PATHOLOGY AND ASYMMETRY: CENTRALIZER RIGIDITY FOR PARTIALLY HYPERBOLIC DIFFEOMORPHISMS

DANJIJE DAMJANOVIĆ, AMIE WILKINSON, AND DISHENG XU

Abstract. We discover a rigidity phenomenon within the volume-preserving partially hyperbolic diffeomorphisms with 1-dimensional center. In particular, for smooth, ergodic perturbations of certain algebraic systems— including the discretized geodesic flows over hyperbolic manifolds of dimension at least 3 and linear toral automorphisms with simple spectrum and exactly one eigenvalue on the unit circle— if the abelianization of the smooth centralizer has sufficiently high rank, then the centralizer contains a smooth flow. In dimension 3, we obtain a global dichotomy: for an ergodic partially hyperbolic diffeomorphism \( f \) that preserves an orientable foliation by circles, either the centralizer is virtually trivial, or it contains a smooth flow (in which case, up to a finite cover, \( f \) is a smooth isometric extension of an area-preserving Anosov diffeomorphism).

At the heart of this work are two very different rigidity phenomena. The first phenomenon was discovered in \([2, 3]\): for a class of volume-preserving partially hyperbolic systems including those studied here, the disintegration of volume along the center foliation is either equivalent to Lebesgue or atomic. The other phenomenon is the rigidity associated to several commuting partially hyperbolic diffeomorphisms with very different hyperbolic behavior transverse to a common center foliation \([25]\).

We introduce a variety of techniques in the study of higher rank, abelian partially hyperbolic actions: most importantly, we demonstrate a novel geometric approach to building new partially hyperbolic elements in hyperbolic Weyl chambers using Pesin theory and leafwise conjugacy, while we also treat measure rigidity for circle extensions of Anosov diffeomorphisms and apply normal form theory to upgrade regularity of the centralizer.

To the memory of Anatole Katok.

Contents

1. Introduction 3

Discretized geodesic flows 4

Toral automorphisms 4

The secret sauce 6

Higher rank abelian actions 7

1.1. Acknowledgements 8

1.2. Structure of this paper 8

2. Statements of the main results and discussion 9
2.1. The general formulations

2.2. Partially hyperbolic diffeomorphisms and center foliations

2.3. Lebesgue disintegration and large centralizer

2.4. A global dichotomy in dimension 3

2.5. Prior results

3. Preliminaries

3.1. Regularity of maps and foliations

3.2. Lyapunov exponents and the Oseledec splitting

3.3. Some useful properties of commuting maps

3.4. More on partial hyperbolicity

3.5. Pesin theory and Lyapunov charts

3.6. Normal forms for uniformly contracting foliations

3.7. Thermodynamic formalism

3.8. Partially hyperbolic higher rank abelian actions

4. Proofs of Theorems 3 and 5

5. Proofs of Theorems 7 and 8

5.1. Proof of Theorem 8

5.2. Proof of Theorem 7

6. Proof of Theorem 6: the Cartan case

6.1. The groups $G$ and $G_0$

6.2. The projection $\pi$

6.3. Uniform hyperbolicity on the horizontal distribution

6.4. $G, G_0$ have the same hyperbolic Weyl chamber picture

6.5. Estimates for elements in the same Weyl chamber

6.6. Proof of Proposition 42

6.7. Integrability of the horizontal distribution and topological rigidity

6.8. Absolutely continuity of $W^\sigma_f$: volume and equilibrium states

6.9. Absolute continuity of $W^\sigma_f$: cocycle rigidity of higher rank partially hyperbolic actions

6.10. Absolute continuity of $W^\sigma_f$: uniqueness of the measure of maximal entropy

7. Proof of Theorem 6: non-Cartan case

7.1. Some basic properties
# 1. Introduction

The centralizer of a diffeomorphism \( f : M \to M \) is the set of diffeomorphisms \( g \) that commute with \( f \) under composition: \( f \circ g = g \circ f \). Put another way, the centralizer of \( f \) is the group of symmetries of \( f \), where “symmetries” is meant the classical sense: coordinate changes that leave the dynamics of the system unchanged. The centralizer of \( f \) always contains the integer powers of \( f \) and typically not more, at least conjecturally \([80, 81]\). By contrast, a diffeomorphism belonging to a smooth flow has large centralizer, containing a 1-dimensional Lie group.

To date, the study of smooth centralizers has mainly focused in two directions: showing that the typical map commutes only with its powers; and classifying the manifolds and/or dynamics that can support abelian centralizers of sufficiently high rank. In this paper we aim at describing the centralizers of all diffeomorphisms in a small neighborhood of a given map, for specific classes of maps. This relates to one of the classical questions in perturbation theory: if a diffeomorphism belongs to a smooth flow, which perturbations also belong to a smooth flow? We answer this question fully for algebraic geodesic flows in negative curvature.

More generally, we start with certain diffeomorphisms with exceptionally large centralizer – containing a 1-dimensional Lie group – and consider what happens when these diffeomorphisms are perturbed. We find that for such perturbed systems, if the centralizer gets large enough, as measured by the rank of its abelianization\(^1\), then in fact it must be exceptionally large.

