RIGIDITY OF SOME ABELIAN-BY-CYCLIC SOLVABLE GROUP ACTIONS ON $\mathbb{T}^N$

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ABSTRACT. In this paper, we study a natural class of groups that act as affine transformations of $\mathbb{T}^N$. We investigate whether these solvable, "abelian-by-cyclic," groups can act smoothly *and nonaffinely* on $\mathbb{T}^N$ while remaining homotopic to the affine actions. In the affine actions, elliptic and hyperbolic dynamics coexist, forcing *a priori* complicated dynamics in nonaffine perturbations. We first show, using the KAM method, that any small and sufficiently smooth perturbation of such an affine action can be conjugated smoothly to an affine action, provided certain Diophantine conditions on the action are met. In dimension two, under natural dynamical hypotheses, we get a complete classification of such actions; namely, any such group action by C^r diffeomorphisms can be conjugated to the affine action by $C^{r-\varepsilon}$ conjugacy. Next, we show that in any dimension, C^1 small perturbations can be conjugated to an affine action via $C^{1+\varepsilon}$ conjugacy. The method is a generalization of the Herman theory for circle diffeomorphisms to higher dimensions in the presence of a foliation structure provided by the hyperbolic dynamics.

CONTENTS

1. INTRODUCTION

This paper is motivated by an attempt to understand the action on the 2-torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ generated by the diffeomorphisms

$$g_0(x, y) = (2x + y, x + y), \quad g_1(x, y) = (x + \rho, y), \quad g_2(x, y) = (x, y + \rho), \quad \rho \in \mathbb{R}.$$

The map g_0 is a hyperbolic linear automorphism, and g_1, g_2 are translations. They satisfy the group relations

$$g_0 g_1 = g_1^2 g_2 g_0, \quad g_0 g_2 = g_1 g_2 g_0, \quad g_1 g_2 = g_2 g_1,$$

and no other relations if ρ is irrational. Broadly stated, our aim is classify all diffeomorphisms g_0, g_1, g_2 satisfying these relations and no other.

To place the problem in a more general context, in this paper, we establish rigidity properties of certain solvable group actions on the torus $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$, for $N > 1$. The solvable groups Γ considered here are the finitely presented, torsion-free, *abelian-by-cyclic* (ABC) groups, which admit a short exact sequence

$$0 \hookrightarrow \mathbb{Z}^d \to \Gamma \to \mathbb{Z} \to 0.$$

All such groups are of the form $\Gamma = \Gamma_B$, where $B = (b_{ij})$ is an integer valued, $d \times d$ matrix with $\det(B) \neq 0$, and

(1.1)

$$\Gamma_B = \mathbb{Z} \ltimes \mathbb{Z}^d = \left\langle g_0, g_1, \ldots, g_d \mid g_0 g_i = \left(\prod_{j=1}^{d} g_j^{b_{ji}}\right) g_0, \quad [g_i, g_j] = 1, \quad i, j = 1, 2, \ldots, d \right\rangle.$$

ABC groups have been studied intensively in geometric group theory, as they present the first case in the open problem to classify finitely generated, non-nilpotent solvable groups up to quasi-isometry. The classification problem for ABC groups has been solved in [FM1] (the non-polycyclic case, $|\det B| > 1$) and [EFW1, EFW2] (the polycyclic case, $|\det B| = 1$), where the authors also revealed close connections between the geometry of these groups and dynamics [FM2, EF]. Here we consider actions of polycyclic ABC groups.

A C^r action α of a finitely generated group Γ with generators $g_1, \ldots, g_k$ on a closed manifold M is a homomorphism $\alpha \colon \Gamma \to \mathrm{Diff}^r(M)$, where $\mathrm{Diff}^r(M)$ denotes the group of orientation-preserving, C^k diffeomorphisms of M. The action is determined completely by $\alpha(g_1), \ldots, \alpha(g_k)$. The polycylic ABC groups admit natural affine actions on tori, as follows.

Up to rearranging the standard basis for $\mathbb{R}^d$, every matrix $B \in \mathrm{SL}(d; \mathbb{Z})$ can be written in the form

$$B = \begin{pmatrix} \bar{B} & 0 \\ 0 & I_{d-N} \end{pmatrix},$$

for some $N \leq d$, where $\bar{B} = (\bar{b}_{ij}) \in \mathrm{SL}(N; \mathbb{Z})$, and I_{d-N} is the $(d - N) \times (d - N)$ identity matrix, chosen to be maximal. In this paper, we restrict our attention to the cases where $d = KN + 1$, for some $K \geq 1$. Then

$$\Gamma_B = \Gamma_{\bar{B},K} := \langle g_0, g_{i,k}, \ i = 1, \ldots, N, \ k = 1, \ldots, K \mid [g_{i,k}, g_{j,\ell}] = 1,$$

(1.2)

$$g_0 g_{i,k} = \left(\prod_{j=1}^{N} g_{j,k}^{\bar{b}_{ji}}\right) g_0, \quad i, j = 1, \ldots, N, \ k, \ell = 1, \ldots, K \Big\rangle.$$

Note that $\Gamma_{\bar{B}} = \Gamma_{\bar{B},1}$.

In the affine actions of $\Gamma_{\bar{B},K}$ we consider, the element g_0 acts on $\mathbb{T}^N$ by the automorphism $x \mapsto \bar{A}x \pmod{\mathbb{Z}^N}$ induced by $\bar{A} \in \mathrm{SL}(N, \mathbb{Z})$, and the elements $g_{i,k}, i = 1, \ldots, N, \ k = 1, \ldots, K$ act as translations $x \mapsto x + \rho_{i,j} \pmod{\mathbb{Z}^N}$, where $\rho_{i,j} \in \mathbb{R}^N$. Thus if we denote this action by $\alpha = \alpha_{\bar{B},K} \colon \Gamma_{\bar{B},K} \to \mathrm{Diff}^r(\mathbb{T}^N)$, we have

$$\alpha(g_0)(x) = \bar{A}x, \quad \text{and} \quad \alpha(g_{i,k})(x) = x + \rho_{i,j}.$$

The group relations in $\Gamma_{\bar{B},K}$ restrict the possible values of $\rho_{i,j}$; we describe precisely these restrictions in the next subsection. We will see that for a typical $\bar{A}$, the affine

actions define a finite dimensional space of distinct (i.e. nonconjugate) actions on the torus.

Given such a group $\Gamma_{\bar{B},K}$ with the associated affine action $\bar{\alpha}$ we investigate whether there exist other actions $\alpha \colon \Gamma_{\bar{B},K} \to \mathrm{Diff}^r(\mathbb{T}^N)$ that are homotopic to $\bar{\alpha}$ but not conjugate to $\bar{\alpha}$ in the group $\mathrm{Diff}^r(\mathbb{T}^N)$. If there are no such actions, or if such actions are proscribed in some manner, then the group is colloquially said to be *rigid* (a much more precise definition is given below).

The main rigidity results of this paper can be grouped into two classes: local and global. Loosely speaking, local rigidity results concern those C^r actions that are C^r perturbations of the affine action, and global results concern actions where C^r closeness to the affine action is not assumed (although other restrictions might be present).

We obtain local rigidity results for the actions of $\Gamma_{\bar{B}} = \Gamma_{\bar{B},1}$, which imply similar results for $\Gamma_{\bar{B}} = \Gamma_{\bar{B},K}$, $K \geq 1$. To each action α on $\mathbb{T}^N$ sufficiently C^r close to an affine action for some large r, we define an $N \times N$ *rotation matrix* $\boldsymbol{\rho}(\alpha)$. Under suitable hypotheses on $\boldsymbol{\rho}(\alpha)$, if the columns of this matrix satisfy a simultaneous Diophantine condition, then α is smoothly conjugate to the affine action with rotation matrix $\boldsymbol{\rho}(\alpha)$. The fact that the action α is a smooth perturbation of an affine action is crucial.

More generally, for each affine action $\bar{\alpha}$ of $\Gamma_{\bar{B},K}$ there are K rotation matrices $\boldsymbol{\rho}_1(\bar{\alpha}), \ldots, \boldsymbol{\rho}_K(\bar{\alpha})$. In the section on global rigidity, we consider actions α of $\Gamma_{\bar{B},K}$ for which $\bar{B}$ acts as an Anosov diffeomorphism, but the rotation matrices $\boldsymbol{\rho}_i(\alpha)$ are not *a priori* well-defined. Under relatively weak additional assumptions on the action, we obtain that the collection of $\boldsymbol{\rho}_i(\alpha)$ can be defined and and forms a complete invariant of the action, up to *topological* conjugacy. We then establish conditions under which this topological conjugacy is smooth. In particular, if K is sufficiently large (depending on the spectrum of $\bar{A}$ and the Anosov element $\alpha(g_0)$), then for almost every set of rotation matrices $\boldsymbol{\rho}_1(\alpha), \ldots, \boldsymbol{\rho}_K(\alpha)$, the conjugacy is smooth.

Before stating these results, we describe precisely the space of affine actions of $\Gamma_{\bar{B},K}$ we consider.

1.1. **The affine actions of $\Gamma_{\bar{B},K}$.** The following proposition can be verified directly using the group relation (3.1). The ABC group relation imposes a restriction to the rotation vectors as follows.

Proposition 1.1. *Let $\bar{A}, \bar{B} \in \mathrm{SL}(N, \mathbb{Z})$, and suppose that $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \ldots, \boldsymbol{\rho}_K$ are real-valued, $N \times N$ matrices such that each $\boldsymbol{\rho} = \boldsymbol{\rho}_i$ satisfies:*

$$(1.3) \qquad\qquad\qquad \bar{A}\boldsymbol{\rho} = \boldsymbol{\rho}\bar{B} \bmod \ \mathbb{Z}^{N \times N}.$$

Denote by $\rho_{i,j}$ the j-th column of $\boldsymbol{\rho}_i$. Then the affine maps

$$\bar{\alpha}(g_0)(x) := \bar{A}x \ (\mathrm{mod}\,\mathbb{Z}^N), \quad \text{and} \quad \bar{\alpha}(g_{i,j})(x) := x + \rho_{i,j} \ (\mathrm{mod}\,\mathbb{Z}^N)$$

define an action $\bar{\alpha} = \bar{\alpha}_K(\bar{A}, \boldsymbol{\rho}) \colon \Gamma_{\bar{B},K} \to \mathrm{SL}(N, \mathbb{Z}) \ltimes \mathbb{R}^N$ *on* $\mathbb{T}^N$.

Conversely, if $\alpha \colon \Gamma_{\bar{B},K} \to \mathrm{SL}(N, \mathbb{Z}) \ltimes \mathbb{R}^N$ *is an action on* $\mathbb{T}^N$ *with*

$$\alpha(g_0)(x) = \bar{A}x \ (\mathrm{mod} \, \mathbb{Z}^N), \quad \text{and} \quad \alpha(g_i)(x) = x + \beta_{i,j} \ (\mathrm{mod} \, \mathbb{Z}^N),$$

for some vectors $\beta_{i,j} \in \mathbb{R}^N$, *then for each* $i = 1, \ldots, K$, *the matrix* $\boldsymbol{\rho}_i$ *whose columns are formed by the* $\beta_{i,j}$ *satisfies* (1.3).

We will give more details about the set of rotation vectors satisfying (1.3) in Proposition B.1, Lemma B.2 and Theorem B.4 in Appendix B.

We further focus on the case $K = 1$. For $\bar{A}, \bar{B} \in \mathrm{SL}(N, \mathbb{Z})$ and $\boldsymbol{\rho} \in M_N(\mathbb{T})$, where $M_N(\mathbb{T})$ denotes $N \times N$ matrices with entries in $\mathbb{T}$, we denote by $\bar{\alpha}(\bar{A}, \boldsymbol{\rho})$ the action $\bar{\alpha}$ on $\mathbb{T}^N$ defined in Proposition 1.1. Let

$$\mathrm{Aff}(\Gamma_{\bar{B}}, \bar{A}) := \{\bar{\alpha}(\bar{A}, \boldsymbol{\rho}) : \boldsymbol{\rho} \text{ satisfies } (1.3)\}.$$

The next proposition describes faithful affine actions.

Proposition 1.2. *The action* $\alpha(\bar{A}, \boldsymbol{\rho}) \in \mathrm{Aff}(\Gamma_{\bar{B}}, \bar{A})$ *is faithful if and only if* $\bar{A}$ *is not of finite order, and the column vectors* $\rho_1, \ldots, \rho_N$ *of* $\boldsymbol{\rho}$ *are linearly independent over* $\mathbb{Z}$; *that is, if there exists* $(p_1, \ldots, p_N) \in \mathbb{Z}^N$ *with* $\sum_{i=1}^N p_i \rho_i = 0 \mod \mathbb{Z}^N$, *then* $p_1 = \cdots = p_N = 0$.

For $\bar{A} \in \mathrm{SL}(N, \mathbb{Z})$ not of finite order, we thus define the set of faithful affine actions by

$$\mathrm{Aff}_\star(\Gamma_{\bar{B}}, \bar{A}) := \{\bar{\alpha}(\bar{A}, \boldsymbol{\rho}) \in \mathrm{Aff}(\Gamma_{\bar{B}}, \bar{A}) : \bar{\alpha}(\bar{A}, \boldsymbol{\rho}) \text{ is faithful}\}.$$

Proposition 1.2 reduces the problem of finding faithful actions to solving the equation (1.3) for solution $\boldsymbol{\rho}$ with linearly independent columns over $\mathbb{Z}$. We will provide more details in Appendix B. It may happen that the set $\mathrm{Aff}_\star(\Gamma_{\bar{B}}, \bar{A})$ is empty, for instance, when $\bar{A}, \bar{B} \in \mathrm{SL}(2, \mathbb{Z})$ with $\mathrm{tr}\bar{A} \neq \mathrm{tr}\bar{B}$ (c.f. Proposition B.1(1)). It is useful to keep in mind the following special case of abundant faithful actions. Suppose $\bar{A} = \bar{B} \in \mathrm{SL}(N, \mathbb{Z})$ has simple spectrum, then all $\boldsymbol{\rho}$ of the form $\boldsymbol{\rho} = \sum_{i=1}^N a_i \bar{A}^{i-1}$ for $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$ satisfy (1.3) (c.f. Lemma B.2). The action is not faithful when a is a rational vector and is faithful when a is a Diophantine vector (c.f. Lemma B.3).

Proposition 1.3. *Given* $\bar{A} \in \mathrm{SL}(N, \mathbb{Z})$, *two actions* $\bar{\alpha}_1, \bar{\alpha}_2 \in \mathrm{Aff}_\star(\Gamma_{\bar{B}}, \bar{A})$ *are conjugate by a homeomorphism homotopic to identity if and only if* $\bar{\alpha}_1 = \bar{\alpha}_2$.

Thus actions in $\mathrm{Aff}_\star(\Gamma_{\bar{B}}, \bar{A})$ may not be locally rigid even among affine actions. Returning to our original example, let

(1.4) $$\bar{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \bar{B} = \bar{A}$$

and

$$\Gamma_{\bar{A}} = \langle g_0, g_1, g_2 \mid g_0 g_1 g_0^{-1} = g_1^2 g_2, \quad g_0 g_2 g_0^{-1} = g_1 g_2, \quad [g_1, g_2] = 1 \rangle.$$

Fix $a_0, a_1 \in \mathbb{T}^1$, and let $\boldsymbol{\rho}(a_0, a_1) := \begin{pmatrix} a_0 + 2a_1 & a_1 \\ a_1 & a_0 + a_1 \end{pmatrix}$. Then

$$\{\bar{\alpha}(\bar{A}, \boldsymbol{\rho}(a_0, a_1)) : a_0, a_1 \in \mathbb{T}^1\}$$

defines a 2-parameter family of non-conjugate actions on $\mathbb{T}^2$; in the next subsection we explain that these are all such affine actions.

Thus the matrix $\boldsymbol{\rho}$ is a complete invariant of the faithful affine representations $\bar{\alpha}(\bar{A}, \boldsymbol{\rho})$ of $\Gamma_{\bar{B}}$. The columns of $\boldsymbol{\rho}$ are rotation vectors of the corresponding translations. We will show that these rotation vectors, and hence the invariant $\boldsymbol{\rho}$, extend continuously to a neighborhood of the affine representations in such a way that $\boldsymbol{\rho}$ gives a complete invariant under smooth conjugacy, under the hypotheses that the columns of $\boldsymbol{\rho}$ satisfy a simultaneous Diophantine condition.

Further properties of the affine representations are discussed in Appendix C, which also contains the proofs of the results in this section.

1.2. **Local rigidity of $\Gamma_{\bar{B},K}$ actions.** An action $\alpha \colon \Gamma \to \mathrm{Diff}^r(M)$ is $C^{r,k,\ell}$ *locally rigid* if any sufficiently C^k small C^r perturbation $\tilde{\alpha}$ is C^ℓ conjugate to α, i.e., there exists a diffeomorphism h of M, C^ℓ close to the identity, that conjugates $\tilde{\alpha}$ to α: $h \circ \alpha(g) = \tilde{\alpha}(g) \circ h$ for all $g \in \Gamma$. The paper of Fisher [Fi] contains background and an excellent overview of the local rigidity problem for general group actions.

Local rigidity results for solvable group actions are relatively rare. In [DK], Damjanovic and Katok proved $C^{\infty,k,\infty}$, for some large k, local rigidity for $\mathbb{Z}^n$ ($n \geq 2$) (abelian) higher rank partially hyperbolic actions by toral automorphisms, by introducing a new KAM iterative scheme. In [HSW] and [W1], the authors proved local rigidity for higher rank ergodic nilpotent actions by toral automorphisms on $\mathbb{T}^N$, for any even $N \geq 6$. Burslem and Wilkinson in [BW] studied the solvable Baumslag-Solitar groups

$$BS(1, n) := \langle a, b \mid aba^{-1} = b^n; n \geq 2 \rangle)$$

acting on $\mathbb{T}^1$ and obtained a classification of such actions and a global rigidity result in the analytic setting. Asaoka in [A1, A2] studied the local rigidity of the action on $\mathbb{T}^N$ or $\mathbb{S}^N$ of non-polycyclic abelian-by-cyclic groups, where the cyclic factor is uniformly expanding. There is a recent paper [W2] by Zhenqi Wang, who proves local rigidity for certain solvable Lie group actions (without any constraints), which can be considered as continuous time counterpart of our actions in the sense that they are solvable and there are only two chambers.

Unless assumptions are made on the action (or the manifold), solvable group actions are typically not locally rigid but can enjoy a form of partial local rigidity: that is, local rigidity subject to constraints that certain invariants be preserved. The simplest example occurs in dimension 1, where the rotation number of a single C^2 circle diffeomorphism supplies a complete topological invariant, provided that it is irrational,

and a complete smooth invariant, provided it satisfies a Diophantine condition. This result extends to actions of higher rank abelian groups on $\mathbb{T}^1$, under a simultaneous Diophantine assumption on the rotation numbers of the generators of the action [M]. In fact, these results are not just local in nature but apply to all diffeomorphisms of the circle [FK].

For higher dimensional tori, even local rigidity results of this type are scarce, one problem being the lack of invariants analogous to the rotation number. One result in this direction is by Damjanovic and Fayad [DF], who proved local rigidity of ergodic affine $\mathbb{Z}^k$ actions on the torus that have a rank-one factor in their linear part, under certain Diophantine conditions.

Definition 1.1. *A collection of vectors $v_1, \ldots, v_m \in \mathbb{R}^N$ is* simultaneously Diophantine *if there exist $\tau > 0$ and $C > 0$ such that*

$$(1.5) \qquad \max_{1 \leq i \leq m} |\langle v_i, n \rangle| \geq \frac{C}{\|n\|^\tau}, \qquad \forall\, n \in \mathbb{Z}^N \setminus \{0\}.$$

We denote by $\mathrm{SDC}(C, \tau)$ *the set of $(v_1, \ldots, v_m)$ satisfying (1.5).*

For example, the matrix $\rho \, \mathrm{Id}_N$ is simultaneously Diophantine if ρ is a Diophantine number. It is known that for any for fixed $\tau > N - 1$, the simultaneous Diophantine vectors

$$\bigcup_{C > 0} \mathrm{SDC}(C, \tau)$$

form a full Lebesgue measure subset of $\mathbb{T}^{N \times m}$ ([P]).

