$ would be unbounded as $n \to \infty$. But $g^n(\mathcal{C})$ is a leaf of $\mathcal{W}^c_g$, a contradiction.

Finally, $g$ is not isotopic to an Anosov diffeomorphism, since $g$ is isotopic to the linear map $f_{0,0}$, which has 1 as an eigenvalue. This completes the proof of Proposition 1.3.

## 2.6 Proof of Proposition 2.11

Use "$f_a$" to denote $f_{a,0}$, "$u_a$" to denote $u_{a,0}$, and "$e(a)$" for $e^u(a,0)$. Differentiating (18), we have

$$e''(a) = \int_{\mathbf{T}^3} \left( \frac{\psi'(w)u_a(w) + a\psi'(w)\frac{\partial u_a(w)}{\partial a}}{1 - a\psi'(w)u_a(w)} \right)^2 + \frac{2\psi'(w)\frac{\partial u_a(w)}{\partial a} + a\psi'(w)\frac{\partial^2 u_a(w)}{\partial a^2}}{(1 - a\psi'(w)u_a(w))}\, dw,$$

and setting $a = 0$, we obtain

$$e''(0) \;=\; \int_{\mathbf{T}^3} (\psi'(w)u_0(w))^2 + 2\psi'(w)\frac{\partial u_a(w)}{\partial a}|_{a=0}\, dw. \qquad (22)$$

The map $f_0 = f_{0,0}$ is linear. It is easy to see that $u_0(w)$ is the constant function $u_0(w) = u_0 = (w_0 \cdot v_0)/(m-1) \neq 0$

For $a \in \mathbf{R}$, $w = (x, y, z) \in \mathbf{T}^3$, and $u \in \mathbf{R}$, let

$$\gamma(a, w, u) = \frac{c + u}{m + a\psi'(w)(c + u)},$$

where $c = w_0 \cdot v_0$. Note that $\gamma(a, w, u_a(f_a^{-1}(w))) = u_a(w)$ and that for $|a|$ sufficiently small,

$$u_a(w) = \lim_{n\to\infty} \gamma(a, w, \gamma(a, f_a^{-1}(w), \ldots \gamma(a, f_a^{-n}(w), 0) \cdots)).$$

We compute:

$$\frac{\partial \gamma}{\partial a}(a, w, u) = \frac{-\psi'(w)(c+u)^2}{(m + a\psi'(w)(c+u))^2},$$

and set $a = 0$ to get:

$$\frac{\partial \gamma}{\partial a}(0, w, u) \quad = \quad \frac{-\psi'(w)(c+u)^2}{m^2}.$$

Similarly,

$$\frac{\partial \gamma}{\partial x}(0, w, u) = \frac{\partial \gamma}{\partial z}(0, w, u) = \frac{\partial \gamma}{\partial y}(0, w, u) = 0,$$

and

$$\frac{\partial \gamma}{\partial u}(0, w, u) = \frac{1}{m}.$$

We want to evaluate $\frac{\partial u_a(w)}{\partial a}|_{a=0}$. Since $u_a(w) = \gamma(a, w, u_a(f_a^{-1}(w)))$, the chain rule yields:

$$\begin{aligned}
\frac{\partial u_a(w)}{\partial a}|_{a=0} &= \frac{\partial}{\partial a}\gamma(a, w, u_a(f_a^{-1}(u)))|_{a=0} \\
&= \frac{\partial \gamma}{\partial a}(0, w, u_0(f_0^{-1}(w))) + \frac{\partial \gamma}{\partial u}(0, w, u_0(F_0^{-1}(w))) \cdot \frac{\partial u_a(f_a^{-1}(w))}{\partial a}|_{a=0}) \\
&= \frac{-\psi'(w)(c + u_0(f_0^{-1}(w)))^2}{m^2} + \frac{1}{m}\frac{\partial u_a(f_a^{-1}(w))}{\partial a}|_{a=0}).
\end{aligned}$$

Recall that $u_0$ is the constant function $u_0(w) = u_0$, and $u_0 = (c + u_0)/m$, so this expression simplifies to:

$$\frac{\partial u_a(w)}{\partial a}|_{a=0} \quad = \quad -\psi'(w)u_0^2 + \frac{1}{m}\frac{\partial u_a(f_a^{-1}(w))}{\partial a}|_{a=0}. \tag{23}$$

Iterating (23) gives

$$\frac{\partial u_a(w)}{\partial a}|_{a=0} \quad = \quad -\psi'(w)u_0^2 - \frac{\psi'(w_{-1})u_0^2}{m} - \frac{\psi'(w_{-2})u_0^2}{m^2} - \cdots$$

where $w_{-j} = f_a^{-j}(w)$. Hence

$$\int_{\mathbf{T}^3} \psi'(w) \frac{\partial u_a(w)}{\partial a}|_{a=0}\, dw \quad = \quad -\sum_{j\geq 0} \int_{\mathbf{T}^3} \frac{\psi'(w)\psi'(w_{-j}) u_0^2}{m^j}\, dw.$$

The $j = 0$ term of this sum is $-\int \psi'(w)^2 u_0^2\, dw$. Pulling out this term, we have

$$\int_{\mathbf{T}^3} (\psi'(w) u_0(w))^2 + 2\psi'(w) \frac{\partial u_a(w)}{\partial a}|_{a=0}\, dw \quad =$$

$$= \quad -\int_{\mathbf{T}^3} \psi'(w)^2 u_0^2\, dw - 2\sum_{j\geq 1} \int_{\mathbf{T}^3} \frac{\psi'(w)\psi'(w_{-j}) u_0^2}{m^j}\, dw. \qquad (24)$$

Notice that

$$\int_{\mathbf{T}^3} \psi'(w)\psi'(w_{-j}) u_0^2\, dw \quad = \quad u_0^2 \int_0^1 \int_0^1 \int_0^1 \psi'(z)\psi'(z - r_j x - s_j y)\, dx\, dy\, dz,$$

where $r_j$ and $s_j$ are integers, and $r_j = 0$ if and only if $j = 0$, in which case $s_j = 0$ as well. Thus, for $j \geq 1$,

$$\begin{aligned} \int_{\mathbf{T}^3} \psi'(w)\psi'(w_{-j}) u_0^2\, dw \quad &= \quad u_0^2 r_j^{-1} \int_0^1 \int_0^1 \psi'(z)\psi(z - r_j x - s_j y))|_{x=0}^{x=1}\, dy\, dz \\ &= \quad 0. \end{aligned}$$

Combining this with equations (22) and (24), we have:

$$\begin{aligned} e''(0) \quad &= \quad -\int_{\mathbf{T}^3} \psi'(w)^2 u_0^2\, dw - 2\sum_{j\geq 1} \int_{\mathbf{T}^3} \frac{\psi'(w)\psi'(w_{-j}) u_0^2}{m^j}\, dw \\ &= \quad -u_0^2 \int_0^1 \psi'(z)^2\, dz, \end{aligned}$$

completing the proof.

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