To fix notation, let \( \mathcal{G} \) be a group of homeomorphisms, for example the space \( \text{Diff}^r(M) \) of \( C^r \) diffeomorphisms of a closed manifold \( M \). For \( f \in \mathcal{G} \), denote by \( \mathcal{Z}_\mathcal{G}(f) \) the centralizer of \( f \) in \( \mathcal{G} \):

\[
\mathcal{Z}_\mathcal{G}(f) := \{ g \in \mathcal{G} : g \circ f = f \circ g \}.
\]

We say that \( f \in \mathcal{G} \) has trivial centralizer in \( \mathcal{G} \) if the centralizer of \( f \) consists of the iterates of \( f \):

\[
\mathcal{Z}_\mathcal{G}(f) = \langle f \rangle := \{ f^n : n \in \mathbb{Z} \} \cong \mathbb{Z},
\]

\(^1\)Recall that the abelianization of a group \( G \) is the quotient \( G^{ab} = G/[G,G] \). The rank of \( G^{ab} \) is the cardinality of a maximal linearly independent subset. The rank of \( G^{ab} \) is \( \geq \ell \) if and only if \( G \) has a subgroup that surjects onto \( \mathbb{Z}^{\ell} \).
and virtually trivial centralizer if \( Z_G(f) \) contains \( < f > \) as a finite index subgroup. Note that if \( f \) has virtually trivial centralizer, then so does its nontrivial iterates.

**Discretized geodesic flows.** The context in which our main results are easiest to state and prove is that of perturbations of discretized geodesic flows in negative curvature. Let \( X \) be a closed, negatively curved locally symmetric manifold with \( \dim X > 2 \), for example, a compact hyperbolic 3-manifold. Denote by \( T^1 X \) the unit tangent bundle of \( X \) and by \( \psi_t \) the geodesic flow \( \psi_t : T^1 X \to T^1 X \) over \( X \). The flow \( \psi_t \) preserves the canonical Liouville probability measure on \( T^1 X \), which we denote by \( \text{vol} = \text{vol}_{T^1 X} \). Any element \( \psi_t \) of this flow commutes with any other element, and thus

\[
Z_{\text{Diff}^\infty(T^1 X)}(\psi_t) \supseteq \{ \psi_s : s \in \mathbb{R} \} \cong \mathbb{R}.
\]

Our first result concerns volume-preserving perturbations of the discretized flow — that is, the time-\( t_0 \) map \( \psi_{t_0} \), for a fixed \( t_0 \neq 0 \). Such a perturbation \( f \in \text{Diff}^\infty_{\text{vol}}(T^1 X) \) will not necessarily embed in a flow: for example, any perturbation with a hyperbolic periodic point cannot embed in a flow, and such perturbations are plentiful. The upshot of this result is that if such a perturbation does not embed in a flow, then it has virtually trivial centralizer.

**Theorem 1.** Let \( X \) be a closed, negatively curved locally symmetric manifold with \( \dim X > 2 \), and let \( \psi_t : T^1 X \to T^1 X \) be the associated geodesic flow. Fix \( t_0 \neq 0 \), and suppose \( f \in \text{Diff}^\infty_{\text{vol}}(T^1 X) \) is a \( C^1 \)-small perturbation of \( \psi_{t_0} \). Then either \( f \) has virtually trivial centralizer in \( \text{Diff}^\infty(T^1 X) \) or \( f \) embeds into a smooth, volume-preserving flow (and thus \( Z_{\text{Diff}^\infty(T^1 X)}(f) \supseteq \mathbb{R} \)). Moreover, in the latter case, the centralizer \( Z_{\text{Diff}^\infty(T^1 X)}(f) \) is virtually \( \mathbb{R} \).

Thus for perturbations of these special flows, up to finite index subgroups, the centralizer is either \( \mathbb{Z} \) or \( \mathbb{R} \). The proof of Theorem 1 uses Mostow rigidity in a central way. We do not know whether the same result holds for perturbations of discretized Anosov flows, in particular for perturbations of discretized geodesic flows over hyperbolic surfaces.

**Question 1.** Do the same conclusions of Theorem 1 hold for the volume-preserving perturbations of the time-\( t_0 \) map of an arbitrary volume-preserving Anosov flow?

For related results and further discussion, see the forthcoming paper \cite{89}.

We remark that virtually trivial cannot be replaced by trivial in the conclusion of Theorem 1. Indeed for any \( t_0 \in \mathbb{R} \), Burslem shows in \cite{17} Theorem 1.3] that the time-\( t_0/2 \) map \( \psi_{t_0/2} \) can be \( C^\infty \) approximated by \( f \in \text{Diff}^\infty_{\text{vol}}(T^1 X) \) with trivial centralizer. Then map \( f^2 \) has virtually trivial, but not trivial, centralizer and \( C^\infty \)-approximates \( \psi_{t_0} \).

**Toral automorphisms.** Linear automorphisms of tori present a rich family of algebraic systems with notable rigidity properties. Any orientation-preserving automorphism of the torus \( \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d \) lifts to a linear automorphism of \( \mathbb{R}^d \) preserving \( \mathbb{Z}^d \), which can be represented by a matrix \( A \in \text{SL}(d, \mathbb{Z}) \). For such a matrix \( A \) we write \( T_A : \mathbb{T}^d \to \mathbb{T}^d \)

---

\(^2\) If general, one says that a property holds virtually for a group \( G \) if \( G \) contains a finite index subgroup \( H \) with that property.
to denote the associated toral automorphism. Since \( A \) has determinant 1, the map \( T_A \) preserves the Lebesgue-Haar measure on \( \mathbb{T}^d \), which we again denote by \( \operatorname{vol}(= \operatorname{vol}_{\mathbb{T}^d}) \).