Definition 1.2. *Given a homeomorphism $f : \mathbb{T}^N \to \mathbb{T}^N$ homotopic to the identity and preserving a probability measure μ, the vector*

$$(1.6) \qquad \rho_\mu(f) := \int_{\mathbb{T}^N} (\tilde{f}(x) - x) \, d\mu, \ \mathrm{mod}\ \mathbb{Z}^N,$$

where $\tilde{f} \colon \mathbb{R}^N \to \mathbb{R}^N$ is any lift of f, is independent of the choice of lift $\tilde{f}$. We call $\rho_\mu(f)$ the rotation vector *of f with respect to μ.*

Our main local rigidity result is:

Theorem 1.4. *For any $\bar{A}, \bar{B} \in \mathrm{SL}(N, \mathbb{Z})$ and any $C, \tau > 0$, there exist $\varepsilon > 0$ and $\ell \in \mathbb{N}$ such that for any $\boldsymbol{\rho} \in \mathrm{SDC}(C, \tau)$ satisfying (1.3) the following holds. Let $\alpha : \Gamma_{\bar{B}} \to \mathrm{Diff}^\infty(\mathbb{T}^N)$ be any representation satisfying*

(1) $\alpha(g_0)$ is homotopic to $\bar{\alpha}(\bar{A}, \boldsymbol{\rho})(g_0) = \bar{A}$;
(2) $\max_{1 \leq i \leq N} \|\alpha(g_i) - \bar{\alpha}(\bar{A}, \boldsymbol{\rho})(g_i)\|_{C^\ell} < \varepsilon$;
(3) there exist $\alpha(g_i)$-invariant probability measures μ_i, $i = 1, \ldots, N$, such that the matrix formed by the rotation vectors $(\rho_{\mu_1}(\alpha(g_1)), \ldots, \rho_{\mu_N}(\alpha(g_N)))$ is equal to $\boldsymbol{\rho}$.

Then there exists a C^∞ diffeomorphism h that is C^1 close to identity such that $h \circ \alpha = \bar{\alpha} \circ h$. Moreover, the measure $\mu = h_*^{-1}\mathrm{Leb}$, where Leb is Haar measure on $\mathbb{T}^N$, is the unique α-invariant measure and thus satisfies $\rho_\mu(T_i) = \rho_{\mu_i}(T_i) = \rho_i, \ i = 1, \ldots, N$.

We remark that there is no assumption on $\bar{A}$ and $\bar{B}$ other than being in $\mathrm{SL}(N, \mathbb{Z})$, and no smallness or hyperbolicity assumption on $\alpha(g_0)$. The reason is that in the proof we first apply the KAM scheme to the $\mathbb{Z}^N$ part of the action to find the conjugacy h then prove that h also conjugates the $\alpha(g_0)$ to $\bar{A}$ using the group relation and the ergodicity of the $\mathbb{Z}^N$ action (c.f. Section 2). On the other hand, the assumption $\boldsymbol{\rho} \in \mathrm{SDC}(C, \tau)$ satisfying (1.3) puts some constraints on the choice of $\bar{A}$ and $\bar{B}$, since it may happen that the affine action is not even faithful for some choices of $\bar{A}$ and $\bar{B}$.

We will prove in Appendix B that the simultaneously Diophantine condition is actually satisfied by a large class of matrices $\boldsymbol{\rho}$ and $\bar{A}, \bar{B}$ satisfying (1.3). One special case is when $\bar{A} = \bar{B} \in \mathrm{SL}(N, \mathbb{Z})$ has simple spectrum and $\boldsymbol{\rho} = \sum_{i=1}^{N} a_i \bar{A}^{i-1}, \ a_i \in \mathbb{R}$, $i = 1, \ldots, N$. The columns of the matrix $\boldsymbol{\rho}$ are simultaneously Diophantine if the nonvanishing a_i's form a Diophantine vector (c.f. Lemma B.3).

Remark 1.1. *We remark that the faithfulness (guaranteed by the Diophantine condition) of the action is necessary for smooth conjugacy. For instance, consider $\rho = 1/2$ in (1.4) and $\alpha(g_i) = \bar{\alpha}(g_i), \ i = 1, 2$, and for any $\varepsilon > 0$,*

$$\alpha(g_0) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \varepsilon \begin{bmatrix} \sin(4\pi x) \\ \sin(4\pi x) \end{bmatrix}.$$

One can verify that this gives rise to a $\Gamma_{\bar{A}}$ action. We will see in Theorem 1.5 that for sufficiently small ε, there exists a bi-Hölder conjugacy h satisfying $h \circ \alpha = \bar{\alpha} \circ h$. However, the conjugacy h is not C^1. Indeed, 0 is a fixed point for both $\alpha(g_0)$ and $\bar{A}$. The derivative $D_0\alpha(g_0) = \bar{A} + 4\pi\varepsilon \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ has determinant 1 but different trace than $\bar{A}$ for $\epsilon \neq 0$, so it is not conjugate to $\bar{A}$.

Since there are still faithful affine actions which do not satisfy the SDC conditions, for which our KAM scheme does not work, it would be interesting to explore the local rigidity of those actions. In the 2D case, [HX] gives the the topological classification for all such actions and smooth classification for faithful actions by volume preserving diffeomorphisms.

1.3. **Global rigidity.** The proof of the above local rigidity theorem is an application of the KAM techniques for $\mathbb{Z}^N$ actions initiated by Moser [M] in the context of $\mathbb{Z}^N$ actions by circle diffeomorphisms. The KAM technique is essentially perturbative. It is natural to ask if our solvable group action is rigid in the nonperturbative sense, i.e. whether it is globally rigid. A class of actions of a group, not necessarily close to a algebraic actions, is called *globally rigid* if any action from this class is conjugate to an

algebraic one. There is a nonperturbative global rigidity theory for circle maps known as *Herman-Yoccoz theory*. For abelian group actions by circle diffeomorphisms, the global version of Moser's theorem was proved by Fayad and Khanin [FK]. These global rigidity results rely on the Denjoy theorem stating that a C^2 circle diffeomorphism with irrational rotation number is topologically conjugate to the irrational rotation by the rotation number.

In the higher dimensional case, there is no corresponding Herman-Yoccoz theory for diffeomorphisms of $\mathbb{T}^N$ isotopic to rotations. The reason is that rotation vectors are not well-defined in general. Even when rotation vectors are uniquely defined, they are not the complete invariants for conjugacy analogous to rotation numbers for circle maps. In particular, the obvious analogue of the topological conjugacy given by the Denjoy theorem does not exist for diffeomorphisms of $\mathbb{T}^N$, $N > 1$.

On the other hand, by a theorem of Franks (Theorem 3.1 below), Anosov diffeomorphisms of $\mathbb{T}^N$ are topologically conjugate to toral automorphisms. A diffeomorphism $f \colon M \to M$ is called *Anosov* if there exist constants C and $0 < \lambda < 1$ and for each $x \in M$ a splitting of the tangent space $T_x M = E^s(x) \oplus E^u(x)$ such that for every $x \in M$, we have

- $D_x f E^s(x) = E^s(f(x))$ and $D_x f E^u(x) = E^u(f(x))$,
- $\|D_x f^n v\| \leq C \lambda^n \|v\|$ for $v \in E^s(x) \setminus \{0\}$ and $n \geq 0$, and
  $\|D_x f^n v\| \leq C \lambda^{-n} \|v\|$ for $v \in E^u(x) \setminus \{0\}$ and $n \leq 0$.

As the starting point of a global rigidity result of our $\Gamma_{\bar{A}}$ action, we assume $\alpha(g_0)$ acts by an Anosov diffeomorphism homotopic to $\bar{A}$. With the topological conjugacy at hand, the next question is to show the topological conjugacy given by Franks's theorem also linearizes the abelian subgroup action. The new problem that arises is that for toral diffeomorphisms homotopic to identity the rotation vector is in general not well-defined, and it only makes sense to talk about the rotation set. When there is more than one vector in the rotation set, the diffeomorphism cannot be conjugate to a translation.

1.3.1. *Topological conjugacy.* The case $N = 2$ admits a fairly complete understanding of the topological picture of ABC actions. In particular, the next result classifies the ABC group actions on $\mathbb{T}^2$ up to topological conjugacy when g_0 acts by an Anosov diffeomorphism and the g_i, for $i \geq 1$ are not too far from translations, in a sense that we make precise.

Theorem 1.5. *Let* $\bar{A}, \bar{B} \in \mathrm{SL}(2, \mathbb{Z})$ *be linear Anosov and* $\alpha \colon \Gamma_{\bar{B}} \to \mathrm{Diff}^r(\mathbb{T}^2)$, $r > 1$, *be a representation satisfying*

(1) $\alpha(g_0)$ *is Anosov and homotopic to* $\bar{A}$;

(2) *the sub-action generated by $\alpha(g_1), \ldots, \alpha(g_N)$ has sub-linear oscillation (see Definition 1.3 below) in the case of $\mathrm{tr}\bar{A}=\mathrm{tr}\bar{B}$ and c-slow oscillation in the case of $\mathrm{tr}\bar{A} \neq \mathrm{tr}\bar{B}$ where c is in Remark 1.3.*

Then there exist $\boldsymbol{\rho}$ satisfying (1.3) and a unique bi-Hölder homeomorphism $h\colon \mathbb{T}^2 \to \mathbb{T}^2$ homotopic to the identity satisfying

$$h \circ \alpha = \bar{\alpha}(\bar{A}, \boldsymbol{\rho}) \circ h.$$

Remark 1.2. *In this theorem faithfulness of the action is not necessary, since we do not need the rotation vectors of $\bar{\alpha}(g_i)$ to be irrational.*

The assumption on the sub-linear oscillation is removed in [HX] by introducing an Anosov foliation Tits' alternative. Here we give the statement and refer the readers to [HX] for the proof.

Theorem 1.6 ([HX]). *Suppose that $\alpha : \Gamma_{\bar{B}} \to \mathrm{Diff}(\mathbb{T}^2)$ is such that:*

(1) *$\bar{B} \in \mathrm{SL}(2,\mathbb{Z})$ is an Anosov linear map (i.e. $\bar{B}$ has eigenvalues of norm different than one.)*
(2) *The diffeomorphism $\alpha(g_0)$ is Anosov and homotopic to $\bar{A}$.*

Then α is topologically conjugate to an affine action of $\Gamma_{\bar{B}}$ as in Proposition 1.1 up to finite index. More concretely, there exist a finite index subgroup $\Gamma' < \Gamma_{\bar{B}}$ and $h \in \mathrm{Homeo}(\mathbb{T}^2)$ such that $h\alpha(\Gamma')h^{-1}$ is an affine action.

With the notion of c-slow oscillation, we also obtain the following result for general N.

Theorem 1.7. *Suppose $N > 2$. Given hyperbolic matrices $\bar{A}, \bar{B} \in \mathrm{SL}(N,\mathbb{Z})$, there exists $0 \leq c < 1$ such that the following holds. Let $\alpha\colon \Gamma_{\bar{B}} \to \mathrm{Diff}^r(\mathbb{T}^N)$, $r > 1$, be a representation satisfying*

(1) *$\alpha(g_0)$ is Anosov and homotopic to $\bar{A}$,*
(2) *the sub-action generated by $\alpha(g_1), \ldots, \alpha(g_N)$ has c-slow oscillation (see Definition 1.3 below).*

Then there exist $\boldsymbol{\rho}$ satisfying (1.3) and a unique bi-Hölder homeomorphism $h\colon \mathbb{T}^N \to \mathbb{T}^N$ homotopic to the identity with

$$h \circ \alpha = \bar{\alpha}(\bar{A}, \boldsymbol{\rho}) \circ h.$$

Remark 1.3. *The constant c in Theorem 1.7 can be made explicit as follows. Suppose $\bar{A}$ has eigenvalues $\lambda_1^u, \ldots, \lambda_k^u$ and $\lambda_1^s, \ldots, \lambda_\ell^s$, $k \geq 1$, $\ell \geq 1$, $k + \ell = N$, (complex eigenvalues and repeated eigenvalues are allowed), ordered as follows*

(1.7) $$|\lambda_\ell^s| \leq \ldots \leq |\lambda_1^s| < 1 < |\lambda_1^u| \leq \ldots \leq |\lambda_k^u|.$$

We introduce similar quantities for $\bar{B}$

(1.8) $$|\mu_{\ell'}^s| \leq \ldots \leq |\mu_1^s| < 1 < |\mu_1^u| \leq \ldots \leq |\mu_{k'}^u|, \quad \ell' + k' = N.$$

Then c can be chosen to be any number satisfying

(1.9) $$0 \leq c < \min\left\{\frac{\ln|\lambda_1^u|}{\ln|\mu_{k'}^u|}, \frac{\ln|\lambda_1^s|}{\ln|\mu_{\ell'}^s|}\right\}.$$

We next introduce the concept of sublinear deviation and c-slow deviation. Let $T \in \mathrm{Diff}_0(\mathbb{T}^N)$ and let $\tilde{T}\colon \mathbb{R}^N \to \mathbb{R}^N$ be a lift of T. Denote by π_i the projection to the i-th component of a vector in $\mathbb{R}^N$. Define the *oscillation* $\mathrm{Osc}(\tilde{T})$ of $\tilde{T}$ by:

$$\mathrm{Osc}(\tilde{T}) := \max_{x,i}\{\pi_i(\tilde{T}(x) - x)\} - \min_{x,i}\{\pi_i(\tilde{T}(x) - x)\}.$$

It is easy to see that Osc is independent of the choice of the lift. We define $\mathrm{Osc}(T) = \mathrm{Osc}(\tilde{T})$.

Definition 1.3. (1) *For given $c \in [0, 1)$, we say that the abelian group action $\beta\colon \mathbb{Z}^N \to \mathrm{Diff}_0^r(\mathbb{T}^N)$ is of c-slow oscillation if*

$$\limsup_{\|p\|\to\infty} \frac{\mathrm{Osc}(\beta(p))}{\|p\|^c} < \infty.$$

(2) *We say the action β is of bounded oscillation if it is of 0-slow oscillation.*

(3) *We say the action β has sublinear oscillation if*

$$\limsup_{\|p\|\to\infty} \frac{\mathrm{Osc}(\beta(p))}{\|p\|} = 0.$$

Let us motivate the definition of c-slow oscillation a bit. In the circle map case, the existence and uniqueness of rotation number relies crucially on the fact that the graph in $\mathbb{R}^2$ of every lifted orbit stays within distance 1 of a straight line, and the rotation number is simply the slope of the line. This fact is also important in the study of Euler class and bounded cohomology for groups acting on circle [Gh]. We say a diffeomorphism $f\colon \mathbb{T}^N \to \mathbb{T}^N$ is of *bounded deviation* if there exists $\rho \in \mathbb{T}^N$ and a constant $C > 0$, such that

$$\|\tilde{f}^n(x) - x - n\rho\|_{C^0} \leq C, \quad \forall\, n \in \mathbb{Z}.$$

Being of bounded deviation implies that each orbit of $\tilde{f}$ stays within bounded distance of the line $\mathbb{R}\rho$. The concept of bounded deviation was first introduced by Morse, who called it of class A, in the case of geodesic flows on surfaces of genus greater than 1 [Mo]. It was later shown by Hedlund that globally minimizing geodesics for an arbitrary smooth metric on $\mathbb{T}^2$ are also of bounded deviation [He]. A generalization to Gromov hyperbolic spaces can be found in [BBI]. In the one-dimensional case, all circle maps are of bounded deviation, from which follows immediately the existence of the rotation number.

Being of bounded deviation does *not* however guarantee the existence of a conjugacy to a rigid translation. In the one dimensional case, a circle map with irrational rotation number is only known to be semi-conjugate to a rotation. Denjoy's counter-example shows that the semi-conjugacy cannot be improved to a conjugacy without further assumptions. In the two dimensional case, it is known [Ja] that for a conservative pseudo-rotation of bounded oscillation, the rotation vector being totally irrational is equivalent to the existence of a semi-conjugacy to the rigid translation. Examples of diffeomorphisms on $\mathbb{T}^2$ of bounded deviation can be found in [MS], which are higher dimensional generalizations of Denjoy's examples on $\mathbb{T}^1$.

It is easy to see that bounded deviation implies c-slow oscillation with $c = 0$. Sub-linear oscillation occurs in first passage percolation (see Section 4.2 of [ADH]) where paths minimizing a cost defined for random walks on $\mathbb{Z}^2$ have c-slow oscillation with a power law $c \leq 3/4$ and conjecturally $c = 2/3$.

1.3.2. *Smooth conjugacy.* The conjugacy h in Theorem 1.5 and 1.7 is only known to be Hölder. It is natural to ask if we can improve the regularity. In hyperbolic dynamics, there is a periodic data rigidity theory for Anosov diffeomorphisms, which implies in the two-dimensional case that if the regularity of h is known to be C^1, then h is in fact as smooth as the Anosov diffeomorphism $\alpha(g_0)$ (see Theorem 5.7 below).

So the problem is now to find sufficient conditions for our action to ensure that the conjugacy h is C^1. The invariant foliation structure given by the Anosov diffeomorphism enables us to generalize the Herman-Yoccoz theory for circle maps to the higher dimensional setting.

To obtain higher regularity of the conjugacy, we consider a slightly different class of ABC groups $\Gamma_{\bar{B},K}$ for some $K \geq 1$.

We introduce the following condition:
$$(\star)\ \boldsymbol{\rho}\ \textit{rationally generates}\ \mathbb{T}^N,$$

meaning: the set $\left\{ \sum_{i=1}^{N} p_i \boldsymbol{\rho}_i \bmod \mathbb{Z}^N \mid (p_1, \ldots, p_N) \in \mathbb{Z}^N \right\}$ is dense in $\mathbb{T}^N$, where $\boldsymbol{\rho}_i$ denotes the ith column of $\boldsymbol{\rho}$.

Theorem 1.8. *Let $\bar{A}, \bar{B} \in \mathrm{SL}(2, \mathbb{Z})$ with $\mathrm{tr}\bar{A}=\mathrm{tr}\bar{B}$. Given an Anosov diffeomorphism $A \colon \mathbb{T}^2 \to \mathbb{T}^2$ homotopic to $\bar{A} \in \mathrm{SL}(2, \mathbb{Z})$, there is a C^1 open set $\mathcal{O}$ of Anosov diffeomorphisms containing A, and a number K_0 such that for any integer $K \geq K_0$, there exists a full measure set $\mathcal{R}_{2,K} \subset (\mathbb{T}^2)^K$ such that the following holds.*

Let $\alpha \colon \Gamma_{\bar{B},K} \to \mathrm{Diff}^r(\mathbb{T}^2)$ be a representation satisfying:

(1) *$\alpha(g_0) \in \mathcal{O}$,*
(2) *the sub-action generated by $\alpha(g_{1,1}), \alpha(g_{2,1})$ has sub-linear oscillation, and assume in addition that $\boldsymbol{\rho}$ given by Theorem 1.5 satisfies $(\star)$.*

(3) *for some* $i\colon \{1,\ldots,K\} \to \{1,2\}$, *the rotation vectors* $(\rho_{i(1),1},\ldots,\rho_{i(K),K})$ *lie in* $\mathcal{R}_{2,K}$, *where* $\rho_{j,k}$ *is the rotation vector of* $\alpha(g_{j,k})$ *with respect to an invariant probability measure* $\mu_{j,k}$, $j = 1,2$ *and* $k = 1,\ldots,K$.