In the hyperbolic case where \( A \) has no eigenvalues on the unit circle, the automorphism \( T_A \) has a strong topological rigidity property known as structural stability: any perturbation of \( T_A \) in \( \text{Diff}^1(\mathbb{T}^d) \) is topologically conjugate to \( T_A \). The centralizer of a perturbation \( f \in \text{Diff}^1(\mathbb{T}^d) \) within \( \text{Homeo}^+(\mathbb{T}^d) \) is thus isomorphic to the centralizer of \( T_A \) in \( \text{Homeo}^+(\mathbb{T}^d) \). It is well-known (see Lemma 16) that when \( A \) is irreducible — meaning that its characteristic polynomial is irreducible over \( \mathbb{Z} \) — both \( \mathcal{Z}_{\text{Homeo}^+(\mathbb{T}^d)}(T_A) \) and \( \mathcal{Z}_{\text{SL}(d,\mathbb{Z})}(A) \) are virtually a finitely generated free abelian group whose rank is determined by the number of distinct eigenvalues of \( A \). Of course for a perturbation \( f \in \text{Diff}^r(\mathbb{T}^d) \) of \( T_A \), the centralizer \( \mathcal{Z}_{\text{Diff}^r(\mathbb{T}^d)}(f) \) can be considerably smaller than \( \mathcal{Z}_{\text{Homeo}^+(\mathbb{T}^d)}(f) \): in fact, Palis and Yoccoz showed that for an open and dense set of perturbations \( f \in \text{Diff}^\infty(\mathbb{T}^d) \), the centralizer \( \mathcal{Z}_{\text{Diff}^\infty(\mathbb{T}^d)}(f) \) is trivial [61, 62].

From a dynamical point of view, perturbations of the non-hyperbolic automorphisms are considerably more interesting. When \( A \) has no eigenvalues that are roots of unity, then \( T_A \) is mixing with respect to \( \operatorname{vol} \), and in several cases of interest, stably mixing: any sufficiently smooth, volume-preserving perturbation of \( T_A \) is mixing if \( d \leq 5 \) [70].

We consider a case in which both structural stability and ergodicity are violated in a fairly dramatic fashion, where the generating matrix \( A \in \text{SL}(d,\mathbb{Z}) \) has 1 as an eigenvalue, with multiplicity 1.\(^3\) By conjugating by a toral automorphism, we may assume without loss of generality that \( A = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \). For such \( A \), the map \( T_A = T_B \times \operatorname{id}_T \) admits non-conjugate affine perturbations of the form \( f = B \times R_\theta \), where \( R_\theta(z) = z + \theta \) is a rotation by \( \theta \in \mathbb{T} \) in last factor in \( \mathbb{T}^d = \mathbb{T}^{d-1} \times \mathbb{T} \), and so \( T_A \) is not structurally stable, even within the restricted class of affine transformations. By the same token, these affine perturbations also have large centralizer, commuting with any affine map of the form \( C \times R_\theta \), with \( C \in \mathcal{Z}_{\text{SL}(d-1,\mathbb{Z})}(B) \), and \( \theta \in \mathbb{T} \).

In the case that \( B \) is irreducible, the Dirichlet unit theorem gives that the group \( \mathcal{Z}_{\text{SL}(d-1,\mathbb{Z})}(B) \) is virtually \( \mathbb{Z}^\ell_0 \), where \( \ell_0 = \ell_0(B) := r + c - 1 \), \( r \) is the number of real eigenvalues of \( B \) and \( c \) is the number of pairs of complex eigenvalues of \( B \) (cf. Lemma 16). Setting \( \ell(B) := \max(\ell_0, 2) \), we obtain the following rigidity result for the centralizer of perturbations of \( T_A \).

**Theorem 2.** Let \( f \in \text{Diff}^\infty_{\text{vol}}(\mathbb{T}^d) \) be a \( C^1 \)-small, ergodic perturbation of \( T_B \times \operatorname{id}_T \), where \( B \in \text{SL}(d-1,\mathbb{Z}) \) is hyperbolic and irreducible. If the abelianization of \( \mathcal{Z}_{\text{Diff}^\infty(\mathbb{T}^d)}(f) \) has rank \( \geq \ell(B) \), then \( \mathcal{Z}_{\text{Diff}^\infty(\mathbb{T}^d)}(f) \) is virtually \( \mathbb{Z}^\ell \times \mathbb{T} \), for some \( \ell \in [1, \ell_0(B)] \).

**Remark 1.** Theorem 2 has a stronger formulation as a rigidity result, stated in Theorem 4 in the next section.

**Remark 2.** For similar problems on nilmanifolds, cf. our subsequent paper [23].

\(^3\) The case where \( A \in \text{SL}(d,\mathbb{Z}) \) has exactly one eigenvalue of modulus 1 can be treated by similar methods.
Remark 3. The ergodicity assumption in Theorem 2 is mild. In fact it is proved by Burns and Wilkinson in [14] and F. Rodriguez Hertz, M. A. Rodriguez Hertz and Ures in [72] that ergodicity (indeed, mixing) holds open and densely among the partially hyperbolic diffeomorphisms with 1-dimensional center in \( \text{Diff}_{\text{vol}}^\infty(\mathbb{T}^d) \) (for precise definitions and more details, see Section 3.4). In particular, for any neighborhood \( U \) of \( T_A \), there is a \( C^1 \)-open set \( U_0 \subset U \) such that every \( f \in U_0 \) is ergodic.

Both Theorems 1 and 2 are consequences of more general results, which we state in Section 2.

The secret sauce. While it does not appear in the statements, there is a hidden concept behind Theorems 1 and 2: pathological foliations. Both the discretized geodesic flows and the toral automorphisms we discuss above preserve smooth, 1-dimensional foliations, in the first case, the foliation by orbits of the flow, and in the second, the foliation by circles tangent to the last factor in \( \mathbb{T}^{d-1} \times \mathbb{T} \).