Then there exists a unique $C^{r-\varepsilon}$ *conjugacy* h *conjugating the action* α *to an affine action for* ε *arbitrarily small.*

Theorem 1.9. *Given a hyperbolic* $\bar{A} \in \mathrm{SL}(N,\mathbb{Z})$, $N > 2$, *with simple real spectrum, there exist a* C^1 *neighborhood* $\mathcal{O}$ *of* $\bar{A}$, *a number* $0 \le c < 1$ *and a number* K_0, *such that for any integer* $K > K_0$, *there exists a full measure set* $\mathcal{R}_{N,K} \subset (\mathbb{T}^N)^K$ *such that the following holds.*

Let $\alpha\colon \Gamma_{\bar{B},K} \to \mathrm{Diff}^r(\mathbb{T}^N)$ *be a representation satisfying*

(1) $\alpha(g_0) \in \mathcal{O}$;
(2) *the sub-action generated by* $\alpha(g_{1,1}),\ldots,\alpha(g_{N,1})$ *has* c-*slow oscillation and assume in addition that* $\boldsymbol{\rho}$ *given by Theorem 1.7 satisfies* $(\star)$;
(3) *for some* $i\colon \{1,\ldots,K\} \to \{1,\ldots,N\}$, *the rotation vectors* $(\rho_{i(1),1},\ldots,\rho_{i(K),K})$ *lie in* $\mathcal{R}_{N,K}$, *where* $\rho_{j,k}$ *is the rotation vector of* $\alpha(g_{j,k})$ *with respect to an invariant probability measure* $\mu_{j,k}$, $j = 1,\ldots,N$ *and* $k = 1,\ldots,K$.

Then there is a unique $C^{1,\nu}$ *conjugacy* h *conjugating* α *to an affine action for some* $\nu > 0$.

In dimension 3, the regularity of the conjugacy can be improved applying the work of Gogolev in [G2] (Theorem 5.8 below).

Corollary 1.10. *Under the same assumptions as Theorem 1.9, suppose in addition that* $N = 3$ *and* $r > 3$. *Then the conjugacy* $h \in C^{r-3-\varepsilon}$, *for arbitrarily small* ε. *Moreover, there exists a* $\kappa \in \mathbb{Z}$ *such that if* $r \notin (\kappa, \kappa+3)$, *then* $h \in C^{r-\varepsilon}$.

Further relaxation of the assumptions of Theorem 1.9 and Corollary 1.10 is possible, snd we discuss this in Section 6.3. In particular, in many cases the condition on the C^1 closeness of A to $\bar{A}$ can be relaxed.

For the $N > 2$ case, the elliptic dynamics techniques in two dimensions carry over completely. However, there are two new obstructions that come from the hyperbolic dynamics. On the one hand, a conjugacy between two Anosov diffeomorphisms sends (un)stable leaves to (un)stable leaves. On the other hand the affine foliations parallel to the *eigenspaces* of $\bar{A}$ might not be sent to A-invariant foliations with smooth leaves. Adding to the difficulty is the fact that the regularity of the weakest stable and unstable distributions are low (only Hölder in general). These issues present an obstacle to developing a theory of periodic data rigidity as strong as the two-dimensional setting. The most general result [G1, GKS] in this direction for $N > 2$ states that *if A and $\bar{A}$*

are C^1 close and have the same periodic data, then the conjugacy h is C^{1+} (i.e. Dh and Dh^{-1} are Hölder).

The paper is organized as follows. We prove the local rigidity Theorem 1.4 in Section 2. All the remaining sections are devoted to the proof of the global rigidity results. In Section 3, we prove that there is a common conjugacy (Theorem 1.5 and 1.7). In Section 4, we prepare techniques from elliptic dynamics and hyperbolic dynamics. In Section 5, we state and prove the main propositions needed for the proof of Theorem 1.8 and 1.9. In Section 6, we prove the main Theorems 1.8 and 1.9. In Appendix A, we give the proof of the number theoretic result Theorem 5.3. In Section C, we prove the results about affine actions stated in Section 1.1.

2. LOCAL RIGIDITY: PROOFS

In this section, we prove Theorem 1.4. Here is a sketch. Given representation $\alpha \colon \Gamma_{\bar{B}} \to \mathrm{Diff}^\infty(\mathbb{T}^N)$ with $(\rho_{\mu_1}(\alpha(g_1)), \ldots, \rho_{\mu_N}(\alpha(g_N))) = \boldsymbol{\rho} \in \mathrm{SDC}(C, \tau)$, where μ_i is a invariant probability measure of $\alpha(g_i)$, we can proceed as in [M] using the KAM method to show that the abelian subgroup action can be smoothly conjugated to rigid translations. Using the group relation, we can further show that this conjugacy also conjugates the diffeomorphism $\alpha(g_0)$ to a linear one.

The following proposition is proved by the standard KAM iteration procedure.

Proposition 2.1 (KAM for abelian group actions)**.** *Given $C, \tau > 0$, there exist $\ell \geq 1$ and $\varepsilon_0 > 0$ such that the following holds.*

Let $T_1, \ldots, T_m \in \mathrm{Diff}_0^\infty(\mathbb{T}^N)$ be commuting diffeomorphisms with $m > 1$. Suppose there exist T_k-invariant measures μ_k such that the rotation vectors $\rho_{\mu_k}(T_k)$, $k = 1, \ldots, m$, satisfy the simultaneous Diophantine condition with constants C, τ, and

$$\max_{1 \leq k \leq m} \|T_k - \mathrm{id} - \rho_{\mu_k}(T_k)\|_{C^\ell} < \varepsilon_0.$$

Then there exists a C^∞ diffeomorphism h that is C^1 close to the identity such that

$$h \circ T_k(x) = h(x) + \rho_{\mu_k}(T_k), \quad x \in \mathbb{T}^N, \quad k = 1, \ldots, m.$$

Moreover the invariant measure $\mu = h_*^{-1}\mathrm{Leb}$, where Leb is Haar measure on $\mathbb{T}^N$, satisfies $\rho_\mu(T_k) = \rho_{\mu_k}(T_k)$, $k = 1, \ldots, m$.

Proof. The proof of this lemma is essentially the same as Moser [M]. A proof was sketched by F. Rodriguez-Hertz in the case of $m = 2, N = 2$ (see Theorem 6.5 of [R1]). It is not difficult to adapt the proof to the case $N > 2, m \geq 2$. The only complexity in the case $N > 1$ is caused by the fact that the rotation vector is in general not uniquely defined. Here we give a sketch of the KAM iteration procedure to explain how to incorporate the rotation vector and invariant measure.

Given $T_1, \ldots T_m$, we want to find a conjugacy h as stated. The strategy of the KAM iteration scheme is to find a sequence $\{h^{(n)}\}, n \geq 0$ of diffeomorphisms such that $\lim_n \mathscr{H}_n \to h$ in the C^1 topology, where $\mathscr{H}_n = h^{(n)} \circ \cdots \circ h^{(1)}$. This limit h is *a priori* only C^1, but a standard argument then shows that h is smooth. Let $h^{(0)} = id$, and for $k = 1, \ldots, m$, let $T_k^{(n)}(x) := \mathscr{H}_n T_k \mathscr{H}_n^{-1}(x)$, for $n \geq 0$. Let $R_k^{(n)} \colon \mathbb{T}^N \to \mathbb{R}^N$ be defined by the equation $T_k^{(n)}(x) = x + \rho_{\mu_k}(T_k) + R_k^{(n)}(x)$.

We show that for each k, the sequence $R_k^{(n)}$ converges to zero in the C^1 topology as $n \to \infty$. When $T_k^{(n-1)}$ is known from the previous step, each $h^{(n)}$ is found by solving the linearization of the equation $h^{(n)} T_k^{(n-1)} = h^{(n)} + \rho_{\mu_k}(T_k)$. Since the equation is not solved exactly in each step, the conjugated map $T_k^{(n)} = h^{(n)} T_k^{(n-1)} (h^{(n)})^{-1}$ is not yet the translation $x \mapsto x + \rho_{\mu_k}(T_k)$ but is closer to it than $T_k^{(n-1)}$ is. The standard KAM method consists mainly of two ideas: the solvability of each linearized equation under the Diophantine condition up to some loss of derivatives, and the convergence of the procedure due to the quadratic smallness of $R_k^{(n+1)}$ compared with $R_k^{(n)}$. The key observation of [M] that will also be important here is that the commutativity enables us to solve for one $h^{(n)}$ simultaneously for all $k = 1, \ldots, m$ assuming the SDC.

Step 1: the cohomological equation and commutativity.

Write $T_k(x) = x + \rho_k + R_k(x)$ for $k = 1, \ldots, m$, where $\rho_k = \rho_{\mu_k}(T_k)$ is the rotation vector of T_k with respect to the given measure μ_k. For the sake of iteration later, we will also label $T_k = T_k^{(0)}, R_k = R_k^{(0)}$ and $\mu_k = \mu_k^{(0)}$. The vector ρ_k will be kept constant independent of the super-script.

The conjugacy equation $h T_k = h + \rho_k$ gives

$$x + \rho_k + R_k^{(0)}(x) + H(x + \rho_k + R_k^{(0)}(x)) = x + H(x) + \rho_k, \text{ where } h(x) = x + H(x),$$

whose linearization is

$$(2.1) \qquad\qquad H(x + \rho_k) - H(x) = -R_k^{(0)}(x).$$

Taking Fourier expansions $H(x) = \sum_{n \in \mathbb{Z}^N} \hat{H}_n e^{2\pi i \langle n, x\rangle}$ and $R_k^{(0)}(x) = \sum_{n \in \mathbb{Z}^N} \hat{R}_{k,n}^{(0)} e^{2\pi i \langle n, x\rangle}$, we get for $n \neq 0$

$$(2.2) \qquad\qquad \hat{H}_n(e^{2\pi i \langle \rho_k, n\rangle} - 1) = -\hat{R}_{k,n}^{(0)}.$$

The commutativity condition $T_k T_j = T_j T_k$ gives

$$x + \rho_j + R_j^{(0)}(x) + \rho_k + R_k^{(0)}(x + \rho_j + R_j^{(0)}(x)) = x + \rho_k + R_k^{(0)}(x) + \rho_j + R_j^{(0)}(x + \rho_k + R_k^{(0)}(x)),$$

whose linearization is

$$(2.3) \qquad\qquad R_k^{(0)}(x + \rho_j) - R_k^{(0)} = R_j^{(0)}(x + \rho_k) - R_j^{(0)}.$$

In terms of Fourier coefficients,

$$(2.4) \qquad \hat{R}_{k,n}^{(0)}(e^{i2\pi\langle \rho_j, n\rangle} - 1) = \hat{R}_{j,n}^{(0)}(e^{i2\pi\langle \rho_k, n\rangle} - 1), \quad n \in \mathbb{Z}^N \setminus \{0\}.$$

The key point is that the commutativity equation (2.4) implies that the solution of the cohomological equation (2.2) for some k also solves the same equation for all the other $j \neq k$.

Step 2: the Fourier cut-off and solving the cohomological equation.

We next show how to solve the cohomological equation (2.2). By the simultaneous Diophantine condition, there exists $C > 0$ such that for each $n \in \mathbb{Z}^N \setminus \{0\}$, there exists $k = k(n) \in \{1, \ldots, m\}$ such that $|\langle \rho_k, n\rangle| \geq \frac{C}{\|n\|^\tau}$ where $\rho_k = \rho_{\mu_k}(T_k)$ is the rotation vector.

We take a Fourier cutoff so that we can control higher order derivatives via lower order derivatives. For $J^{(0)} \in \mathbb{N}$, we solve for $\hat{H}_n$ with $|n| < J^{(0)}$ so that we have $\|\bar{H}^{(1)}\|_{C^{\ell+\tau}} \leq C(J^{(0)})^\tau \|\bar{H}^{(1)}\|_{C^\ell}$, where $\bar{H}^{(1)}(x) := \sum_{|n| \leq J^{(0)}} \hat{H}_n e^{2\pi i\langle n, x\rangle}$. Solving (2.2) for $k = k(n)$, we get

$$\hat{H}_n = -(e^{2\pi i\langle \rho_{k(n)}, n\rangle} - 1)^{-1} \hat{R}_{k(n),n}, \text{ for all } |n| \leq J^{(0)}.$$

Denoting $h^{(1)}(x) = x + \bar{H}^{(1)}(x)$, we get the estimate $\|\bar{H}^{(1)}\|_{C^\ell} \leq C\|R^{(0)}\|_{C^{\ell+\tau}}$ by the SDC.

From equation (2.2) and (2.4), we get that $\bar{H}^{(1)}$ solves the following equation, for all k:

$$(2.5) \qquad \bar{H}^{(1)}(x + \rho_k) - \bar{H}^{(1)}(x) = -\Pi_{J^{(0)}} R_k^{(0)} + \hat{R}_{k,0}^{(0)},$$

where $\Pi_{J^{(0)}}$ denotes the projection to Fourier modes with $|n| < J^{(0)}$, and the constant $\hat{R}_{k,0}^{(0)}$ is the 0th Fourier coefficient of $R_k^{(0)}$.

Step 3: the iteration.

Further introduce, for $k = 1, \ldots, m$,

$$T_k^{(1)} = h^{(1)} T_k^{(0)} (h^{(1)})^{-1} = x + \rho_k + R_k^{(1)}, \quad \text{and } \mu_k^{(1)} = h_*^{(1)} \mu_k^{(0)},$$

where $R_k^{(1)}$ is defined as follows. Expanding the expression $h^{(1)} T_k^{(0)} = T_k^{(1)} h^{(1)}$, we get for all k:

$$x + \rho_k + R_k^{(0)} + \bar{H}^{(1)}(x + \rho_k + R_k^{(0)}) = x + \bar{H}^{(1)}(x) + \rho_k + R_k^{(1)} \circ h^{(1)}.$$

Comparing with (2.5), we obtain for all k:
(2.6)
$$R_k^{(1)} = (\bar{H}^{(1)}(x + \rho_k + R_k^{(0)}) - \bar{H}^{(1)}(x + \rho_k)) \circ (h^{(1)})^{-1} + (R_k^{(0)} - \Pi_{J^{(0)}} R_k^{(0)}) \circ (h^{(1)})^{-1} + \hat{R}_{k,0}^{(0)}.$$

Since the conjugation by $h^{(1)}$ does not change the rotation vector, we have

$$\rho_k = \rho_{\mu_k}(T_k) = \rho_{\mu_k^{(0)}}(T_k^{(0)}) = \int_{\mathbb{T}^N} \tilde{T}^{(0)} x - x \, d\mu_k^{(0)} = \int_{\mathbb{T}^N} \tilde{T}^{(1)} x - x \, d\mu_k^{(1)} = \rho_{\mu_k^{(1)}}(T_k^{(1)});$$

from the equation $\rho_k = \int \tilde{T}_k^{(1)}(x) - x \, d\mu_k^{(1)}$, we get $\int R_k^{(1)} \, d\mu_k^{(1)} = 0$, so that the jth component $j = 1, \ldots, N$ of $R_k^{(1)}$ vanishes at some point x_j. We see from (2.6) that $\hat{R}_{k,0}^{(0)}$ is bounded by the C^0 norm of the first two terms on the RHS. The remainder $R_k^{(1)}$ consists of the quadratically small error discarded when deriving (2.1), as well as the higher Fourier modes with $|n| \geq J^{(0)}$ in $R_k^{(0)}$. We thus obtain from (2.6) that

$$(2.7) \qquad \|R_k^{(1)}\|_{C^1} \leq C\|\bar{H}^{(1)}\|_{C^2}\|R_k^{(0)}\|_{C^1} + C\|(R_k^{(0)} - \Pi_{J^{(0)}} R_k^{(0)})\|_{C^1}.$$

We set $\varepsilon^{(1)} = \max_k \|R_k^{(1)}\|_{C^1}$.

The standard KAM method in [M] then applies by repeating the above procedure for infinitely many steps, during which we shall let $J^{(n)} \to \infty$, $\varepsilon^{(n)} \to 0$. The loss of derivative in (2.7) is handled in the standard way using the quadratic smallness on the RHS of (2.7). Higher order derivative estimates are obtained by interpolation between the C^1 estimate in (2.7) and $C^{\ell+\tau}$ estimate due to the Fourier cut-off for some large ℓ. In the limit, we get the conjugacy h in the statement. Since a collection of translations satisfying the simultaneous Diophantine condition is uniquely ergodic on the torus, we get the common invariant measure μ must equal $h_*^{-1}\mathrm{Leb}$.

We next prove the final statement on the rotation vectors. On the one hand, we have by the conjugacy that $\rho_{\mu_k}(T_k) = \rho_{h_*\mu_k}(\bar{T}_k)$ and $\rho_{\mathrm{Leb}}(\bar{T}_k) = \rho_\mu(T_k)$. On the other hand, since we have $\bar{T}_k(x) = x + \rho_{\mu_k}(T_k)$, we get $\rho_{\mathrm{Leb}}(\bar{T}_k) = \rho_{\mu_k}(T_k)$ by definition of rotation vectors. This completes the proof.

$\square$

Now we are ready to prove local rigidity.

Proof of Theorem 1.4 (local rigidity). Let $T_i = \alpha(g_i)$ and let

$$\bar{T}_i = \bar{\alpha}(\bar{A}, \boldsymbol{\rho})(g_i) : x \mapsto x + \rho_{\mu_i}(T_i), \;\; i = 1, \ldots, N.$$

Using the commutativity of the T_j and the simultaneous Diophantine condition, we apply Proposition 2.1 to construct h that simultaneously conjugates T_i to $\bar{T}_i$:

$$h \circ T_i = \bar{T}_i \circ h, \;\; i = 1, \ldots, N.$$

We then compose with h^{-1} on the right and h on the left on both sides of the group relation $AT_i = (\prod_{j=1}^N T_j^{b_{ji}})A$ to get

$$hAh^{-1}\bar{T}_i = (\prod_{j=1}^N \bar{T}_j^{b_{ji}})hAh^{-1};$$

in other words,

$$(2.8) \qquad hAh^{-1}(x + \rho_j) = hAh^{-1}(x) + \sum_{j=1}^{N} b_{ji}\rho_j, \text{ mod } \mathbb{Z}^N, \quad i = 1, \ldots, N.$$

We introduce the function $F(x) = hAh^{-1}(x) - \bar{A}x$ defined from $\mathbb{T}^n$ to $\mathbb{T}^n$. We can choose a homotopy connecting h to the identity under which F is homotopic to $A - \bar{A}$. Since A is homotopic to $\bar{A}$, the image of $A - \bar{A}$ is homotopic to a point. Therefore we can treat F as a continuous function from $\mathbb{T}^n$ to $\mathbb{R}^n$. Combined with (1.3), equation (2.8) then gives $F(x + \rho_i) = F(x)$, mod $\mathbb{Z}^n$, $i = 1, \ldots, N$. Continuity of F implies that $F(x + \rho_i) - F(x)$ is a constant integer vector. We may choose $n \in \mathbb{Z}$ such that $n\rho_i$ mod $\mathbb{Z}^n$ is arbitrarily close to zero, and so by the continuity of F, this constant integer vector has to be zero. We thus obtain that $F(x + \rho_i) = F(x)$. The Diophantine property of the vectors $\rho_1, \ldots, \rho_N$ implies that the action generated by the $\bar{T}_i$ on $\mathbb{T}^N$ is ergodic with respect to Leb. Since the function $F(x)$ is invariant, there is a vector $F_0 \in \mathbb{R}^N$ such that $F(x) = hAh^{-1}(x) - \bar{A}x = F_0$ almost everywhere (but in fact everywhere, since F is continuous).

To kill this constant vector F_0, we introduce the translation $t(x) = x + (\mathrm{id} - \bar{A})^{-1}F_0$. It is easy to check that t conjugates $\bar{A}x + F_0$ and $\bar{A}x$, i.e. $\bar{A}t(x) + F_0 = t(\bar{A}x)$. Composing the above h with t, we get the conjugacy in the statement of the theorem. $\qquad \square$

3. The existence of the common conjugacy

In this section, we prove Theorem 1.7. We will use the following result of Franks [Fr].