Transverse to the leaves of these foliations, the dynamics is hyperbolic, and so the theory of normally hyperbolic foliations developed in [39] applies. In particular, the perturbations of these examples considered in Theorems 1 and 2 also preserve 1-dimensional foliations with smooth leaves, homeomorphic as foliations to the unperturbed smooth foliations (see Section 3.4 for a detailed discussion). The measure-theoretic properties of these center foliations are well-studied and play a key role in our proofs.

By a standard procedure, the volume \( \text{vol} \) can be locally disintegrated along the leaves of a foliation \( \mathcal{F} \) to obtain in each foliation chart a measurable family of measures, supported on the local leaves (or plaques) \( \mathcal{F}_{\text{loc}} \) of the foliation. Each plaque \( \mathcal{F}_{\text{loc}}(x) \) of a foliation, being a \( C^1 \) embedded disk, also carries a natural measure class \( \text{vol}_{\mathcal{F}_{\text{loc}}(x)} \) associated to leafwise volume, or length in the case of 1-dimensional leaves. If the foliation is \( C^1 \) (i.e. has \( C^1 \) foliation charts), then the disintegration of \( \text{vol} \) and leafwise volume are equivalent, meaning they have the same sets of measure zero.

When, as is typically the case in our perturbed examples, the foliation is not \( C^1 \), anything goes, at least \( a \text{ priori} \). The two extremal cases are:

- **Lebesgue disintegration**, where the disintegrated and leafwise volume are equivalent. A foliation \( \mathcal{F} \) of \( M \) has Lebesgue disintegration if for every set \( Z \subset M \):

  \[
  \text{vol}(Z) = 0 \iff \text{vol}_{\mathcal{F}_{\text{loc}}(x)}(Z) = 0, \quad \text{for vol-a.e.} \ x \in M.
  \]

- **atomic disintegration**, where the disintegrated volume is atomic. A foliation \( \mathcal{F} \) of \( M \) has atomic disintegration if there exists a set \( B \subset M \) and \( k \geq 1 \) such that

  \[
  \text{vol}(M \setminus B) = 0 \quad \text{and} \quad \#\{B \cap \mathcal{F}_{\text{loc}}(x)\} \leq k, \quad \text{for vol-a.e.} \ x \in M.
  \]

If a foliation fails to have Lebesgue disintegration with respect to volume, we call it **pathological**, a concept first considered by Shub and Wilkinson in [79]. This concept plays an important role in our paper. In brief, pathological disintegration is associated with small centralizer and Lebesgue disintegration with large centralizer (at least in the group of homeomorphisms).
Higher rank abelian actions. Another key role in our proofs is played by higher rank abelian group actions with some hyperbolicity. In particular, for the diffeomorphisms $f \in G$ we consider here, the centralizer $Z_G(f)$ turns out to be virtually abelian, and in the context of Theorem 2 and related results discussed below, the maximal rank of an abelian subgroup is assumed to be at least 2. Manifolds that admit higher rank abelian actions with some hyperbolicity conjecturally have special structure. Indeed in the structurally stable setting, where elements of the action are actually hyperbolic (i.e. Anosov) conjecturally the only higher rank actions are essentially algebraic (the Katok-Spatzier global rigidity conjecture, [26] and references therein).

In our setting, the diffeomorphism $f$ is partially hyperbolic (see Section 2.2 for a definition), admitting a hyperbolic splitting transverse to a 1-dimensional invariant center foliation. Thus $Z_G(f)$ contains at least one partially hyperbolic element: $f$ itself (plus its nontrivial iterates). One important novelty of our argument, and perhaps the most subtle part of the proofs here, is to show that if $Z_G(f)$ is large enough, then it contains many independent partially hyperbolic elements, at least as many as in the unperturbed system (Section 6). Here “independent” means that the partially hyperbolic elements lie in different hyperbolic Weyl chambers associated to the action of the centralizer group, and “many” means that there are partially hyperbolic elements in each chamber.

The task of reconstructing many independent hyperbolic elements in an action from a single hyperbolic element is a challenging problem and one of the main obstacles to proving the Katok-Spatzier global rigidity conjecture for Anosov actions in full generality. This problem has been resolved so far only for the case of Anosov higher rank actions on nilmanifolds. In this setting, Rodriguez Hertz and Wang [74] proved that a Anosov diffeomorphism in a higher-rank Abelian action forces the existence of Anosov elements in every Weyl chamber; together with [32], this proves the Katok–Spatzier conjecture on nilmanifolds. The proof in [74] makes use of the Franks–Manning conjugacy on nilmanifolds and fine analytic properties of the dynamics of Anosov diffeomorphisms, in particular exponential rates of mixing.

The actions considered here have a hyperbolic part and a 1-dimensional nonhyperbolic, central part. The hyperbolic part is, on a topological level, a maximal Anosov action – considerably simpler than the actions considered in [32, 74]. On the other hand, the methods in these works are not available to us: the central part of our actions obstruct conjugacy to a linear system, and the dynamics of the systems are potentially not even mixing. What is available instead is a leaf conjugacy to a linear system, that is, a topological conjugacy modulo the center dynamics. Starting from the leaf conjugacy and using maximality of the action, we build up the partial hyperbolicity of other elements in the action. Our arguments are geometric rather than analytic in nature and employ a range of techniques.

Our arguments extend further a recently-developed theory for partially hyperbolic abelian group actions (see [25] and references therein). Existence of many partially hyperbolic elements in the large rank centralizer allows us to obtain full description of the centralizer up to a smooth conjugacy. One important idea, also employed in [13], is to use Pesin theory in the presence of invariant conefields and uniform estimates to upgrade a
uniformly expanded topological foliation $W^\#$ to a foliation with smooth leaves. To carry out such an argument requires precise control over the Hölder exponent of leaf conjugacies, something established relatively recently in [68]. These arguments occupy the bulk of Sections 6 and 7.

We also rely on the geometric method to establish cohomological rigidity over higher rank abelian actions, which is the technique developed in [50], [52] and later in [20]. In the context of a single partially hyperbolic diffeomorphism, the geometric method is explained in [87]. The main feature of the geometric method in the situation we consider here is that unlike other techniques, it is applicable to non-algebraic actions and non-smooth cocycles. We use the geometric method in Section 6 to obtain leafwise cohomological rigidity. This further allows us to also derive a measure rigidity type of result: for the conservative partially hyperbolic systems we consider here, the volume is the measure of maximal entropy. This in particular eliminates the possibility of having a pathological center foliation in the case when the centralizer has full rank.

These arguments allow us to show that if the rank of the centralizer of $f$ is sufficiently high, then $f$ commutes with a volume-preserving flow. To upgrade the regularity of this flow from continuous to smooth, we employ a variety of arguments, including Livsic regularity techniques from [87] and recently-developed results in the theory of normal forms due to Kalinin.

1.1. Acknowledgements. We thank Federico Rodriguez Hertz, Curtis McMullen, Yakov Pesin and Zhiren Wang for useful discussions and Andy Hammerlindl and Dennis Sullivan for corrections to an earlier manuscript. We are grateful to Boris Kalinin for explaining to us the details of his recent results in normal form theory, which are used in this paper. Damjanović was supported by Swedish Research Council grant VR2015-04644. Wilkinson was supported by NSF Grant DMS-1402852.

1.2. Structure of this paper. In Section 2 we state our main results in the more general context of partially hyperbolic diffeomorphisms with 1 dimensional center foliations and discuss prior results. Section 3 contains background information and some new techniques used in the proofs of our main results. In Section 4, we prove the main results about discretized geodesic flows (Theorems 3 and 5), and in Section 5, we prove our main global rigidity results in dimension 3 (Theorems 7 and 8). Theorem 6 is a disintegration dichotomy that is the driver behind our main result, Theorem 4. The proof of Theorem 6 occupies Sections 6 and 7. In the first section, we restrict ourselves to the case whether the centralizer satisfies a particularly strong maximality condition called the maximal Cartan condition (see Section 3.8), and the second contains the modifications necessary to treat the general case. Finally, in Section 8, we prove Theorem 4. The Appendix contains the statement of a result from another work that we use in this paper.
2. Statements of the main results and discussion

2.1. The general formulations. In this section we state the following more general versions of the rigidity results for centralizers, which immediately imply Theorems 1 and 2.

**Theorem 3.** Let \( X \) be a closed, negatively curved locally symmetric manifold with \( \dim X > 2 \), and let \( \psi_1 : T^1 X \to T^1 X \) be the geodesic flow. Then there exists \( r_0 = r_0(X) \geq 1 \) such that for all \( r > r_0 \), and any \( t_0 \neq 0 \), if \( f \in \text{Diff}^\infty_{\text{vol}}(T^1 X) \) is sufficiently \( C^1 \) close to \( \psi_{t_0} \), then either the diffeomorphism \( f \) has virtually trivial centralizer in \( \text{Diff}^r(T^1 X) \), or \( Z_{\text{Diff}^r(T^1 X)}(f) = Z_{\text{Diff}^r_{\text{vol}}(T^1 X)}(f) \) is virtually \( \mathbb{R} \) for any \( s \geq r \). In the latter case \( f \) embeds into a \( C^\infty \), volume preserving flow.

**Remark 4.** If \( X \) is a compact real hyperbolic manifold, then \( r_0(X) = 1 \); otherwise \( r_0(X) = 2 \).

Let \( g : \mathbb{T}^{d-1} \to \mathbb{T}^{d-1} \) be a diffeomorphism. An isometric (circle) extension of \( g \) is a map \( f = g_\rho : \mathbb{T}^{d-1} \times \mathbb{T} \to \mathbb{T}^{d-1} \times \mathbb{T} \) of the form
\[
g_\rho(x, y) = (g(x), y + \rho(x)),
\]
where \( \rho : \mathbb{T}^{d-1} \to \mathbb{T} \) is a function taking values in the circle \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \). The map \( g_\rho \) is a \( C^r \) diffeomorphism if and only if \( g \) and \( \rho \) are \( C^r \) and preserves volume if and only if \( \rho \) does.

The simplest examples of isometric extensions are products \( g \times R_\theta \), where \( R_\theta(y) = y + \theta \) is a rotation. In this case \( \rho \equiv \theta \) is a constant function. It is easy to check that there exists \( \beta : \mathbb{T}^{d-1} \to \mathbb{T} \) such that \( \text{id}_\beta \circ g_\rho = (g \times R_\theta) \circ \text{id}_\beta \) if and only if \( \rho \) satisfies the cohomological equation
\[
\rho = -\beta \circ g + \beta + \theta.
\]
In this case we say that \( \rho \) is cohomologous to a constant \( \theta \). If \( g \in \text{Diff}^2_{\text{vol}}(\mathbb{T}^{d-1}) \) is Anosov, then \( g_\rho \) is ergodic if and only if \( \rho \) is not cohomologous to a rational constant, and \( g_\rho \) is stably ergodic if and only if \( \rho \) is not cohomologous to a constant \( 15 \).