Theorem 3.1. *If $A : \mathbb{T}^N \to \mathbb{T}^N$ is an Anosov diffeomorphism, then A is topologically conjugate to a hyperbolic toral automorphism induced by $A_* : H_1(\mathbb{T}^N, \mathbb{Z}) \to H_1(\mathbb{T}^N, \mathbb{Z})$.*

This result has been generalized to the infranilmanifold case by Manning. It is also known ([KH] Theorem 19.1.2) that the conjugacy h is bi-Hölder; i.e. both h and h^{-1} are Hölder continuous.

Proof of Theorem 1.7. Suppose we are given an action $\alpha : \Gamma_{\bar{B}} \to \mathrm{Diff}^r(\mathbb{T}^N)$ such that $\alpha(g_0) = A$ is Anosov and homotopic to $\bar{A}$, and $\alpha(g_i) = T_i$, $i = 1, \ldots, N$ has c-slow oscillation, where c satisfies (1.9). For $p = (p_1, \ldots, p_n) \in \mathbb{Z}^N$, we use the notation T^p to denote $\prod_{i=1}^{N} T_i^{p_i}$. By Theorem 3.1, there is a homeomorphism h such that $hAh^{-1} = \bar{A}$. Let $R_i := hT_ih^{-1}$, for $i = 1, \ldots, N$. We will show that $R_i(x) = x + \rho_i$ where ρ_i is the rotation vector of T_i. We lift h to $\tilde{h} : \mathbb{R}^N \to \mathbb{R}^N$ and decompose

$$\tilde{h}(x) = x + g(x), \quad \tilde{h}^{-1}(x) = x + g_-(x), \quad \tilde{T}^p(x) = x + \Delta T_p(x),$$

for $p \in \mathbb{Z}^N$ where $g(x)$, $g_-(x)$ and $\Delta T_p(x)$ are $\mathbb{Z}^N$-periodic.

For $p \in \mathbb{Z}^N$ and $t \in \mathbb{T}^N$, we have

$$
\begin{aligned}
\tilde{R}^p(x) &= \tilde{h}\tilde{T}^p\tilde{h}^{-1}(x) \\
&= \tilde{T}^p\tilde{h}^{-1}(x) + g(\tilde{T}^p\tilde{h}^{-1}(x)) \\
&= \tilde{h}^{-1}(x) + \Delta T_p(\tilde{h}^{-1}(x)) + g(\tilde{T}^p\tilde{h}^{-1}(x)) \\
&= x + \Delta T_p(\tilde{h}^{-1}(x)) + g_-(x) + g(\tilde{T}^p\tilde{h}^{-1}(x)).
\end{aligned}
$$

Since both g_- and g are uniformly bounded, it follows that if $\{T^p\}$ has c-slow oscillation, then so does $\{R^p\}$. From the group relation, we obtain for $p \in \mathbb{Z}^N$ and $n \in \mathbb{Z}$,

$$
(3.1) \quad \bar{A}^n\tilde{R}^p(x) = \tilde{R}^{(\bar{B}^t)^n p}\bar{A}^n(x) + Q_{p,n}, \quad \bar{A}^n(\tilde{R}^p(x) - x) = (\tilde{R}^{(\bar{B}^t)^n p} - \mathrm{id})\bar{A}^n(x) + Q_{p,n},
$$

where $Q_{p,n}$ is an integer vector in $\mathbb{Z}^N$ depending on p, n and the choice of the lifts.

For each $\tilde{R}^p$, $p \in \mathbb{Z}^N$, we take the Fourier expansion $\tilde{R}^p(x) - x = \sum_{k \in \mathbb{Z}^N} \hat{R}_k(p)e^{2\pi i \langle k, x \rangle}$, where the coefficient for $k \neq 0$ is

$$
\hat{R}_k(p) = \int_{\mathbb{T}^N} (\tilde{R}^p(x) - x)e^{-2\pi i \langle k, x \rangle} \, dx.
$$

The condition that $\{R^p\}$ has c-slow oscillation implies that there exist C, P such that when $\|p\| \geq P$, we have $\|\hat{R}_k(p)\| \leq C\|p\|^c$, uniformly for all $k \neq 0$. From equation (3.1) we obtain that for all $k \in \mathbb{Z}^N \setminus \{0\}$,

$$
(3.2) \quad \hat{R}_k(p) = \bar{A}^{-n}\hat{R}_{(\bar{A}^t)^{-n}k}((\bar{B}^t)^n p).
$$

We next consider the splitting of $\mathbb{R}^N$ into $\bar{\mathcal{W}}^u(0) \oplus \bar{\mathcal{W}}^s(0)$, the direct sum decomposition into unstable and stable eigenspaces of $\bar{A}$. Each $\hat{R}_k(p)$ is a vector, so we write $\hat{R}_k(p) = (\hat{R}_k(p))^u + (\hat{R}_k(p))^s$ where $(\hat{R}_k(p))^{u,s} \in \bar{\mathcal{W}}^{u,s}(0)$. Applying $\bar{A}^n$ we get $\bar{A}^n\hat{R}_k(p) = \bar{A}^n(\hat{R}_k(p))^u + \bar{A}^n(\hat{R}_k(p))^s$ with the estimate $\|\bar{A}^n(\hat{R}_k(p))^u\| \geq |\lambda_1^u|^n\|(\hat{R}_k(p))^u\|$. Doing this decomposition to the equation (3.2), we obtain the following estimate for $\|(\bar{B}^t)^n p\| \geq P$:

$$
\|(\hat{R}_k(p))^u\| \leq \frac{1}{|\lambda_1^u|^n}\|\hat{R}_{(\bar{A}^t)^{-n}k}((\bar{B}^t)^n p)\| \leq C\frac{\|(\bar{B}^t)^n p\|^c}{|\lambda_1^u|^n} \leq C\|p\|^c \left(\frac{|\mu_{k'}^u|^c}{|\lambda_1^u|}\right)^n \to 0
$$

as $n \to \infty$, if $c < \frac{\ln|\lambda_1^u|}{\ln|\mu_{k'}^u|}$. Similarly, letting $n \to -\infty$ and projecting to the $\bar{\mathcal{W}}^s(0)$ in the above argument, we get that the projection of $\hat{R}_k(p)$ to $\bar{\mathcal{W}}^s(0)$ is also 0. Therefore $\hat{R}_k(p) = 0$ for all $k \neq 0$. This implies that each $R^p(x) - x$, $p \in \mathbb{Z}^N$, is a constant. Since a conjugacy does not change the rotation vector, we have $R_i(x) = x + \rho_i$, where ρ_i the rotation vector of T_i, $i = 1, \ldots, N$. Next we have $R^p(x) = x + \boldsymbol{\rho}p$, $p \in \mathbb{Z}^N$. This completes the proof. $\qquad\square$

4. Preliminaries: elliptic and hyperbolic dynamics

In this section, we explain and develop techniques from elliptic and hyperbolic dynamics that we will use to prove our main results. We first introduce the framework of Herman-Yoccoz-Katznelson-Ornstein for obtaining regularity of the conjugacy of circle maps and generalize it to abelian group actions on $\mathbb{T}^N$. Next, we state facts about Anosov diffeomorphisms, including the invariant foliation structure and its regularity properties.

4.1. **Elliptic dynamics: the framework of Herman-Yoccoz-Katznelson-Ornstein.**
In this section, we generalize to abelian group actions the framework of Herman-Yoccoz theory for circle maps after Katznelson-Ornstein.

Definition 4.1. *Let $\mathcal{F}$ be a continuous foliation of $\mathbb{T}^N$ by one-dimensional uniformly C^1 leaves $\mathcal{F}(x)$, $x \in \mathbb{T}^N$, and let $k \geq 1$.*

   (1) *We denote by $\mathcal{H}^k = \mathcal{H}^k(\mathbb{T}^N)$ the group of C^k diffeomorphisms on $\mathbb{T}^N$, $k \in \mathbb{N}$.*
   (2) *We denote by $\mathcal{H}^k_{\mathcal{F}}$ the subgroup of diffeomorphisms in $\mathcal{H}^k$ preserving the foliation $\mathcal{F}$; i.e., $f\mathcal{F}(x) = \mathcal{F}(f(x))$, $\forall\ f \in \mathcal{H}^k_{\mathcal{F}}$ and $\forall\ x \in \mathbb{T}^N$.*
   (3) *The C^k norm $\|\cdot\|_{C^k(\mathcal{F})}$ on $C^k(\mathbb{T}^N, \mathbb{R}^N)$ along the foliation $\mathcal{F}$ is defined as follows. For $\varphi \in C^k(\mathbb{T}^N, \mathbb{R}^N)$, let*

$$\|\varphi\|_{C^k(\mathcal{F})} := \sum_{i=0}^{k} \sup_x \|(D_x^i\,(\varphi|_{\mathcal{F}})\,\|,$$

   where the norm inside the summand on the right hand side is the operator norm induced by the Euclidean metric restricted to the leaves of $\mathcal{F}$.

4.1.1. *Generalization of the framework of Herman after Katznelson-Ornstein.* The following statement about circle maps was known to Herman [H]:

Suppose that $f \in \mathcal{H}^k(\mathbb{T}^1)$ takes the form $f = h^{-1}(h(x) + \rho)$, where $h \colon \mathbb{T}^1 \to \mathbb{T}^1$ is a homeomorphism and $\rho \notin \mathbb{Q}$. Then $h \in \mathcal{H}^k$ if and only if the iterates $\{f^j\}_{j \in \mathbb{Z}}$ are uniformly bounded in $\mathcal{H}^k$.

Following Katznelson-Ornstein [KO], we generalize this statement to abelian subgroup actions.

Definition 4.2. *A collection of m vectors $\rho_1, \ldots, \rho_m \in \mathbb{T}^N$ is said to* rationally generate $\mathbb{T}^N$ *if $\{\sum_{i=1}^m p_i\rho_i,\ p_i \in \mathbb{Z},\ i = 1, \ldots, m\}$ is dense on $\mathbb{T}^N$.*

Given commuting diffeomorphisms $T_1, \ldots, T_m$ on $\mathbb{T}^N$ and $p = (p_1, \ldots, p_m) \in \mathbb{Z}^m$, we will use the abbreviation $T^p = \prod_{i=1}^m T_i^{p_i}$.

Proposition 4.1. *Suppose that for some $k > 0$, the maps $T_i \in \mathcal{H}^k(\mathbb{T}^N)$, $i = 1, \ldots, m$, commute. Suppose also that there exists a homeomorphism $h: \mathbb{T}^N \to \mathbb{T}^N$ such that $T_i = h^{-1}\bar{T}_i h$, where $\bar{T}_i(x) = x + \rho_i, \mathrm{mod}\, \mathbb{Z}^N$, $i = 1, 2, \ldots, m$, and $\rho_1, \ldots, \rho_m$ rationally generate $\mathbb{T}^N$. Fix a lift $\tilde{h}: \mathbb{T}^N \to \mathbb{R}^N$ of h.*

Then the following equality holds for all x:

$$(4.1) \qquad \tilde{h}(x) = \mathrm{const.} + \lim_{n \to \infty} \frac{1}{(2n+1)^N} \sum_{\|p\|_{\ell^\infty} \leq n} \left(\tilde{T}^p(x) - \boldsymbol{\rho}p \right).$$

where $p = (p_1, \ldots, p_m) \in \mathbb{Z}^m$, $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_m) \in \mathbb{T}^{N \times m}$ and $\tilde{T}^p$ is the lift of T^p satisfying $\tilde{T}^p(0) = \tilde{h}^{-1}(\tilde{h}(0) + \boldsymbol{\rho}p)$.

Proof of Proposition 4.1. From $T^p = h^{-1}\bar{T}^p h$, we get $T^p = h^{-1}(h(x) + \boldsymbol{\rho}p)$. We next fix the lift $\tilde{T}^p$ of T^p that satisfies $\tilde{T}^p(0) = \tilde{h}^{-1}(\tilde{h}(0) + \boldsymbol{\rho}p)$ to obtain

$$\tilde{T}^p(x) - \boldsymbol{\rho}p - \tilde{h}(x) = (\tilde{h}^{-1} - \mathrm{id}) \circ (\tilde{h}(x) + \boldsymbol{\rho}p).$$

Averaging over all $p \in \mathbb{Z}^m$ with $\|p\|_{\ell^\infty} \leq n$, and letting $n \to \infty$, we get

$$\tilde{h}(x) = -\int_{\mathbb{T}^N} (\tilde{h}^{-1}(x) - x)dx + \lim_{n \to \infty} \frac{1}{(2n+1)^N} \sum_{\|p\|_{\ell^\infty} \leq n} \left(\tilde{T}^p(x) - \boldsymbol{\rho}p \right),$$

where to get the integral, we use the fact that the affine action of $\mathbb{Z}^m$ via the rigid translations $\bar{T}_i$, $i = 1, \ldots, m$ is ergodic with respect to Lebesgue, combined with a version of the Birkhoff ergodic theorem for abelian group actions (c.f. Theorem 1.1. of [L]). $\qquad \square$

Corollary 4.2. *Let the abelian group $\mathcal{A} = \{T^p: p \in \mathbb{Z}^m\}$ $(< \mathcal{H}^k(\mathbb{T}^N))$ and the conjugacy h be as in Proposition 4.1.*

  (1) *Let $\bar{\mathcal{F}} = \{\bar{\mathcal{F}}(x), x \in \mathbb{T}^N\}$ be an affine foliation of $\mathbb{T}^N$ by parallel lines. Let $\mathcal{F}$ be the (topological) foliation of $\mathbb{T}^N$ whose leaves are $\mathcal{F}(x) = h^{-1}(\bar{\mathcal{F}}(h(x)))$, $x \in \mathbb{T}^N$.*
  (2) *Assume the leaves $\mathcal{F}(x)$ of the foliation $\mathcal{F}$ are uniformly C^1. Note that this implies that $\mathcal{A} < \mathcal{H}_{\mathcal{F}}^k(\mathbb{T}^N)$.*

If the set $\{T^p(x) - x: p \in \mathbb{Z}^m\} \subset C^k(\mathbb{T}^N, \mathbb{R}^N)$ is precompact in the $\|\cdot\|_{C^k(\mathcal{F})}$ norm, then h is uniformly C^k along the leaves of $\mathcal{F}$. Moreover, in the case of $k = 1$, we also have that h^{-1} is uniformly C^1 along the leaves of $\bar{\mathcal{F}}$.

The proof of Corollary 4.2 is given in Section 4.1.2.

Given a continuous increasing function $\psi(x): \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\psi(0) = 0$, we say that a function $f: (X, d) \to (X', d')$ between two metric spaces has *modulus of continuity*

ψ at a point $x_0 \in X$, if there exists a constant $C > 0$ such that

$$d'(f(x_0), f(y)) \leq C\psi(d(x_0, y)),$$

for any $y \in X$ sufficiently close to x_0.

Proposition 4.3. *Let the abelian group $\mathcal{A}$, the conjugacy h, and the foliations $\mathcal{F}$, $\bar{\mathcal{F}}$ be as in Corollary 4.2. Assume that $\{T^p(x) - x \colon p \in \mathbb{Z}^m\} \subset C^1(\mathbb{T}^N, \mathbb{R}^N)$ is uniformly bounded in the $\|\cdot\|_{C^1(\mathcal{F})}$ norm and that the mapping*

$$\boldsymbol{\rho}p \ (\mathrm{mod}\ \mathbb{Z}^N) \mapsto \|T^p(x) - x\|_{C^1(\mathcal{F})}$$

has modulus of continuity ψ at $\boldsymbol{\rho}p = 0$. Then both $\|D\left(h|_{\mathcal{F}}\right)\|$ and $\|D\left(h^{-1}|_{\bar{\mathcal{F}}}\right)\|$ have modulus of continuity ψ with respect to the Euclidean metric.

The proof of Proposition 4.3 is given in Section 4.1.3.

4.1.2. *Proof of Corollary 4.2.* We only prove the case of $k = 1$. A similar argument gives the continuity of higher derivatives.

We denote the nth Birkhoff average on the right hand side of of (4.1) by $\tilde{S}_n$, so (4.1) can be rephrased as $\tilde{h} = \lim_{n \to \infty} \tilde{S}_n$ up to an additive constant. Since $\mathcal{A}$ is assumed to be pre-compact in $\mathcal{H}_{\mathcal{F}}^1$ and the pointwise convergence is given by (4.1), we have that $\{D\tilde{S}_n|_{\mathcal{F}} : n \geq 1\}$ is precompact in the C^0 operator norm, by Theorem 5.35 of [AB], which states that the convex hull of compact sets is compact in a completely metrizable locally convex space. This shows that h is differentiable along $\mathcal{F}$ and any subsequential limit of $\{D\tilde{S}_n|_{\mathcal{F}} : n \geq 1\}$ is $Dh|_{\mathcal{F}}$.

We have proved that $Dh|_{\mathcal{F}}$ is continuous. To show that $Dh^{-1}|_{\bar{\mathcal{F}}}$ is also continuous, by the implicit function theorem, it is enough to show that $\|D_x h|_{\mathcal{F}}\|$ is bounded away from zero. Fix $\varepsilon > 0$ such that $\|D_{x_0} h|_{\mathcal{F}}\| < \varepsilon$ for some x_0. Then the same inequality holds in a small neighborhood $B(x_0)$ of x_0. By the ergodicity of T^p, there exist finitely many p_i, $i = 1, \ldots, n$ such that $\cup_{i=1}^n T^{p_i}(B(x_0)) = \mathbb{T}^N$. This, combined with the equation

$$D_{T^p(x)} h|_{\mathcal{F}} \cdot D_x T^p|_{\mathcal{F}} = D_x h|_{\mathcal{F}},$$

implies that there exists a constant C independent of p such that

(4.2) $\|D_x h|_{\mathcal{F}}\| < C\varepsilon,$

for all $x \in \mathbb{T}^N$

Consider a leaf $\mathcal{F}(x)$ and $x' \in \mathcal{F}(x)$. We lift the leaf to the universal cover and consider the image of the segment between x and x' under $\tilde{h}$, i.e. the line segment between $\tilde{h}(x)$ and $\tilde{h}(x')$. Since $\tilde{h}(x) = x + g(x)$ where $g(x)$ is $\mathbb{Z}^N$ periodic, choosing x and x' far apart on $\mathcal{F}(x)$ we can make $\|\tilde{h}(x') - \tilde{h}(x)\| \geq 1$. Fix such a choice of x, x'. There is a C^1 curve $\gamma_{x,x'} \subset \mathcal{F}(x)$ connecting x to x' with length bounded by a constant

$C_{x,x'}$. The image $h(\gamma_{x,x'})$ is a C^1 curve connecting $h(x)$ and $h(x')$ with length larger than 1. Inequality (4.2) implies that

$$1 \le \|\tilde{h}(x) - \tilde{h}(x')\| \le \int_{\gamma_{x,x'}} \|Dh|_{\mathcal{F}}\| \le \varepsilon C_{x,x'}.$$

This implies that $\epsilon > C_{x,x'}^{-1}$, and so $\|Dh|_{\mathcal{F}}\|$ is uniformly bounded below, completing the proof. $\qquad\square$

4.1.3. *Proof of Proposition 4.3.* We first introduce a translation-invariant distance $d_{\mathcal{F}}^1$ on $\mathcal{A}$ that is equivalent to the C^1 norm as follows (c.f. [K]). Let B_1 be the set of $\phi \in C(\mathbb{T}^n, \mathbb{R})$ with $\|\phi\|_{C^1(\mathcal{F})} = 1$. For $f, g \in \mathcal{H}_{\mathcal{F}}^k$, we introduce $d_{\mathcal{F}}^1(f, g) := d_{\mathcal{F}}^1(fg^{-1}, \mathrm{id})$ and

$$d_{\mathcal{F}}^1(f) := \log(\max\{\Phi(f), \Phi(f^{-1})\}) + \sup_x \|f(x) - x\|,$$

where $\Phi(f) := \sup_{\phi \in B_1} \|\phi \circ f\|_{C^1(\mathcal{F})}$. To verify the triangle inequality, we note that
(4.3)

$$\Phi(fg) = \sup_{\phi \in B_1} \|\phi \circ (fg)\|_{C^1(\mathcal{F})} = \sup_{\phi \in B_1} \left\| \frac{1}{\|\phi \circ f\|_{C^1(\mathcal{F})}} \phi \circ (fg) \right\|_{C^1(\mathcal{F})} \cdot \|\phi \circ f\|_{C^1(\mathcal{F})}$$

$$\le \sup_{\psi \in B_1} \|\psi \circ g\|_{C^1(\mathcal{F})} \sup_{\phi \in B_1} \|\phi \circ f\|_{C^1(\mathcal{F})} = \Phi(f)\Phi(g).$$

The chain rule implies that $d_{\mathcal{F}}^1$ is equivalent to the C^1 distance $\sup_x \|f(x) - x\|_{C^1(\mathcal{F})}$.