If \( g = T_{A_0} \) is an irreducible hyperbolic automorphism and \( g_\rho \) is ergodic, then for all \( s \geq 1 \), the centralizer of \( g_\rho \) in \( \text{Diff}^s(\mathbb{T}^d) \) contains \( Z^{\ell_0(A_0)} \times \mathbb{T} \), where \( \ell_0(A_0) > 0 \) is defined in the introduction. 4 Our first result addresses perturbations of these maps.

**Theorem 4.** For every irreducible hyperbolic matrix \( A_0 \in \text{SL}(d-1, \mathbb{Z}) \), there exists \( r_0 = r_0(A_0) > 0 \) with the following property.

Fix a \( C^\infty \) function \( \rho_0 : \mathbb{T}^{d-1} \to \mathbb{T} \), and let \( f_0 := (T_{A_0})_{\rho_0} \). Suppose that \( f \in \text{Diff}^\infty_{\text{vol}}(\mathbb{T}^d) \) is a \( C^1 \)–small, ergodic perturbation of \( f_0 \). If, for some \( r > r_0 \), the abelianization of \( Z_{\text{Diff}^s(\mathbb{T}^d)}(f) \) has rank \( \geq \ell(A_0) \), then there exists a \( C^\infty \) function \( \rho : \mathbb{T}^{d-1} \to \mathbb{T} \) and an Anosov diffeomorphism \( g \in \text{Diff}^\infty_{\text{vol}}(\mathbb{T}^{d-1}) \) such that \( f \) is \( C^\infty \) conjugate to the isometric extension \( g_\rho \). In particular, \( Z_{\text{Diff}^s(\mathbb{T}^d)}(f) \) is virtually \( \mathbb{Z}^\ell \times \mathbb{T} \), for all \( s \geq 1 \) and some \( \ell \in [1, \ell_0(A_0)] \).

4In fact the centralizer is virtually \( Z^{\ell_0(A_0)} \times \mathbb{T} \). This follows from Theorem 4 but also has a more elementary proof using cocycle rigidity of the centralizer of \( g \).
Remark 5. The value of $r_0$ in Theorem 4 is given by $r_0(A_0) = \max(\lambda_\sigma^u, \mu_\sigma^u, 2)$, where $\lambda^u, \mu^u$ (resp. $\lambda^s, \mu^s$) are the top and bottom unstable (resp. stable) Lyapunov exponents of $A_0$.

Remark 6. The $\ell \in [1, \ell_0(A_0)]$ part of the conclusion is “sharp” in the sense that for any $d \geq 3$, there exist $f = g_\rho$ such that $\mathcal{Z}(f)$ is $\mathbb{Z} \times \mathbb{T}$ and such that $\mathcal{Z}(f)$ is virtually $\mathbb{Z}^{\ell_0(A_0)} \times \mathbb{T}$. This still leaves the following question.

Question 2. For $\ell \in (1, \ell_0(A_0))$, does there exist a perturbation $f$ as in Theorem 4 with $\mathcal{Z}_{\text{Diff}}(\mathbb{T}^d)(f)$ virtually $\mathbb{Z}^{\ell} \times \mathbb{T}$?

Remark 7. In sufficiently low dimension, Theorems 2 and 4 give a dichotomy between virtually trivial centralizer and large centralizer. In particular, if we assume in addition to the hypotheses of Theorem 4 that $A_0 \in \text{SL}(d-1, \mathbb{Z})$ satisfies one of the following conditions:

- $d = 3$ or $4$;
- $d = 5$ and $A_0$ has at least one pair of complex roots;
- $d = 6$ and $A_0$ has two pairs of complex roots; or
- $d = 7$ and $A_0$ has three pairs of complex roots;

then $\ell_0(A_0) = 1$, and particular the dichotomy in Theorem 6 reduces to the following: if $f \in \text{Diff}_\text{vol}^\infty(\mathbb{T}^d)$ is a $C^1$--small, ergodic perturbation of $f_0$, then $\mathcal{Z}_{\text{Diff}}(\mathbb{T}^d)(f)$ is either virtually trivial for some (hence for all) $s > r_0$ or virtually $\mathbb{Z} \times \mathbb{T}$ for all $s \geq 1$.

Before stating the rest of the main results in this paper, we define partial hyperbolicity and some related concepts.

2.2. Partially hyperbolic diffeomorphisms and center foliations. Let $M$ be a complete Riemannian manifold, and let $h \in \text{Diff}(M)$. A dominated splitting for $h$ is a direct sum decomposition of the tangent bundle $TM = E^1 \oplus E^2 \oplus \cdots \oplus E^k$ such that

- the bundles $E^i$ are $Dh$-invariant: for every $i \in \{1, \ldots, k\}$ and $x \in M$, we have $D_x h(E^i(x)) = E^i(h(x));$ and
- $Dh_{\mid E^i}$ dominates $Dh_{\mid E^{i+1}}$: there exists $N \geq 1$ such that for any $x \in M$ and any unit vectors $u \in E^{i+1}$, and $v \in E^i$:
  $$\|D_x h^N(u)\| \leq \frac{1}{2}\|D_x h^N(v)\|.$$ 

The property of a splitting being dominated is independent of choice of equivalent metric (and independent of choice of metric in the case where $M$ is compact). A dominated splitting is always continuous. If $M$ is compact and $h'$ is $C^1$ close to $h$ with a dominated splitting, then $h'$ also has a dominated splitting, which varies continuously with $h'$ in the $C^1$ topology.
A $C^1$ diffeomorphism $f : M \to M$ of a complete Riemannian manifold $M$ is **partially hyperbolic** if there is a dominated splitting $TM = E^s \oplus E^c \oplus E^u$ and $N \geq 1$ such that for any $x \in M$, and any choice of unit vectors $v^s \in E^s(x)$ and $v^u \in E^u(x)$, we have
\[
\max\{\|D_x f^N(v^s)\|, \|D_x f^{-N}(v^u)\|\} < 1/2.
\]
We will always assume the bundles $E^s$ and $E^u$ are nontrivial. If $E^c$ is trivial then $f$ is **Anosov**.

A flow $\varphi : M \times \mathbb{R} \to M$ is **Anosov** if for some $t_0 \neq 0$, the time-$t_0$ map $\varphi_{t_0}$ is partially hyperbolic, with the center bundle $E^c = \mathbb{R}\dot{\varphi}$ tangent to the orbits of the flow. If $\varphi$ is Anosov, then the time-$t$ map $\varphi_t$ is partially hyperbolic for every $t \neq 0$. An example of an Anosov flow is the geodesic flow over a closed, negatively curved manifold, such as those considered in Theorem 5.

Isometric circle extensions of Anosov diffeomorphisms, such as the diffeomorphisms considered in Theorem 4, are also partially hyperbolic, with $E^c$ tangent to the vertical foliation by circles $\{\{x\} \times \mathbb{T} : x \in \mathbb{T}^{d-1}\}$ (see, e.g. [15]).

If $M$ is a closed manifold, then partial hyperbolicity is open property in the $C^1$ topology on $\text{Diff}^1(M)$. Thus the $C^1$-small perturbations considered in Theorems 3 and 4 are also partially hyperbolic.

If $f$ is partially hyperbolic and $C^r$, $1 \leq r \leq \infty$, then the bundles $E^s$ and $E^u$ are tangent to foliations $W^s$ and $W^u$, known respectively as the stable and the unstable foliations of $f$. These foliations have $C^r$ leaves but are typically only Hölder continuous. For a more detailed discussion of foliation regularity, see Section 3.1.

We say a $Df$-invariant distribution $E$ is **integrable** if there exists an $f$-invariant foliation $W = \{W(x)\}_{x \in M}$ with $C^1$ leaves everywhere tangent to the bundle $E$. Thus $E^u$ and $E^s$ are integrable. The center bundle $E^c$ is not always integrable (see [73]), but in many examples of interest, such as the time-one map of an Anosov flow and its perturbations, or perturbations of an isometric extension of Anosov map, the theory of normally hyperbolic foliations developed in [39] implies that $E^c$ is integrable, as are the bundles $E^{cs} = E^c \oplus E^s$ and $E^{cu} = E^c \oplus E^u$. In particular, for those $f$ considered in this paper, $E^c$ is integrable, tangent to a **center foliation** $W^c$. Our main results can be recast in terms of the measure theoretic properties of center foliations, as follows.

### 2.3. Lebesgue disintegration and large centralizer.
As mentioned in the introduction, some of the key ingredients in proofs of Theorems 3 and 4 are the following **dichotomy results** which link the disintegration of volume along the center foliation with the structure of the centralizer.

For volume-preserving perturbations of the discretized geodesic flow on a negatively curved locally symmetric space, we have

**Theorem 5.** Let $X$ be a negatively curved locally symmetric space with $\dim X > 2$, and let $\psi_t: T^1X \to T^1X$ be the geodesic flow. Fix $t_0 \neq 0$, and suppose $f \in \text{Diff}_\text{vol}^1(T^1X)$ is a $C^1$-small perturbation of $\psi_{t_0}$. Then either the volume $\text{vol}$ has **Lebesgue disintegration** along $W^c_f$, or $f$ has virtually trivial centralizer in $\text{Diff}(T^1X)$.
For perturbations of an isometric extension of a hyperbolic toral automorphism, we have

**Theorem 6.** Let \( f_0 : \mathbb{T}^d \to \mathbb{T}^d \) and \( \mathcal{I} \) be as in Theorem 4, and let \( f \in \text{Diff}^2_{\text{vol}}(\mathbb{T}^d) \) be a \( C^1 \)–small, ergodic perturbation of \( f_0 \). Then either the volume has Lebesgue disintegration along \( \mathcal{W}_f \), or \( \mathbb{Z}_{\text{Diff}^2(\mathbb{T}^d)}(f) \) is virtually \( \mathbb{Z}^\ell \) for some \( \ell < \ell(A_f) \).

2.4. A global dichotomy in dimension 3. In [2] and [3], Avila, Viana and Wilkinson study volume preserving, partially hyperbolic systems in lower dimensions and prove that in many situations, there is a dichotomy between large centralizer and atomic disintegration along the center foliation.

By combining results of [2] and [3] with results of this paper, we obtain a global dichotomy for centralizers.

**Theorem 7.** Let \( M \) be a closed 3-manifold, and let \( f \in \text{Diff}^\infty_{\text{vol}}(M) \) be a partially hyperbolic, ergodic diffeomorphism. Assume that there exists a foliation \( \mathcal{W}_c \) invariant under \( f \), whose leaves are all circles (such a foliation is necessarily tangent to \( E^c \)). Then one of the following holds:

- up to a fourfold cover, the centralizer of \( f \) in \( \text{Diff}^r(M) \) is virtually \( \mathbb{Z} \times \mathbb{T} \) for all \( r \geq 1 \); or
- \( f \) has virtually trivial centralizer in \( \text{Diff}^1(M) \).