By the assumption on the modulus of continuity ψ, the map from $\boldsymbol{\rho}p \in \mathbb{T}^N$ to $C^1(\mathcal{F})$ via $\boldsymbol{\rho}p \mapsto T^p$ is continuous in the C^1 norm at the point $\boldsymbol{\rho}p = 0$. By the translation invariance of the $d_{\mathcal{F}}^1$ norm, it is continuous at every point $\boldsymbol{\rho}p$ mod $\mathbb{Z}^N$. From the compactness of $\mathbb{T}^N$ we obtain that $\{T^p(x), \ p \in \mathbb{Z}^m\}$ is pre-compact in $\mathcal{H}_{\mathcal{F}}^1$. If then follows from Corollary 4.2 that the functions $Dh|_{\mathcal{F}}$ and $Dh^{-1}|_{\bar{\mathcal{F}}}$ are continuous. Differentiating the expression $T^p(x) = h^{-1}(h(x) + \boldsymbol{\rho}p)$ along the leaf $\mathcal{F}(x)$, we get that

$$D_x T^p|_{\mathcal{F}} - \mathrm{id}|_{\mathcal{F}} = \left(D_{(h(x)+\boldsymbol{\rho}p)} h^{-1}|_{\bar{\mathcal{F}}} - D_{h(x)} h^{-1}|_{\bar{\mathcal{F}}} \right) \cdot D_x h|_{\mathcal{F}}.$$

Since the LHS satisfies the modulus of continuity ψ by assumption, i.e.

$$\|D_x T^p|_{\mathcal{F}} - \mathrm{id}|_{\mathcal{F}}\|_{C^0} \le C\psi(\|\boldsymbol{\rho}p\|),$$

for all p with $\|\boldsymbol{\rho}p\|$ small, we get

$$\|D_{(h(x)+\boldsymbol{\rho}p)} h^{-1}|_{\bar{\mathcal{F}}} - D_{h(x)} h^{-1}|_{\bar{\mathcal{F}}}\|_{C^0} \le C\psi(\|\boldsymbol{\rho}p\|)(\min_x \|D_x h|_{\mathcal{F}}\|)^{-1}.$$

Hence $Dh^{-1}|_{\bar{\mathcal{F}}}$ has modulus of continuity ψ. To get the same modulus of continuity for $Dh|_{\mathcal{F}}$, we use $Dh|_{\mathcal{F}} \cdot Dh^{-1}|_{\bar{\mathcal{F}}} = \mathrm{id}|_{\bar{\mathcal{F}}}$. $\qquad\square$

4.2. **Hyperbolic dynamics: invariant foliations of Anosov diffeomorphisms.**
In this section, we recall some results from hyperbolic dynamics. Our statements
concern the the unstable objects $\mathcal{W}^u$ and E^u; the stable analogues also hold.

Definition 4.3. *A C^1 diffeomorphism $A \colon \mathbb{T}^N \to \mathbb{T}^N$ is a Anosov diffeomorphism with
simple Mather spectrum if there exists a DA-invariant splitting of the tangent space*

$$T_x\mathbb{T}^N = E^s_\ell(x) \oplus \ldots \oplus E^s_1(x) \oplus E^u_1(x) \oplus \ldots \oplus E^u_k(x), \quad k + \ell = N, \ k, \ell \geq 1$$

and numbers

$$\underline{\mu}^s_\ell \leq \bar{\mu}^s_\ell < \ldots < \underline{\mu}^s_1 \leq \bar{\mu}^s_1 < 1 < \underline{\mu}^u_1 \leq \bar{\mu}^u_1 < \ldots < \underline{\mu}^s_k \leq \bar{\mu}^s_k$$

such that for some constant $C > 1$,

$$\frac{1}{C}(\underline{\mu}^{u,s}_i)^n \leq \frac{\|DA^n v\|}{\|v\|} \leq C(\bar{\mu}^{u,s}_i)^n, \quad \forall v \in E^{u,s}_i \setminus \{0\},$$

where $i = 1, \ldots, \ell$ for s and $i = 1, \ldots, k$ for u.

The next result is classical (see [HPS]).

Proposition 4.4. *For any C^r, $r > 1$ Anosov diffeomorphism $A : \mathbb{T}^N \to \mathbb{T}^N$ with
simple Mather spectrum, the strong invariant distribution $E^u_{i\leq} := E^u_i \oplus \ldots \oplus E^u_k$ is
uniquely integrable, tangent to a foliation $\mathcal{W}^u_{i\leq}$ of $\mathbb{T}^N$ whose leaf $\mathcal{W}^u_{i\leq}(x)$ passing through
x is $C^r, x \in \mathbb{T}^N$. This gives rise to a flag of strong unstable foliations*

$$\mathcal{W}^u_k(x) \subset \mathcal{W}^u_{(k-1)\leq}(x) \subset \ldots \subset \mathcal{W}^u_{2\leq}(x) \subset \mathcal{W}^u_{1\leq}(x) := \mathcal{W}^u(x), \quad x \in \mathbb{T}^N,$$

*where each of the inclusions is proper and $\mathcal{W}^u_{i\leq}$ sub-foliates $\mathcal{W}^u_{(i-1)\leq}$ with C^r leaves for
$i = 2, \ldots, k$.*

It is known that simple Mather spectrum is an open property in the C^1 topology.
In particular, if $\bar{A}$ is a toral automorphism with simple real spectrum, then an Anosov
diffeormophism that is C^1 close to $\bar{A}$ has simple Mather spectrum.

Proposition 4.5 (Hölder regularity of the invariant distribution, Theorem 19.1.6 of
[KH]). *For each i, the distribution $E^u_i(x)$ is Hölder in the base point x.*

The Holder exponent depends only on the expansion and contraction rates $\bar{\mu}^{u,s}_i$ and
$\underline{\mu}^{u,s}_i$.

We denote the weak unstable bundles for A by $E^u_{\leq i} := E^u_1(x) \oplus E^u_2(x) \oplus \cdots \oplus E^u_i(x)$,
and that of $\bar{A}$ by $\bar{E}^u_{\leq i}(x) := \bar{E}^u_1(x) \oplus \bar{E}^u_2(x) \oplus \cdots \oplus \bar{E}^u_i(x)$, $i = 1, \ldots, k$, $x \in \mathbb{T}^N$. Denote
the unstable foliation of A by $\mathcal{W}^u$ and that of $\bar{A}$ by $\bar{\mathcal{W}}^u$.

Proposition 4.6 (Lemma 6.1-6.3 of [G1]). *Consider A a C^r Anosov diffeomorphism
that is C^1 close to a linear toral automorphism $\bar{A}$ with simple real spectrum, and the
bi-Hölder conjugacy h given by Theorem 3.1 with $h \circ A = \bar{A} \circ h$. Then*

(1) h preserves the unstable foliation: $h(\mathcal{W}^u(x)) = \bar{\mathcal{W}}^u(h(x))$, for all $x \in \mathbb{T}^N$;
(2) each weak unstable distribution $E^u_{\leq i}$ is uniquely integrable, tangent to a foliation $\mathcal{W}^u_{\leq i}$ of $\mathbb{T}^N$, whose leaf $\mathcal{W}^u_{\leq i}(x)$ passing through $x \in \mathbb{T}^N$ is C^{1+};
(3) each distribution $E^u_{i,j} := \bar{E}^u_{\geq i} \cap E^u_{\leq j}$, $i \leq j$, is uniquely integrable, tangent to a foliation with C^{1+} leaves;
(4) h preserves the weak unstable foliations: $h(\mathcal{W}^u_{\leq i}(x)) = \bar{\mathcal{W}}^u_{\leq i}(h(x))$, for $i = 1, \ldots, k$ and all $x \in \mathbb{T}^N$.

We remark that the item (1) does not require any C^1 closeness of A to $\bar{A}$, and it holds under the same assumption as Theorem 3.1.

From Proposition 4.6 we obtain a flag of weak foliations

$$\mathcal{W}^u_1 \subset \mathcal{W}^u_{\leq 2} \subset \ldots \subset \mathcal{W}^u_{\leq k-1} \subset \mathcal{W}^u_{\leq k} := \mathcal{W}^u,$$

where each of the inclusions is proper and $\mathcal{W}^u_{\leq(i-1)}$ sub-foliates $\mathcal{W}^u_{\leq i}$ with C^{1+} leaves for $i = 2, \ldots, k$. This flag is preserved by the conjugacy h.

When the weak distributions E^u_i are known to be uniquely integrable, we have the following proposition, which is proved by a standard graph transform technique.

Proposition 4.7. (1) Each weak unstable leaf $\mathcal{W}^u_{\leq i}$ in item (2) of Proposition 4.6 is subfoliated by $\mathcal{W}^u_i$, whose leaves are uniformly C^r.
(2) The weakest unstable leaf $\mathcal{W}^u_1(x)$ is C^{1+}, and its tangent distribution $E^u_1(x)$ is Hölder.

5. Elliptic regularity with the help of an invariant distribution

In this section, we show how to get regularity of the conjugacy h by combining elliptic dynamics within invariant distributions with hyperbolic dynamics.

5.1. Elliptic dynamics within invariant distributions.
We start with a few preparatory lemmas. Recall that we use the abbreviation $T^p = \prod_{i=1}^m T_i^{p_i}$, $p = (p_1, \ldots, p_m) \in \mathbb{Z}^m$ where T_i, $i = 1, \ldots, m$ are commuting diffeomorphisms.

Lemma 5.1. Let $\mathcal{F}$ be an orientable foliation of $\mathbb{T}^N$ with uniformly C^1 leaves of dimension one. Suppose a homeomorphism h conjugates the abelian group $\{T^p, \ p \in \mathbb{Z}^m\} < \mathrm{Diff}_0(\mathbb{T}^N)$ to translations $\{\bar{T}^p(x) = x + \boldsymbol{\rho}p, \ p \in \mathbb{Z}^m\}$ and sends the foliation $\mathcal{F}$ to an affine foliation $\bar{\mathcal{F}}$. Denote by $E(x)$ the one-dimensional distribution that is tangent to the leaf $\mathcal{F}(x)$. Then the distribution $E(x)$ is invariant under the DT^p; that is,

$$D_x T^p \left(E(x)\right) = E(T^p(x)),$$

for all $p \in \mathbb{Z}^m$ and $x \in \mathbb{T}^N$.

Proof. The straight line foliation $\bar{\mathcal{F}}$ is invariant under translations, so after the conjugation the foliation $\mathcal{F}$ is also invariant under $\{T^p, \ p \in \mathbb{Z}^m\}$. The lemma follows directly by differentiating the equation $T^p\mathcal{F}(x) = \mathcal{F}(T^p(x))$ along the leaves. $\square$

Lemma 5.2. *Suppose the conjugacy h in the previous Lemma 5.1 is bi-Hölder. Then for each $p \in \mathbb{Z}^m \setminus \{0\}$, all the Lyapunov exponents of T^p with respect to any invariant probability measure are zero.*

Proof. Suppose there is an invariant measure μ with at least one nonzero exponent. Without loss of generality, assume that this exponent is negative. Pesin theory implies that through μ-a.e. x, there are local stable manifolds, which are smoothly embedded disks on which T^n contracts distances at an exponential rate. Thus for two points y, y' on the same local stable manifold of a point x, we have $\|T^{np}(y) - T^{np}(y')\|$ converges to zero exponentially fast.

On the other hand, using the bi-Hölder conjugacy h we have

$$\|T^{np}(y) - T^{np}(y')\| = \|h^{-1}(n\boldsymbol{\rho}p + h(y)) - h^{-1}(n\boldsymbol{\rho}p + h(y'))\| \geq \text{const.}\|h(y) - h(y')\|^\eta.$$

where η is the Hölder exponent of h^{-1}, which gives a contradiction. $\square$

5.2. **A quantitative Kronecker theorem.** We will need the following number theoretic result, whose proof is postponed to the Appendix.

Theorem 5.3. *Let $N, K \in \mathbb{N}$ be given. Then there exists a full measure set $\mathcal{O}$ in the set $\mathcal{M}_{N \times K}(\mathbb{T})$ of matrices of $N \times K$ such that for all $M \in \mathcal{O}$, the following holds.*

For any small $\epsilon > 0$, there exists a constant C such that for any $y \in \mathbb{T}^N$ and any $n \in \mathbb{N}$ there exist $q \in \mathbb{Z}^N$, $p \in \mathbb{Z}^K$ satisfying $\|p\| < n$, such that the following inequality holds

$$\|Mp - q - y\| \leq Cn^{-\frac{K}{N}+\epsilon}.$$

This theorem is a quantitative version of the classical Kronecker approximation theorem. When $K = 1$, this is the classical Dirichlet's simultaneous Diophantine approximation theorem where we can set $\epsilon = 0$. The $N = 1$ case was proved in [K].

This theorem inspires the following definition.

Definition 5.1. *Suppose the m vectors $\rho_1, \ldots, \rho_m \in \mathbb{T}^N$ rationally generate $\mathbb{T}^N$, and consider the set of finite linear combinations*

$$(5.1) \qquad S := S(\rho_1, \ldots, \rho_m) = \left\{ \sum_{i=1}^m p_i\rho_i \bmod \mathbb{Z}^N \mid p_i \in \mathbb{Z}, \quad i = 1, \ldots, m \right\}.$$

For each element $\gamma \in S$, we denote by $\|\gamma\|_w$ the word length $\|\gamma\|_w := \|p\|_{\ell_1}$, where $p = (p_1, \ldots, p_m) \in \mathbb{Z}^m$ and by $\|\gamma\|$ the closest Euclidean distance of γ (mod $\mathbb{Z}^N$) to zero.

We say that S has dimension d *if there exists a constant c such that for any $x \in \mathbb{T}^N$ and any $\ell > 0$ there exists a point $\gamma \in S$ satisfying*

$$\|\gamma\|_w \leq \ell, \quad \|\gamma - x\| \leq c\ell^{-d}.$$

Theorem 5.3 implies that for almost every choice of vector tuple $\rho_1, \ldots, \rho_m \in \mathbb{T}^N$, the set S formed by linear combinations as above has dimension $m/N - \epsilon$ for all $\epsilon > 0$ small.

5.3. **Organization of the proofs of Theorem 1.8 and Theorem 1.9.** To prove Theorems 1.8 and 1.9, we just need to improve the regularity of the conjugacies obtained in Theorems 1.5 and 1.7, respectively. We carry this out in the following propositions.

The first proposition chooses the K_0 in Theorems 1.8 and 1.9.

Proposition 5.4. *Given $\eta \in (0, 1)$ and $d > 2/\eta^2$, there exists K_0 such that the following holds: for all $K > K_0$, there exists a full measure set $\mathcal{R}_{N,K} \subset (\mathbb{T}^N)^K$ such that the set S generated by any tuple of vectors $(\rho_1, \ldots, \rho_K)$ lying in $\mathcal{R}_{N,K}$ is dense on $\mathbb{T}^N$ and has dimension d.*

Proof of Proposition 5.4. To satisfy the inequality $2/d < \eta^2$, we choose $K_0 > 2N/\eta^2$. Applying Theorem 5.3, we get a full measure set in $(\mathbb{T}^N)^K$, $K > K_0$, each point of which generates a set S of dimension $d = K/N - \epsilon$ satisfying $2/d < \eta^2$, where ϵ is arbitrarily small. Next, removing further a zero measure set to guarantee that the vectors rationally generate $\mathbb{T}^N$, we get the full measure set $\mathcal{R}_{N,K}$ as claimed. $\square$

The next proposition gives the choice of η in Proposition 5.4, and will give the C^{1+} regularity of h along the one-dimensional leaves of a foliation after applying Corollary 4.2 and Proposition 4.3.

Proposition 5.5. *Suppose*

(1) *the abelian group $\mathcal{A}(< \mathcal{H}^r)$ is generated by*

$$\{T_{i,j} \mid i = 1, \ldots, N, \ j = 1, \ldots, K, \ T_{i,j}T_{i',j'} = T_{i',j'}T_{i,j}\};$$

(2) *there is an η-bi-Hölder conjugacy h such that $T_{i,j}(x) = h^{-1}(h(x) + \rho_{i,j})$;*

(3) *there is a $\{T_{i,j}\}$-invariant foliation $\mathcal{F}$ into one-dimensional C^1 leaves $\mathcal{F}(x)$ with tangential distributions $E(x)$, $x \in \mathbb{T}^N$ that is η-Hölder in x. Denote by $v(x)$ a unit vector field tangent to $\mathcal{F}(x)$, $x \in \mathbb{T}^N$;*

(4) *the set S generated by the rotation vectors $\rho_{i,j}$ has dimension d, with $2/d < \eta^2$.*

For $\gamma \in S$, we write

$$T_\gamma := \prod_{j=1}^{K} \prod_{i=1}^{N} T_{i,j}^{q_{i,j}},$$

where $q_{i,j} \in \mathbb{Z}$ are the coefficients in the linear combination of γ, i.e. $\gamma = \sum_j \sum_i q_{i,j} \rho_{i,j}$.

Then for all $\gamma \in S$ with $\|\gamma\|$ small enough, we have

$$\|\|D_x T_\gamma v(x)\|_{C^0} - 1| \le \mathrm{const.}\|\gamma\|^\nu$$

where $\nu \le \eta^2 - 2/d$.

We defer the proof to Section 5.4. We next cite the following well-known theorem of Journé.

Theorem 5.6 ([J]). *Suppose $\mathcal{F}^1$, $\mathcal{F}^2$ are two transverse continuous foliations a manifold M with uniformly $C^{n,\nu}$ leaves. Suppose that a continuous function $u : M \to \mathbb{R}$ is uniformly $C^{n,\nu}$ when restricted to each local leaf $\mathcal{F}^1_\varepsilon(x), \mathcal{F}^2_\varepsilon(x)$, $x \in M$. Then u is $C^{n,\nu}$ on M.*

In the 2-dimensional case, we apply Proposition 5.5 and Proposition 4.3 to get that h is C^{1+} along the stable and unstable foliations of the Anosov diffeomorphism A. Applying Theorem 5.6, we get that h is C^{1+} on $\mathbb{T}^2$.

An application of the next result completes the proof of Theorem 1.8. More details of the proof of Theorem 1.8 will be given in Section 6.1.

Theorem 5.7 ([LMM, Ll]). *Suppose f and g are two C^r, $r > 1$, Anosov diffeomorphims $\mathbb{T}^2$ that are topologically conjugated by h, i.e. $f \circ h = h \circ g$. Suppose the periodic data of f and g coincide, namely, $D_{h(x)}f^q$ is conjugate to $D_x g^q$ at every q-periodic point x of g for all $q \in \mathbb{Z}$. Then $h \in C^{r-\varepsilon}$ for ε arbitrarily small.*

The proof of Theorem 1.9 in the $N > 2$ case follows from the same general strategy. However, there is some more work needed to show that the conjugacy h sends the one-dimensional leaves $W_i^{u,s}$ to the straight lines parallel to the eigenvectors of $\bar{A}$. We will give the proof of the C^{1+} regularity of h in Section 6.2.

In dimension three, we get improved regularity (Corollary 1.10) by applying the following result of Gogolev in [G2].

Theorem 5.8 (Addendum 1.2 of [G2]). *Suppose $\bar{A} \in \mathrm{SL}(3, \mathbb{Z})$ has simple real spectrum and $A : \mathbb{T}^3 \to \mathbb{T}^3$ is C^r, $r > 3$ that is C^1 close to $\bar{A}$. Suppose also that $\bar{A}$ and A have the same periodic data, then there exists $h : \mathbb{T}^3 \to \mathbb{T}^3$ in $C^{r-3-\varepsilon}$ with $h \circ A = \bar{A} \circ h$. Furthermore there exists $\kappa \in \mathbb{Z}$, such that if $r \notin (\kappa, \kappa + 3)$, then $h \in C^{r-\varepsilon}$, where ε is arbitrarily small.*

5.4. **Elliptic regularity in the presence of invariant distributions.** In this section, we prove Proposition 5.5.

Proof of Proposition 5.5. Let T_γ and $v(x)$ be as in the statement, and let μ be any ergodic measure of T_γ. We get from Lemma 5.2 that for μ-a.e. x

$$(5.2) \qquad \lim_k \frac{1}{k} \log \|D_x (T_\gamma)^k(x)v(x)\| = \int \log \|D_x T_\gamma(x)v(x)\| \, d\mu = 0.$$

This shows that $\log \|D_x T_\gamma(x)v(x)\|$ vanishes at some point on $\mathbb{T}^N$.

To simplify notation, we reindex the $T_{i,j}$ appearing in T_γ by $T_1, \ldots, T_{\|\gamma\|_w}$, and write $T_\gamma = \prod_{i(\gamma)=1}^{\|\gamma\|_w} T_{i(\gamma)}$. (Due to the commutativity of the $T_{i,j}$'s, the ordering of the (i,j) appearing in $i(\gamma)$ does not matter). We also write $T_{i(\gamma)} x = x_{i(\gamma)}$.

Consider the η-Hölder function $\ell_{i(\gamma)}(x) := \log \|D_x T_{i(\gamma)+1} v(x)\|$. Invariance of the distribution E implies that

$$\log \|D_x T_\gamma v(x)\| = \log \left\| D_x \left(\prod_{i(\gamma)=1}^{\|\gamma\|_w} T_{i(\gamma)} \right) v(x) \right\| = \sum_{i(\gamma)=1}^{\|\gamma\|_w} \log \|D_{x_{i(\gamma)}} T_{i(\gamma)+1} v(x_{i(\gamma)})\|$$

$$= \sum_{i(\gamma)=1}^{\|\gamma\|_w} \ell_{i(\gamma)}(x_{i(\gamma)}).$$

To prove the lemma, it suffices to restrict attention to a neighborhood of $\gamma = 0$. We consider a dyadic decomposition of a small neighborhood of 0 by

$$D_m = \{\gamma \in S \mid c2^{-d(m+1)/2} < \|\gamma\| \le c2^{-dm/2}\},$$

where c is the constant in Definition 5.1. Next, for D_m, we introduce a $c2^{-dm}$-net by defining

$$S_m := \{\gamma \in D_m \mid \|\gamma\|_w \le 2^m\}.$$

The remaining proof is split into two steps. In the first step, we prove the following

Claim 1: *For any $\gamma \in S_m$, we have*

$$|\log \|D_x T_\gamma v(x)\|| \le \text{const.} \|\gamma\|^{\eta^2 - 2/d}, \ \forall \ x \in \mathbb{T}^N.$$

Proof of Claim 1. First, by (5.2), for any given $\gamma \in S_m$, there exists $y \in \mathbb{T}^N$ such that $\log \|D_y T_\gamma v(y)\| = 0$. Next, it follows from the definition of the dimension of the set S that there exists $\delta \in S$ with

$$\|\delta\|_w \le \|\gamma\|_w, \quad \|\delta + \bar{y} - \bar{x}\| \le c\|\gamma\|_w^{-d}, \ \bar{x} = h(x), \ \bar{y} = h(y).$$

We denote $y_\delta = T_\delta y$ and $y_\gamma = T_\gamma y$.

Since h is bi-Hölder, we have for all $i(\gamma) = 0, 1, \dots, \|\gamma\|_w - 1$ and $i(\delta) = 0, 1, \dots, \|\delta\|_w - 1$, the following estimates

$$\|x_{i(\gamma)} - (y_\delta)_{i(\gamma)}\| \leq \text{const.}\|\gamma\|_w^{-d\eta}, \text{ and } \|y_{i(\delta)} - (y_\gamma)_{i(\delta)}\| \leq \text{const.}\|\gamma\|^\eta.$$

Next we estimate $\log \|D_x T_\gamma v(x)\|$ as follows

$$
\begin{aligned}
&|\log \|D_x T_\gamma v(x)\||\\
&= |\log \|D_x T_\gamma v(x)\| - \log \|D_y(T_\gamma T_\delta)v(y)\| + \log \|D_y(T_\delta T_\gamma)v(y)\||\\
&= |\log \|D_x T_\gamma v(x)\| - \log \|D_{T_\delta y} T_\gamma v(T_\delta y)\|\\
&\quad - \log \|D_y T_\delta v(y)\| + \log \|D_{T_\gamma y} T_\delta v(T_\gamma y)\| + \log \|D_y T_\gamma v(y)\||\\
&= |\log \|D_x T_\gamma v(x)\| - \log \|D_{T_\delta y} T_\gamma v(y_\delta)\|\\
&\quad - \log \|D_y T_\delta v(y)\| + \log \|D_{T_\gamma y} T_\delta v(y_\gamma)\||
\end{aligned}
$$

$$
\begin{aligned}
(5.3)\quad &= \Big| \sum_{i(\gamma)=1}^{\|\gamma\|_w - 1} (\ell_{i(\gamma)}(x_{i(\gamma)}) - \ell_{i(\gamma)}((y_\delta)_{i(\gamma)})) + \sum_{i(\delta)=1}^{\|\delta\|_w - 1} (\ell_{i(\delta)}(y_{i(\delta)}) - \ell_{i(\delta)}((y_\gamma)_{i(\delta)})) \Big|\\
&\leq \text{const.}(\|\gamma\|_w \cdot \|\gamma\|_w^{-\eta^2 d} + \|\delta\|_w \|\gamma\|^{\eta^2})\\
&\leq \text{const.}(2^{m(1-\eta^2 d)} + 2^m \|\gamma\|^{\eta^2})\\
&\leq \text{const.}(\|\gamma\|^{2(\eta^2 - 1/d)} + \|\gamma\|^{\eta^2 - 2/d})\\
&\leq \text{const.}\|\gamma\|^{\eta^2 - 2/d}.
\end{aligned}
$$

$\square$

In the second step, we prove the following.

Claim 2: *Suppose for any $\gamma_m \in S_m$, we have $\|\log \|D_x T_{\gamma_m} v(x)\|\|_{C^0} \leq \text{const.}\|\gamma_m\|^\nu$, for some $\nu > 0$ and all x. Then for any $\gamma \in S$, we have $\|\log \|D_x T_\gamma v(x)\|\|_{C^0} \leq \text{const.}\|\gamma\|^\nu$.*

Proof of Claim 2. By the definition of D_m and S_m and Definition 5.1, we get that each annulus D_m in the dyadic decomposition is covered by at least $O(2^{dNm/2})$ balls of radius $c2^{-dm}$ centered at points in S_m.

We claim that for any $\gamma \in S$ with small norm $\|\gamma\|$, there exists a finite number $\kappa(\gamma)$ and $\{\gamma_{m_k}, \ k = 1, 2, \dots, \kappa(\gamma)\}$ satisfying $\gamma_{m_k} \in D_{m_k}$, and $m_{k+1} \geq 2m_k$ and $\gamma = \sum_{k=1}^{\kappa(\gamma)} \gamma_{m_k}$.

The algorithm is as follows. First find m such that $\gamma \in D_m$. Denote this m by m_1 and find $\gamma_{m_1} \in S_{m_1}$ that is closest to γ. The closest distance is bounded by $c2^{-dm_1}$. Next consider the vector $\gamma - \gamma_{m_1}$ and repeat the above procedure to it in place of γ. We see that $\gamma - \gamma_{m_1} \in D_{m_2}$ for some $m_2 \geq 2m_1$. This procedure terminates after finitely

many steps since $\gamma \in S$ is a finite integer linear combination of the rotation vectors ρ_i, $i = 1, \ldots, m$.

Next, let $x_{m_i} = \prod_{j=i}^{\kappa(\gamma)} T_{\gamma_{m_j}}(x)$ and $x_{m_{\kappa(\gamma)+1}} = x$. Then

$$| \log \|D_x T_\gamma v(x)\| | = | \log \|D_x \prod_i T_{\gamma_{m_i}} v(x)\| |$$

$$\leq \sum_i | \log \|D_{x_{m_{i+1}}} T_{\gamma_{m_i}} v(x_{m_{i+1}})\| |$$

$$\leq \text{const.} \sum_{i=1}^{\kappa(\gamma)} \|\gamma_{m_i}\|^\nu.$$

By the construction of D_m and S_m, for all $\gamma \in S$, we have that $\frac{1}{2}\|\gamma_{m_1}\| \leq \|\gamma\| \leq 2\|\gamma_{m_1}\|$, and $\|\gamma_{m_k}\|$ decays exponentially with uniform exponential rate. This gives that $| \log \|D_x T_\gamma v(x)\| | \leq \text{const.}\|\gamma\|^\nu$ for every $\gamma \in S$ close to zero. $\qquad\square$

This completes the proof of Proposition 5.5. $\qquad\square$

6. Proof of the Theorems

In this section, we prove Theorems 1.8 and 1.9.

6.1. **Proof of Theorem 1.8.**

Proof of Theorem 1.8. We first explain how to choose K_0 and the open set $\mathcal{O}$ in the statement of Theorem 1.8. We choose $\mathcal{O}$ to be a C^1 neighborhood of $\bar{A}$ in the set of Anosov diffeomorphisms with simple spectrum.

By Proposition 5.4, in order to determine K_0, it is enough to determine η. Given $\bar{A}$ and an Anosov diffeomorphsm $A \colon \mathbb{T}^2 \to \mathbb{T}^2$ homotopic to $\bar{A}$, Theorem 3.1 provides a a bi-Hölder map h such that $h \circ A = \bar{A} \circ h$. The Hölder regularity of the conjugacy h depends on both the spectrum of $\bar{A}$ and the Mather spectrum of A ([KH] Theorem 19.1.2), and the Hölder regularity of the invariant distribution $E_i(x)$ of the Anosov diffeomorphism A depends on the Mather spectrum of A. We choose η to be the minimum of these Hölder exponents.

For $K > K_0$, Proposition 5.4 supplies a full measure set $\mathcal{R}_{2,K}$ in $\mathbb{T}^{N \times K}$. Given $i \colon \{1, \ldots, K\} \to \{1, 2\}$, if the rotation vectors of $(\rho_{i(1),1}, \ldots, \rho_{i(K),K})$ lie in $\mathcal{R}_{2,K}$, then the set S generated by the set of all rotation vectors $\{\rho_{i,j}, \ i = 1, 2, \ j = 1, \ldots, K\}$ has dimension $d \in (K/2, K)$. For $K > K_0$, we have $2/d < \eta^2$ by Proposition 5.4. Moreover S is dense on $\mathbb{T}^N$.

Consider now an action $\alpha \colon \Gamma_{\bar{B},K} \to \mathrm{Diff}^r(\mathbb{T}^2)$ with $\alpha(g_0) = A \colon \mathbb{T}^2 \to \mathbb{T}^2$ Anosov, and $\alpha(g_{i,k})$, $i = 1,2$, $k = 1,\ldots,K$ generating an abelian subgroup action $(\mathbb{Z}^2)^K \to \mathrm{Diff}^r(\mathbb{T}^2)$. As in the hypotheses of the theorem, assume that the subgroup generated by $\alpha(g_{1,1})$ and $\alpha(g_{2,1})$ has sublinear oscillation. Then applying Theorem 1.5 to the $\Gamma_{\bar{A}}$ action generated by $\alpha(g_0)$, $\alpha(g_{1,1})$ and $\alpha(g_{2,1})$, we get a bi-Hölder map h linearizing the $\Gamma_{\bar{B}}$ action.

We show that the conjugacy h given by Theorem 1.5 also linearizes the whole $\Gamma_{\bar{B},K}$ action α. Indeed, for any diffeomorphism f that commutes with $\alpha(g_{1,1}), \alpha(g_{2,1})$, we have

$$hfh^{-1}(x + \rho_{i,1}) = hfh^{-1}(x) + \rho_{i,1},$$

for $i = 1,2$. Since the rotation vectors $\rho_{i,1}$, $i = 1,2$, rationally generate $\mathbb{T}^N$, by taking Fourier expansions, we get that hfh^{-1} is a rigid rotation by a constant vector that is the rotation vector of f. Thus h conjugates the whole action α to an affine action by rigid translations.

We next apply Proposition 5.5, Corollary 4.2 and Proposition 4.3 to get that the conjugacy h is C^{1+} along the stable and unstable leaves of the Anosov diffeomorphism A. By Theorem 5.6, we get that h is C^{1+} on $\mathbb{T}^2$ and finally by Theorem 5.7, we get that h is $C^{r-\varepsilon}$, for ε sufficiently small. $\qquad\square$

6.2. **Proof of Theorem 1.9, the N dimensional case.** The main difficulty in generalizing the above argument to the N-dimensional case is that it is in general unknown if the one dimensional distributions E_i^u (or E_i^s) that are invariant under DA are also invariant under DT_γ. It is only known that the weakest stable and unstable distributions E_1^u and E_1^s are invariant under DT_γ by Proposition 4.6 (4) and Lemma 5.1.

We cite the following Lemma from [GKS].

Lemma 6.1 (Proposition 2.4 of [GKS])**.** *Let A, $\bar{A}$ and h be as in Proposition 4.6. Suppose h is C^{1+} along $\mathcal{W}_{\leq i}^u$ and $h(\mathcal{W}_j^u(x)) = \bar{\mathcal{W}}_j^u(h(x))$, $1 \leq j \leq i$, then*

$$h(\mathcal{W}_{i+1}^u(x)) = \bar{\mathcal{W}}_{i+1}^u(h(x)), \ x \in \mathbb{T}^N.$$

Using this lemma, we now prove that $h \in C^{1+}$ in the general case $N > 2$.

Proof of Theorem 1.9. The proof follows the strategy of the proof of Theorem 1.8 with small modifications to deal with the high dimensionality.

We first choose K_0 and the open set $\mathcal{O}$ of Anosov diffeomorphisms. Since $\bar{A}$ is assumed to have simple spectrum, it has a C^1 small neighborhood in which the Anosov diffeomorphisms have simple Mather spectrum. We choose such a neighborhood and denote it by $\mathcal{O}$. We will choose K_0 to satisfy $2/d < \eta^2$ using Proposition 5.4, where d is the dimension of the set S generated by the rotation vectors $\rho_{i,j}$ and η is a lower

bound on the Hölder exponent of the conjugacy h and all the distributions $E_i^{u,s}$, for all Anosov diffeomorphisms in $\mathcal{O}$.

Proposition 5.4 then gives a full measure set $\mathcal{R}_{N,K} \subset \mathbb{T}^{N \times K}$. We obtain a bi-Hölder conjugacy h that linearizes the whole action $\alpha \colon \Gamma_{\bar{B},K} \to \mathrm{Diff}^r(\mathbb{T}^N)$ by applying Theorem 1.7 and the argument in the proof of Theorem 1.8.

It remains to improve the regularity of h to C^{1+}. To start, Proposition 4.6 (4) implies that weakest leaves are preserved: $h(\mathcal{W}_1^u(x)) = \bar{\mathcal{W}}_1^u(h(x))$, for all x. Next, we apply Lemma 5.1 to get that the weakest distribution E_1^u is invariant under the abelian group action generated by $\alpha(g_{i,k})$, $i = 1, \ldots, N$, $k = 1, \ldots, K$. Applying Proposition 5.5, Corollary 4.2 and Proposition 4.3, we conclude that h is C^{1+} along the weakest leaves $\mathcal{W}_1^u(x)$. Thus the assumption of the Lemma 6.1 is satisfied with $i = 1$, and we conclude that the second weakest leaves are preserved $h(\mathcal{W}_2^u(x)) = \bar{\mathcal{W}}_2^u(h(x))$. We next apply Lemma 5.1, Proposition 5.5, Corollary 4.2 and Proposition 4.3 to conclude that h is C^{1+} along $\mathcal{W}_2^u$. By Journé's theorem 5.6, we get that h is C^{1+} along the leaves $\mathcal{W}_{\leq 2}^u$.

Applying Lemma 6.1 inductively in i, we conclude h is C^{1+} along the unstable foliation $\mathcal{W}^u$. Similarly, we prove that h is C^{1+} along $\mathcal{W}^s$. Then by Journé's theorem 5.6, we have that $h \in C^{1+}$. $\qquad\square$

6.3. **Alternative assumptions.** In this section, we discuss possible alternative assumptions for Theorem 1.9. Our technique developed in Section 4 relies on the existence of foliations by one dimensional leaves that are invariant under the abelian group action. In our proofs, the foliations are provided by the Anosov diffeomorphism. The foliations being invariant under the abelian group action follows from the existence of a common conjugacy h. In other words, we need that the leaves (straight lines) of the invariant foliation of the toral automorphism $\bar{A}$ are mapped to the leaves of the invariant foliations of A by the conjugacy h^{-1} (Proposition 4.6). This is true when $N = 2$ or in higher dimensions when we assume that A is C^1 close to $\bar{A}$. There are also circumstances under which Proposition 4.6 can be proved without the C^1 smallness assuption. We mention here two main cases.

In [G1], the author considers an Anosov diffeomorphism A homotopic to a linear map $\bar{A}$ with simple Mather spectrum and the property that in each connected component of the Mather spectrum, there lies exactly one eigenvalue of $\bar{A}$. Moreover it is assumed that the invariant distributions $E_i^{u,s}$ form angles less than $\pi/2$ with the corresponding affine distributions $\bar{E}_i^{u,s}$ for the linear map $\bar{A}$. (This assumption guarantees a certain quasi-isometric property of $\mathcal{W}^s$ and $\mathcal{W}^u$). Under these assumptions, the conclusions of Proposition 4.6 hold [G1].

In [FPS], a similar result is shown assuming that A is isotopic to $\bar{A}$ along a path of Anosov diffeomorphisms with simple Mather spectrum.

Appendix A. Proof of Theorem 5.3

The proof was communicated to us by the user Fedja on MathOverflow `http://mathoverflow.net/questions/227817/a-quantitative-kronecker-theorem`.

We need only consider matrices $M = (m_{ij})$, $m_{i,j} \in \mathbb{T}$, $i = 1, \ldots, N$ and $j = 1, \ldots, K$. So $\mathcal{M}_{N \times K}(\mathbb{T})$ is identified with $\mathbb{T}^{N \times K}$ endowed with Lebesgue measure.

Fix a smooth function $\psi \in C^\infty(\mathbb{R})$ with $\mathrm{supp}\psi \subset (-1, 1)$, $\psi \geq 0$ and $\int \psi = 1$. Let ε be fixed. We next introduce $r_n = n^\varepsilon n^{-K/N}$, $n \in \mathbb{N}$, and put $\Psi_n(x) = r_n^{-N} \prod_{i=1}^{N} \psi(x_i/r_n)$ and consider the periodic function $\Phi_{n,y}(x) = \sum_{q \in \mathbb{Z}^N} \Psi_n(x - q - y)$ for each $y \in \mathbb{T}^N$. Then we claim that

Given $\varepsilon > 0$, there exists a $\delta > 0$ such that for each $n \in \mathbb{N}$, there exists a set $\mathcal{U}_n \subset \mathcal{M}_{N \times K}(\mathbb{T})$ with $Leb(\mathcal{U}_n) < n^{-\delta}$, and for each $M \notin \mathcal{U}_n$ and any $y \in \mathbb{T}^N$, there exists $p \in \mathbb{Z}^K$ with $\|p\| \leq n$ and $\Phi_{n,y}(Mp) \neq 0$.

Assuming the claim, considering $n = 2^\ell, \ell \in \mathbb{N}$ and using Borel-Cantelli, we get $Leb(\limsup_n \mathcal{U}_n) = 0$. This means that the probability for M lying in infinitely many $\mathcal{U}_n$ is zero. This completes the proof of the theorem.

It remains to prove the claim. Decompose $\Phi_{n,y}(x)$ into Fourier series $\Phi_{n,y}(x) = \sum_{k \in \mathbb{Z}^N} c_k(n,y) e^{2\pi i \langle k, x \rangle}$. Notice that for each x, there is only one $q \in \mathbb{Z}^N$ such that $x - q - y \in (-1, 1)^N$. It follows that $\|c_k(n,y)\| \leq 1$ for all $k \in \mathbb{Z}^N$ and $c_0(n,y) = 1$ is independent of n, y. Moreover, for each $\ell > 0$, there exists C_ℓ (depending only on ψ) such that $\|c_k(n,y)\| \leq C_\ell r_n^{-N-\ell}/\|k\|^\ell$ due to the C^∞ smoothness of ψ. Next for any matrix $M \in \mathcal{M}_{N \times K}(\mathbb{T})$ write

$$S_n(M,k) := \sum_{\|p\|_\infty \leq n} e^{2\pi i \langle k, Mp \rangle}, \quad \Lambda_{n,y}(M) := \sum_{\|p\|_\infty \leq n} \Phi_{n,y}(Mp) = \sum_{k \in \mathbb{Z}^N} c_k(n,y) S_n(M,k).$$

We get $S_n(M, 0) = n^K$ and for $\beta > 0$ to be determined later

$$\left| \sum_{\|k\| \geq r_n^{-\beta}} c_k(n,y) S_n(M,k) \right| \leq n^K C_\ell r_n^{\beta\ell - N - \ell}.$$

It remains to investigate the sum

$$\Gamma_{n,r_n}(M) := \sum_{0 < \|k\|_\infty \leq r_n^{-\beta}} |S_n(M,k)| = \sum_{0 < \|k\|_\infty \leq r_n^{-\beta}} \left| \sum_{\|p\|_\infty \leq n} e^{2\pi i \langle z, p \rangle} \right|, \quad z = M^t k.$$

We use the fact that $|\sum_{\|p\|_\infty \leq n} e^{2\pi i \langle z, p \rangle}| \leq C \prod_{j=1}^{K}(\min\{n, \|z_j\|^{-1}\})$, where $\|z_j\|$ is the distance from z_j to the nearest integer. Consider a map $F_k: \mathcal{M}_{N \times K}(\mathbb{T}) \to \mathbb{T}^K$ via $F_k(M) = M^t k$, mod $\mathbb{Z}^K$, then F_k pushes forward the Lebesgue measure on $\mathcal{M}_{N \times K}(\mathbb{T})$

to a Lebesgue measure on $\mathbb{T}^K$. We immediately get that

$$\int_{\mathcal{M}_{N \times K}(\mathbb{T})} |S_n(M, k)| \, d\mathrm{Leb} \leq \int_{\mathbb{T}^K} C \prod_{j=1}^K (\min\{n, \|z_j\|^{-1}\}) \, dz \leq C \log^K n,$$

so there exists a set $\mathcal{U}_n \subset \mathcal{M}_{N \times K}(\mathbb{T})$ with $\mathrm{Leb}(\mathcal{U}_n) \leq n^{-\delta}$ such that we have

$$\Gamma_{n, r_n}(M) \leq C n^\delta r_n^{-\beta N} \log^K n, \quad \forall \, M \in \mathcal{M}_{N \times K}(\mathbb{T}) \setminus \mathcal{U}_n.$$

Note that this set $\mathcal{U}_n$ is independent of y since $\Gamma_{n, r_n}(M)$ is. Now we get

$$|\Lambda_{n,y}(M)| \geq n^K - n^K C_\ell r_n^{\beta \ell - N - \ell} - C n^\delta r_n^{-\beta N} \log^K n, \quad \forall \, M \in \mathcal{M}_{N \times K}(\mathbb{T}) \setminus \mathcal{U}_n, \ \forall y \in \mathbb{T}^N.$$

We choose $r_n = n^\varepsilon n^{-K/N}$, and $\beta(> 1)$ and $\delta(> 0)$ sufficiently close to 1 and 0 respectively to satisfy the inequality $(\beta - 1)K + 2\delta < \beta N \varepsilon$ for given ε, and choose ℓ large enough to satisfy $(\beta - 1)\ell > N$. Hence $|\Lambda_{n,y}(M)| \geq \frac{1}{2} n^K$. This completes the proof of the claim hence the theorem. $\square$

Appendix B. Affine action and the simultaneous Diophantine condition

In this section, we discuss the assumption in Theorem 1.4 on $\bar{A}$ and $\boldsymbol{\rho}$. We show here how to ensure that assumption (1.3) and the Diophantine assumption $\boldsymbol{\rho}$ are satisfied simultanously.

Given $\bar{A}$, we solve equation (1.3) for $\boldsymbol{\rho}$. Lifting (1.3) to $\mathbb{R}^N$, we get the following equation

$$(B.1) \qquad\qquad \bar{A}\boldsymbol{\rho} = \boldsymbol{\rho}\bar{B} + \mathbf{P}, \quad \mathbf{P} \in \mathbb{Z}^{N \times N}.$$

As usual, we first set $\mathbf{P} = 0$ and consider the homogeneous equation.

The following facts can be found in [HJ], Theorem 4.4.14.

Proposition B.1. *Suppose $\bar{A}, \bar{B} \in \mathrm{SL}(n, \mathbb{Z})$.*

(1) *If the sets of spectrum of $\bar{A}$ and $\bar{B}$ do not intersect, then the homogeneous equation has zero solution and the inhomogeneous equation (B.1) is solvable with only rational solutions. In this case, the affine action can never be faithful.*

(2) *If the sets of spectrum of $\bar{A}$ and $\bar{B}$ do intersect and either $\bar{A}$ or $\bar{B}$ is diagonalizable over $\mathbb{C}$. Denote the common eigenvalues by $\lambda_1, \ldots, \lambda_k$, the eigenvector for $\bar{A}$ associated to λ_i by $a_{i,1}, \ldots, a_{i,n_i}$ and the eigenvector for $\bar{B}^t$ associated to λ_i by $b_{i,1}, \ldots, b_{i,m_i}$. Then the null space of $\boldsymbol{\rho} \mapsto \bar{A}\boldsymbol{\rho} - \boldsymbol{\rho}\bar{B}$ is the span of*

$$\{a_{i,j} \otimes b_{i,\ell}, \quad j = 1, \ldots, n_i, \ \ell = 1, \ldots, m_i, \quad i = 1, \ldots, k\}.$$

Proof. The first item follows from Theorem 4.4.6 of [HJ] using the properties of Kronecker product.

For the second statement, by Theorem 4.4.14 of [HJ], the null space of the map $\boldsymbol{\rho} \mapsto \bar{A}\boldsymbol{\rho} - \boldsymbol{\rho}\bar{B}$ has dimension $\sum_i n_i \times m_i$. It is clear that each matrix $a_{i,j} \otimes b_{i,\ell}$ where $j = 1, \ldots, n_i$ and $\ell = 1, \ldots, m_i$ lies in the kernal of $\boldsymbol{\rho} \mapsto \bar{A}\boldsymbol{\rho} - \boldsymbol{\rho}\bar{B}$, and these matrices are linearly independent, so we get the second statement. $\square$

In the 2D case, suppose $\mathrm{tr}\bar{A} = \mathrm{tr}\bar{B}$ and $|\mathrm{tr}\bar{A}| > 2$, then $\bar{A}$ and $\bar{B}$ share the same spectrum λ and $1/\lambda$ for some $|\lambda| > 1$. The zero space of $\boldsymbol{\rho} \mapsto \bar{A}\boldsymbol{\rho} - \boldsymbol{\rho}\bar{B}$ is then spanned by $u_{\bar{A}} \otimes u_{\bar{B}^t}$ and $u_{\bar{A}^{-1}} \otimes u_{(\bar{B}^t)^{-1}}$, where $u_{\bar{A}}$ is the eigenvector corresponding to the eigenvalue λ. Similarly for others. If $\mathrm{tr}\bar{A} \neq \mathrm{tr}\bar{B}$, we get that the zero space of $\boldsymbol{\rho} \mapsto \bar{A}\boldsymbol{\rho} - \boldsymbol{\rho}\bar{B}$ is zero.

If some of the $a_{i,j}$ is Diophantine, then the simultaneous Diophantine condition is satisfied automatically. We next focus on the special case of $\bar{A} = \bar{B}$, where the simultaneous Diophantine condition is more explicit. We recall a fact and definition from linear algebra:

Lemma B.2 (Corollary 4.4.15 of [HJ]). *Let $A \in M_N(\mathbb{R})$ where $M_N(\mathbb{R})$ is the set of $N \times N$ matrices with entries in $\mathbb{R}$. The set of matrices in $M_N(\mathbb{R})$ that commute with A is a subspace of $M_N(\mathbb{R})$ with dimension at least N. The dimension is equal to N if and only if M is non-derogatory, i.e. each eigenvalue of A has geometric multiplicity exactly 1. Thus if A is nonderogary, the centralizer $Z(A)$ of A is*

$$Z(A) = \mathrm{span}_{\mathbb{R}}\{\mathrm{id}, A, \ldots, A^{N-1}\}.$$

If $\bar{A}$ is non-derogatory, then for any $\boldsymbol{\rho}$ satisfying $\bar{A}\boldsymbol{\rho} = \boldsymbol{\rho}\bar{A}$, we can thus write each $\boldsymbol{\rho} \in Z(\bar{A})$ as a linear combination $\boldsymbol{\rho} = \sum_{i=1}^{N} a_i \bar{A}^{i-1}$, where $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$.

Lemma B.3. *Let $\boldsymbol{\rho} = \sum a_i \bar{A}^{i-1}$ for some $a = (a_1, \ldots, a_N)$ and $\bar{A}^{i-1} \in \mathrm{SL}(N, \mathbb{Z})$. Suppose the nonvanishing entries of a form a vector $a' \in \mathbb{R}^k$, $1 \leq k \leq N$ satisfying the Diophantine condition: there exist $C, \tau > 0$ such that*

$$|\langle a', m \rangle| \geq \frac{C}{|m|^{\tau}}, \quad \forall\, m \in \mathbb{Z}^k \setminus \{0\}.$$

Then the columns of $\boldsymbol{\rho}$, denoted by $\rho_1, \ldots, \rho_N$, satisfy the simultaneous Diophantine condition for some $C' > 0$, i.e.

(B.2) $$\max_{1 \leq j \leq N}\{|\langle n, \rho_j \rangle|\} \geq \frac{C'}{\|n\|^{\tau}}, \quad \forall\, n \in \mathbb{Z}^N \setminus \{0\}.$$

Proof. Denote by v_j^i is the j-th column of $\bar{A}^{i-1}$, $i, j = 1, \ldots, N$, and by ρ_j the j-th column of $\boldsymbol{\rho}$. Hence we have $\rho_j = \sum_i a_i v_j^i$.

Assume the vector formed by the non vanishing entries of a satisfies the Diophantine condition and denote by $\mathcal{I}$ the set of indices of the non vanishing entries of the vector

a, then we have for each j

$$|\langle n, \rho_j \rangle| = \left| \sum_{i=1}^{N} a_i \langle n, v_j^i \rangle \right| \geq \frac{C}{(\sum_{i \in \mathcal{I}} |\langle n, v_j^i \rangle|)^\tau}$$

(B.3)
$$= \frac{C}{\|n\|^\tau (\sum_{i \in \mathcal{I}} \left| \langle \frac{n}{\|n\|}, v_j^i \rangle \right|)^\tau}$$

$$\geq \frac{C}{\|n\|^\tau (\sum_{i \in \mathcal{I}} \|v_j^i\|)^\tau}$$

if $\langle n, v_j^i \rangle \neq 0$ for some $i \in \mathcal{I}$.

To show that the simultaneous Diophantine condition holds for $\rho_1, \ldots, \rho_N$, it remains to show that for each $u \in \mathbb{S}^{N-1}$, there exist $i \in \mathcal{I}$, $j \in \{1, 2, \ldots, N\}$, such that $\langle u, v_j^i \rangle \neq 0$. This follows from the non-degeneracy of $\bar{A}$. We fix any $i \in \mathcal{I}$, then v_j^i, $j = 1, 2, \ldots, N$, form the matrix $\bar{A}^{i-1}$ which is non-degenerate. Hence the vectors v_j^i, $j = 1, 2, \ldots, N$, are linearly independent. The compactness of $\mathbb{S}^{N-1}$ implies that there does not exists $u \in \mathbb{S}^{N-1}$ that is simultaneously orthogonal to all of v_j^i, $j = 1, 2, \ldots, N$. $\qquad \square$

Next, in order to solve the inhomogeneous equation $\bar{A}\boldsymbol{\rho} = \boldsymbol{\rho}\bar{A} + \mathbf{P}$, it is enough produce a particular solution for given $\mathbf{P} \in \mathbb{Z}^{N \times N}$ in addition to the general solutions to the homogeneous equation. Note that the (B.1) might not be solvable for some $\mathbf{P}$. We have the following result.

Theorem B.4 (Theorem 4.2.22 of [HJ]). *Given matrices $A, B, C \in M_N(\mathbb{R})$. Then there exists some $X \in M_N(\mathbb{R})$ solving the equation $AX - XB = C$ if and only if the matrices $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ are similar.*

So to solve (B.1), the necessary and sufficient condition is the similarity of the matrices $\begin{bmatrix} \bar{A} & \mathbf{P} \\ 0 & \bar{A} \end{bmatrix}$ and $\begin{bmatrix} \bar{A} & 0 \\ 0 & \bar{A} \end{bmatrix}$. Given a particular solution $\boldsymbol{\rho}_*(\mathbf{P})$ of (B.1). If $\bar{A}$ is nonderogatory, then the general solution of (B.1) can be written as

$$\boldsymbol{\rho} = \sum a_i \bar{A}^{i-1} + \boldsymbol{\rho}_*$$

for some $a = (a_1, \ldots, a_N)$, if $\boldsymbol{\rho}_*$ happens to be rational, then $\boldsymbol{\rho}$ is simultaneously Diophantine, if the nonvanishing entries of a form a Diophantine vector.

Appendix C. Affine actions and vanishing Lyapunov exponents

In this appendix, we prove the results in Section 1.1. Proposition 1.1 is verified straightforwardly from the group relation. We prove Proposition 1.2 and Proposition 1.3.

Proof of Proposition 1.2. The proof of $\Longrightarrow$ is easy. We only prove $\Longleftarrow$ here. Suppose the action is not faithful. Then there exist $\gamma_1, \gamma_2 \in \Gamma_{\bar{B}}$ with $\gamma_1 \neq \gamma_2$ but $\alpha(\gamma_1) = \alpha(\gamma_2)$. Using the group relation, we first rewrite γ_i in the form $\gamma_i = g_0^{m_i} g^{p_i}$, $i = 1, 2$, where $g^p = g_1^{p_1} \dots g_N^{p_N}$. We can deduce an equation of the form $\alpha(g_0)^m = \alpha(g^p)$ with $m = m_1 - m_2$ and $p = p_1 - p_2$ from $\alpha(\gamma_1) = \alpha(\gamma_2)$. We pick any rational point x on $\mathbb{T}^N$ and note that $\alpha(g_0)^m x$ is rational but $\alpha(g^p)x$ is irrational unless $p = 0$ by the linaer independence of $\boldsymbol{\rho}$. If $p = 0$, then $m = 0$ since $\bar{A}$ is not of finite order. This implies that $\gamma_1 = \gamma_2$. $\qquad \square$

Proof of Proposition 1.3. Suppose we have two affine actions $\bar{\alpha} = \bar{\alpha}(\bar{A}, \boldsymbol{\rho})$ and $\bar{\alpha}' = \bar{\alpha}(\bar{A}, \boldsymbol{\rho}')$ conjugate by a homeomorphism h of the form $h(x) = x + \tilde{h}(x)$, $x \in \mathbb{T}^N$ where $\tilde{h}$ is $\mathbb{Z}^N$-periodic. We want to show that $\boldsymbol{\rho} = \boldsymbol{\rho}'$. Denote by ρ_j and ρ_j' the j-th column of $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ respectively. We have

$$h(x + \rho_j) = h(x) + \rho_j', \quad j = 1, \dots, N.$$

This is equivalent to

$$\rho_j + \tilde{h}(x + \rho_j) = \tilde{h}(x) + \rho_j', \quad j = 1, \dots, N.$$

Integrating over $\mathbb{T}^N$, we get that $\int_{\mathbb{T}^N} \tilde{h}(x + \rho_j) \, dx = \int_{\mathbb{T}^N} \tilde{h}(x) \, dx$, hence $\rho_j = \rho_j'$. $\quad \square$

Proposition C.1. *Suppose $\bar{B} \in \mathrm{SL}(N, \mathbb{Z})$ has no eigenvalue 1. Then for any action of $\alpha : \Gamma_{\bar{B}} \to \mathrm{Diff}^r(\mathbb{T}^N)$, $r > 1$, all the Lyapunov exponents of $\alpha(g_i)$, $i = 1, 2 \dots, N$, are zero with respect to any invariant measure.*

Proof. We use the following Zimmer amenable reduction theorem. Fix a group action $\alpha \colon \Gamma \to \mathrm{Diff}(M)$, and let $\phi \colon (\alpha, M) \to \mathrm{GL}(N, \mathbb{R})$ be a cocycle, meaning that

$$\phi(\alpha(\gamma_1)\alpha(\gamma_2), x) = \phi(\alpha(\gamma_1), \alpha(\gamma_2)x)\phi(\alpha(\gamma_2), x),$$

for all $x \in M$ and $\gamma_i \in \Gamma$. We say that ϕ is cohomologous to another cocycle ψ if there exists a measurable map $h \colon M \to \mathrm{GL}(N, \mathbb{R})$ such that

$$\phi(\alpha(\gamma), x)h(x) = h(\alpha(\gamma)x)\psi(\alpha(\gamma), x), \quad \forall x \in M, \ \forall \gamma \in \Gamma.$$

Theorem C.2 (Theorem 1.8 of [HuK]). *Let $\alpha \colon \Gamma \to \mathrm{Diff}(M)$ be an amenable group action and $\phi \colon (\alpha, X) \to \mathrm{GL}(N, \mathbb{R})$ a cocycle. Then there exists a cocycle $\psi \colon (\alpha, M) \to \mathrm{GL}(N, \mathbb{R})$ that is cohomologous ϕ and such that there exists a partition of $X = \cup_{i=1}^{2^N} X_i$ and $\psi \colon (\alpha, X_i) \to H_i$, where H_i is one of the 2^n conjugacy classes of maximal amenable subgroups of $\mathrm{GL}(N, \mathbb{R})$ of the form $\begin{bmatrix} A_1 & * & \dots & * \\ 0 & A_2 & * & * \\ 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & 0 & A_k \end{bmatrix}$, each A_i is $n_i \times n_i$ with $\sum_{i=1}^k n_i = N$ and is of the form of a scalar times an orthogonal matrix.*

We next cite the following result on the Weyl chamber of the $\mathbb{Z}^N$ actions on a compact manifold.

Theorem C.3 (Proposition 2.1 of [FKS])**.** *Suppose μ is an ergodic measure for the action $\beta \colon \mathbb{Z}^N \to \mathrm{Diff}^r(M)$, $r > 1$. Then there are finitely many linear functionals $\chi \colon \mathbb{Z}^N \to \mathbb{R}$, a set $\mathcal{P}$ of full measure and a β-invariant measurable splitting of the tangent bundle $T_x M = \oplus E_\chi(x)$, $x \in \mathcal{P}$ such that for all $a \in \mathbb{Z}^N$ and $v \in E_\chi$, the Lyapunov exponent of v is*

$$\lim_{n \to \pm\infty} n^{-1} \log \|D\beta(a^n)(v)\| = \chi(a).$$

With the two results, we give the proof of the proposition. Without loss of generality, we assume μ is an ergodic measure for the action. A general invariant measure can be decomposed into averages of ergodic measures. From the group relation we obtain

$$AT^p = T^{\bar{B}^t p} A.$$

where we have $\alpha(\bar{B}) = A$, $\alpha(g_i) = T_i$ and $T^p = \prod_{i=1}^{N} T_i^{p_i}$, $p = (p_1, \ldots, p_N) \in \mathbb{Z}^N$.

Applying Theorem C.2 to the cocycle $D\alpha$, we get a measurable map $h \colon \mathbb{T}^N \to \mathrm{GL}(N, \mathbb{R})$ such that $D_x \hat{\alpha}(g)(x) := h(\alpha(g)x) D_x \alpha(g) h^{-1}$ is of the form H_i as in Theorem C.2. Thus

(C.1) $$D\hat{A}D\hat{T}^p = D\hat{T}^{\bar{B}^t p} D\hat{A},$$

Since h is only known to be measurable, we denote by Y the zero measure set of points where h is unbounded, and by $X := \mathbb{T}^N \setminus (\cup_{g \in \Gamma} \alpha(g)^{-1}(Y))$, which has full measure. For each $k \in \mathbb{N}$, we introduce the set $X_k := \{x \in X \mid \|h(x)\| \le k, \ \|h(x)^{-1}\| \le k\}$. By Poincaré recurrence, for μ-a.e. $x \in X_k$, the T^p- and $T^{\bar{B}^t p}$-orbits of x will return to X_k infinitely often. We pick such an $x_* \in X_k$ and we get that $A(x_*) \in X_{k'}$ for some k'. Next we apply Poincaré recurrence to both T^p and $T^{\bar{B}^t p}$ to obtain a subsequence $\{n_i\} \subset \mathbb{N}$ such that

$$T^{n_i p} x_* \in X_k, \quad T^{\bar{B}^t n_i p}(Ax_*) \in X_{k'}.$$

This implies $AT^{n_i p} x_* = T^{\bar{B}^t n_i p}(Ax_*) \in X_{k'}$. This gives the estimates

$$\|D\hat{A}(T^{n_i p} x_*)\| = \|h(AT^{n_i p} x_*) DA(T^{n_i p} x_*) h^{-1}(T^{n_i p} x_*)\| \le kk' \|DA\|_{C^0}.$$

Similarly, we estimate

$$\|D\hat{T}^{n_i p}(x_*)\| \le k^2 \|DT^{n_i p}\|_{C^0}, \quad \|D\hat{T}^{\bar{B}^t n_i p}(Ax_*)\| \le k'^2 \|DT^{\bar{B}^t n_i p}\|_{C^0}.$$

By Theorem C.2, since each $\hat{\alpha}(g)$, $g \in \Gamma_{\bar{B}}$ has the form of H_i, we consider only the diagonal blocks. Suppose $D\hat{A}(x)$ has diagonal blocks $\mathsf{a}_1(x), \ldots, \mathsf{a}_j(x)$, and $D\hat{T}_i(x)$ has diagonal blocks $\mathsf{t}_{i,1}(x), \ldots, \mathsf{t}_{i,j}(x)$, where a_ℓ and $\mathsf{t}_{i,\ell}$ are $n_\ell \times n_\ell$ and $\sum_{\ell=1}^{j} n_\ell = N$. Similarly, we denote the diagonal blocks of $D\hat{T}^p$ by $\{\mathsf{t}_\ell^p\}$. We further denote $\lambda(\mathsf{a}_\ell)$ and $\lambda(\mathsf{t}_\ell^p)$ the modulus of the scalar part of a_ℓ and t_ℓ^p respectively.

Equation (C.1) gives the following on the diagonal

$$\text{(C.2)} \qquad \mathsf{a}_\ell \mathsf{t}_\ell^{n_i p} = \mathsf{t}_\ell^{\bar{B}^t n_i p} \mathsf{a}_\ell, \quad \text{and} \quad \lambda(\mathsf{a}_\ell)\lambda(\mathsf{t}_\ell^{n_i p}) = \lambda(\mathsf{t}_\ell^{\bar{B}^t n_i p})\lambda(\mathsf{a}_\ell).$$

We take log and divide by n_i and let $n_i \to \infty$. Since $\lambda(a_j)$ is bounded by $kk'\|DA\|$, we have that $\lim \frac{1}{n} \log(a_j) \to 0$. Let $\mu_{i,j} := \lim_{n_i} \frac{\log \lambda(\mathsf{t}_{i,j}^{n_i p})}{n_i}$, whose existence is given by the ergodic theorem, and denote by M the matrix $(\mu_{i,j})$. We will show below that each row of M gives rise to a Lyapunov functional χ_j, and hence by Theorem C.3 and equation (C.2) we have $\chi_j(p) = \chi_j(\bar{B}^t p)$. Choosing p to be of the form $n(1, 0, \ldots, 0)$, $n(0, 1, 0, \ldots, 0), \ldots, n(0, \ldots, 0, 1)$, we get the following

$$M = M\bar{B}^t, \quad \text{i.e.} \quad M(\bar{B}^t - \text{Id}) = 0.$$

Since $\bar{B}$ does not have eigenvalue 1, the only solution is $M = 0$ so all the Lyapunov exponents $\mu_{i,j}$ are 0.

It remains to show that each row of M is a Lyapunov functional. We apply Theorem C.3 to the abelian group $\{T^p, \ p \in \mathbb{Z}^N\}$. For the linear functional χ and invariant splitting $\oplus E_\chi(x)$ of $\{T^p\}$, we get that the splitting $\oplus h(x)E_\chi(x)$ is invariant under $\{D\hat{T}^p\}$. So for each $v \in E_\chi(x)$, the Lyapunov exponent of T^p at point $x \in X$ along the vector v is given by $\lim \frac{1}{n} \log \|D_x T^{np} v\| = \chi(p)$ and for $h(x)v \in h(x)E_\chi(x)$, the Lyapunov exponent of $\hat{T}^p$ at the point x along the vector $h(x)v$ is also $\chi(p)$. This shows that DT^p and $D\hat{T}^p$ share the same Lyapunov functional. It remains to identify the Lyapunov exponents of each $D\hat{T}_j$ as $\{\mu_{i,j}\}$. Since $D\hat{T}^p$ has the form of H_i in Theorem C.2, we get that the invariant splitting can be constructed explicitly and inductively. We denote by $e_1, \ldots, e_n$ the standard basis vectors of $\mathbb{R}^N$. We first denote $\mathcal{E}_1 = \text{span}\{e_1, \ldots, e_{n_1}\}$. From the normal form in Theorem C.2, it is clear that $\mu_{i,1}$ is the Lyapunov exponent $\lim \frac{1}{n} \log \|D\hat{T}_i(x_*)v\|$ for any $v \in \mathcal{E}_1$. Therefore $\mathcal{E}_1$ is one summand in the splitting $\oplus hE_\chi$ and $h^{-1}\mathcal{E}_1$ is one summand in the splitting $\oplus E_\chi$. The second Lyapunov exponent $\mu_{i,2}$ is found by acting $D\hat{T}_i$ on the quotient $\mathbb{R}^N/\mathcal{E}_1 = (\oplus hE_\chi)/\mathcal{E}_1$, equivalently by acting DT_i on the quotient $\mathbb{R}^N/h^{-1}\mathcal{E}_1 = (\oplus E_\chi)/h^{-1}\mathcal{E}_1$. We denote by $\mathcal{E}_2 = \text{span}\{e_{n_1+1}, \ldots, e_{n_1+n_2}\}$. From the normal form in Theorem C.2, we see that $\mathcal{E}_2/\mathcal{E}_1$ is the invariant subspace for the action of $D\hat{T}_i$ on the quotient $\mathbb{R}^N/\mathcal{E}_1$. This implies that $\mathcal{E}_2/\mathcal{E}_1$ is one summand in the quotient splitting $\mathbb{R}^N/\mathcal{E}_1 = (\oplus hE_\chi)/\mathcal{E}_1$ and equivalently $h^{-1}\mathcal{E}_2/h^{-1}\mathcal{E}_1$ is invariant under the action of DT in the quotient space $\mathbb{R}^N/h^{-1}\mathcal{E}_1 = (\oplus E_\chi)/h^{-1}\mathcal{E}_1$, therefore is a summand in the quotient space. This shows that $\mu_{i,2}$ as the Lyapunov exponent of the quotient action $D\hat{T}_i$ on the quotient space $\mathcal{E}_2/\mathcal{E}_1$ is also the Lyapunov exponent of the quotient action of DT_i on the quotient space $h^{-1}\mathcal{E}_2/h^{-1}\mathcal{E}_1$, therefore is one Lyapunov exponent of DT_i.

Inductively, we find all the Lyapunov exponents $\{\mu_{i,j}\}$. For each j, the vector $(\mu_{1,j}, \ldots, \mu_{N,j})$ gives rise to a Lyapunov functional χ_j.

$\square$

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References

[A1]   Asaoka, Masayuki. *Rigidity of certain solvable actions on the sphere.* Geometry & Topology 16.3 (2012): 1835-1857.

[A2]   Asaoka, Masayuki. *Rigidity of certain solvable actions on the torus.* (2014).

[AB]   Aliprantis, Charalambos D., and Kim Border. *Infinite dimensional analysis: a hitchhiker's guide.* Springer Science & Business Media, 2006.

[ADH]   Auffinger, Antonio, Michael Damron, and Jack Hanson. *50 years of first passage percolation.* arXiv preprint arXiv:1511.03262 (2015).

[BBI]   Burago, Dmitri, Yuri Burago, and Sergei Ivanov. *A course in metric geometry.* Vol. 33. Providence: American Mathematical Society, 2001.

[BW]   Burslem, Lizzie, and Amie Wilkinson. *Global rigidity of solvable group actions on $\mathbb{S}^1$.* Geometry & Topology 8.2 (2004): 877-924.

[DF]   Damjanovic, Danijela, and Bassam Fayad. *KAM rigidity for partially hyperbolic affine $\mathbb{Z}^k$ actions on the torus with a rank one factor.* arXiv preprint arXiv:1308.3546 (2013).

[DK]   Damjanović, Danijela, and Anatole Katok. *Local rigidity of partially hyperbolic actions I. KAM method and $\mathbb{Z}^k$ actions on the torus.* Annals of mathematics (2010): 1805-1858.

[EF]   Eskin, Alex, and David Fisher. *Quasi-isometric rigidity of solvable groups.* Proceedings of the International Congress of Mathematicians. Vol. 3. 2010.

[EFW1] Eskin, Alex, David Fisher, and Kevin Whyte. *Coarse differentiation of quasi-isometries I: Spaces not quasi-isometric to Cayley graphs.* Annals of Mathematics 176.1 (2012): 221-260.

[EFW2] Alex Eskin, David Fisher, Kevin Whyte, *Coarse differentiation of quasi-isometries II: Rigidity for Sol and Lamplighter groups,* Annals of Mathematics, Volume 177, Issue 3, (2013): 869-910

[Fi]   Fisher, David. *Local rigidity of group actions: past, present, future.* arXiv preprint math/0507455 (2005).

[FK]   Fayad, Bassam, and Kostantin Khanin. *Smooth linearization of commuting circle diffeomorphisms.* Annals of Mathematics (2009): 961-980.

[Fr]   Franks, John. *Anosov diffeomorphisms on tori.* Transactions of the American Mathematical Society 145 (1969): 117-124.

[FKS]   Fisher, David, Boris Kalinin, and Ralf Spatzier. *Global rigidity of higher rank Anosov actions on tori and nilmanifolds.* JAMS 26.1 (2013): 167-198.

[FM1]   Farb, Benson, and Lee Mosher. *On the asymptotic geometry of abelian-by-cyclic groups.* Acta Mathematica 184.2 (2000): 145-202.

[FM2]   Farb, Benson, and Lee Mosher. *Problems on the geometry of finitely generated solvable groups.* Crystallographic groups and their generalizations (Kortrijk, 1999), 121-134, Contemp. Math 262.

[FPS]   Fisher, Todd, Rafael Potrie, and Martin Sambarino. *Dynamical coherence of partially hyperbolic diffeomorphisms of tori isotopic to Anosov.* Mathematische Zeitschrift 278.1-2 (2014): 149-168.

[G1]   Gogolev, Andrey. *Smooth conjugacy of Anosov diffeomorphisms on higher dimensional tori.* Journal of Modern Dynamics, 2, no. 4, 645-700 (2008).

[G2]   Gogolev, Andrey. *Bootstrap for local rigidity of Anosov automorphisms on the 3-torus.* arXiv preprint arXiv:1407.7771 (2014).

[Gh]   Ghys, Étienne. *Sur les groupes engendrés par des difféomorphismes proches de l'identité.* Boletim da Sociedade Brasileira de Matemática-Bulletin/Brazilian Mathematical Society 24.2 (1993): 137-178.

[GG]   Gogolev, Andrey, and Misha Guysinsky. C^1 *Differentiable Conjugacy of Anosov Diffeomorphisms on Three Dimensional Torus.* DCDS-A, 22, no. 1/2, 183-200 (2008).

[GKS]   Gogolev, Andrey, Boris Kalinin, and Victoria Sadovskaya. *Local rigidity for Anosov automorphisms,* (appendix by Rafael de la Llave) Mathematical Research Letters, 18, no. 5, 843-858 (2011)

[GS]   Gorodnik, Alexander, and Ralf Spatzier. *Mixing properties of commuting nilmanifold automorphisms.* Acta Mathematica 215.1 (2015): 127-159.

[H]   Herman, Michael Robert. *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations.* Publications Mathématiques de l'Institut des Hautes Études Scientifiques 49.1 (1979): 5-233.

[He]   Hedlund, Gustav A. *Geodesics on a two-dimensional Riemannian manifold with periodic coeffcients.* Annals of Mathematics (1932): 719-739.

[HuK]   Hurder, Steve, and Anatole Katok. *Ergodic theory and Weil measures for foliations.* Annals of Mathematics 126.2 (1987): 221-275.

[HJ]   Horn, Roger A., and Charles R. Johnson. *Topics in matrix analysis.* 1991. Cambridge University Presss, Cambridge 37: 39.

[HP]   Hammerlindl, Andy, and Rafael Potrie. *Classification of partially hyperbolic diffeomorphisms in 3-manifolds with solvable fundamental group.* Journal of Topology 8.3 (2015): 842-870.

[HPS]   Hirsch, M. W., C. C. Pugh, and M. Shub. *Invariant manifolds,* volume 583 of Lecture notes in mathematics. (1977).

[HSW]   Hu, Huyi, Enhui Shi, and Zhenqi Jenny Wang. *Some ergodic and rigidity properties of discrete Heisenberg group actions.* arXiv preprint arXiv:1405.1120 (2014).

[HX]   Hurtado, Sebastian, Jinxin Xue, *A Tits alternative for surface group diffeomorphisms and Abelian-by-Cyclic actions on surfaces containing an Anosov diffeomorphism,* arXiv preprint arXiv:1909.10112 (2019).

[J]   Journé, Jean-Lin. *A regularity lemma for functions of several variables.* Revista Matematica Iberoamericana 4.2 (1988): 187-194.

[Ja]   Jäger, T. *Linearization of conservative toral homeomorphisms.* Inventiones mathematicae 176.3 (2009): 601-616.

[K]   Kra, Bryna. *The conjugating map for commutative groups of circle diffeomorphisms.* Israel Journal of Mathematics 93.1 (1996): 303-316.

[KH]   Katok, Anatole, and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems.* Vol. 54. Cambridge university press, 1997.

[Kl]   Kleppner, Adam. *Measurable homomorphisms of locally compact groups.* Proceedings of the American Mathematical Society 106.2 (1989): 391-395.

[KO]   Katznelson, Yitzhak, and Donald Ornstein. *The differentiability of the conjugation of certain diffeomorphisms of the circle.* Ergodic Theory and Dynamical Systems 9.04 (1989): 643-680.

[KS]   Khanin, K. M., and Ya G. Sinai. *A new proof of M. Herman's theorem.* Communications in Mathematical Physics 112.1 (1987): 89-101.

[KaS]   Kalinin, Boris and Sadovskaya, Victoria, *Linear cocycles over hyperbolic systems and criteria of conformality.* Journal of modern dynamics, Vol 4, No. 3, 2010, 419-441

[KT]   Khanin, Konstantin, and Alexey Teplinsky. *Hermans theory revisited.* Inventiones mathematicae 178.2 (2009): 333-344.

[L]   Lindenstrauss, Elon. *Pointwise theorems for amenable groups.* Inventiones mathematicae 146.2 (2001): 259-295.

[Ll]   de la Llave, Rafael. *Smooth conjugacy and SRB measures for uniformly and non-uniformly hyperbolic systems.* Communications in mathematical physics 150.2 (1992): 289-320.

[LMM] de la Llave, Rafael, Marco, J. M., and R. Moriyon. *Invariants for smooth conjugacy of hyperbolic dynamical systems I-V.* Communications in Mathematical Physics 109, 112, 116 (1987, 1988).

[M]   Moser, Jürgen. *On commuting circle mappings and simultaneous Diophantine approximations.* Mathematische Zeitschrift 205.1 (1990): 105-121.

[Mc]   McCarthy, Anne. *Rigidity of trivial actions of abelian-by-cyclic groups.* Proceedings of the American Mathematical Society 138.4 (2010): 1395-1403.

[Mo]   Morse, Harold Marston. *A fundamental class of geodesics on any closed surface of genus greater than one.* Transactions of the American Mathematical Society 26.1 (1924): 25-60.

[MS]   McSwiggen, Patrick D. *Diffeomorphisms of the torus with wandering domains.* Proceedings of the American Mathematical Society 117.4 (1993): 1175-1186.

[P]   Pöschel, J. *A Lecture on the Classical KAM Theorem.* Proc. Symp. Pure Math. Vol. 69. 2001.

[R1]   Hertz, Federico Rodriguez. *Stable ergodicity of certain linear automorphisms of the torus.* Annals of Math (2005): 65-107.

[R2]   Hertz, F. Rodriguez, *Global rigidity of certain abelian actions by toral automorphisms,* Journal of Modern Dynamics. 10/2006; 1(3).

[RW]   Hertz, Federico Rodriguez, Zhiren Wang, *Global rigidity of higher rank abelian Anosov algebraic actions,* Inventiones mathematicae, October 2014, Volume 198, Issue 1, pp 165-209

[SK]   Sinai, Yakov, and Konstantin Khanin. *Smoothness of conjugacies of diffeomorphisms of the circle with rotations.* Russian Mathematical Surveys 44.1 (1989): 69-99.

[W1]   Wang, Zhenqi Jenny. *Local rigidity of higher rank non-abelian action on tours.* arXiv preprint arXiv:1405.7737 (2014).

[W2]   Wang, Zhenqi Jenny. *New examples of local rigidity of algebraic partially hyperbolic actions.* arXiv preprint arXiv: 1908.09181v3 (2019).

[Y]   Yoccoz, Jean-Christophe. *Analytic linearization of circle diffeomorphisms.* Dynamical systems and small divisors. Springer Berlin Heidelberg, 2002. 125-173.

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