A key ingredient in the proof of Theorem 7 is the following result.

**Theorem 8.** Let \( M \) be a closed 3-manifold and let \( f \in \text{Diff}^\infty_{\text{vol}}(M) \) be a partially hyperbolic ergodic diffeomorphism. Assume that there exists a foliation \( \mathcal{W}_c \) invariant under \( f \), whose leaves are all circles. Then either the volume has Lebesgue disintegration along \( \mathcal{W}_c \) or \( f \) has virtually trivial centralizer in \( \text{Diff}^1(M) \).

2.5. Prior results. As mentioned in the introduction, it is expected that the typical diffeomorphism has small centralizer. Indeed, Smale asked [80, 81] whether the set of \( C^r \) diffeomorphisms with trivial centralizer is generic in \( \text{Diff}^r(M) \). Several works have been devoted to this question in various contexts, going back to Kopell’s solution [55] to the question in the smooth case on the circle: those diffeomorphisms with trivial centralizer contains a \( C^\infty \) open and dense in \( \text{Diff}^\infty(\mathbb{T}) \). The question has also been answered in full generality by Bonatti–Crovisier–Wilkinson in the \( C^1 \) topology: trivial centralizer is generic (but not open) in \( \text{Diff}^1(M) \) and \( \text{Diff}^1_{\text{vol}}(M) \), for any closed manifold \( M \) [7, 8, 9]. See [7] for a discussion of the history of this problem.

In the restricted context of partially hyperbolic systems, stronger results are known in the smooth category: Palis–Yoccoz showed that the set of \( C^\infty \) diffeomorphisms with trivial centralizer contains an open and dense subset of the set of Axiom A diffeomorphisms in \( \text{Diff}^\infty(M) \) possessing at least one periodic sink or source. The conditions have subsequently been relaxed [31, 69]. In another direction, Burslem showed that for a class of \( C^\infty \) partially hyperbolic systems, (including non-volume-preserving perturbations of the systems considered in this paper), there is a residual subset whose centralizer is trivial.
When it comes to (partially) hyperbolic diffeomorphisms whose centralizers contain large rank abelian subgroups of (partially) hyperbolic diffeomorphisms, the general philosophy has been that a rich variety of (partially) hyperbolic dynamics in an abelian group action should be a rare occurrence. Classes of algebraic examples of such abelian actions have been listed in [53] by Katok and Spatzier, who also proved in [54] that such Anosov abelian actions are *locally rigid*: small perturbations of such an action are all smoothly conjugate to unperturbed action. Further local rigidity results for classes of partially hyperbolic abelian actions are found in [21], [86]. Moreover, for Anosov diffeomorphisms, if the centralizer contains a $\mathbb{Z}^2$ subgroup that does not factor onto a virtually $\mathbb{Z}$-action, Katok and Spatzier conjectured that $f$ is then smoothly conjugate to a hyperbolic (infra)nilmanifold automorphism, and in particular it has a full rank centralizer smoothly conjugate to a group of automorphisms. This is known in the literature as the Katok-Spatzier global rigidity conjecture. We refer to [26], [83] and references therein for the history and most recent results in the direction of this conjecture.

In the case of volume preserving partially hyperbolic diffeomorphisms with compact center foliation, in the presence of a large centralizer with sufficiently many partial hyperbolic elements, results in the direction of global rigidity have been obtained in [25]. In particular, it was first discovered in [25] that in the context of higher rank actions there is a close connection between the structure of the centralizer and disintegration of volume along the leaves of the center foliation. The forthcoming paper [23] exploits this connection further by obtaining in some cases stronger global rigidity results. For the case of commuting isometric extensions over hyperbolic toral automorphisms, local rigidity results have been obtained earlier under Diophantine conditions, in [19].

Work of Avila, Viana and Wilkinson [2, 3] establishes a dichotomy for a class of partially hyperbolic diffeomorphisms with 1-dimensional center foliation: either the disintegration of Lebesgue is atomic on the center foliation or volume has Lebesgue disintegration on the center. Moreover, for these maps, if volume has Lebesgue disintegration on the center, then there is a continuous volume-preserving flow commuting with the map. These results apply directly to the systems considered here, and we take them as a starting point. Otherwise, our methods are almost entirely disjoint from those in [2, 3].

3. Preliminaries

3.1. Regularity of maps and foliations. For $r \in (0, 1)$, we say that map between metric spaces is $C^r$ if it is Hölder continuous of exponent $r$. For $r \geq 1$ we say that a map between smooth manifolds is $C^r$ if it is $C^{(r)}$ and the $[r]$th-order derivatives are $C^{r-[r]}$. For $r \geq 0$, a map is $C^{r+}$ if it is $C^{r+\epsilon}$ for some $\epsilon > 0$.

Let $M$ be a manifold of dimension $d \geq 2$. A $k$-dimensional topological foliation $\mathcal{F}$ of $M$ is a decomposition of $M$ into path-connected subsets

$$M = \bigcup_{x \in M} \mathcal{F}(x)$$

\[\text{Under an accessibility assumption. See Section 3.4.6.}\]
called leaves, where \( x \in \mathcal{F}(x) \), and two leaves \( \mathcal{F}(x) \) and \( \mathcal{F}(y) \) are either disjoint or equal, and a covering of \( M \) by coordinate neighborhoods \( \{ U_\alpha \} \) with local coordinates \( (x^1_\alpha, \ldots, x^d_\alpha) \) with the following property. For \( x \in U_\alpha \), denote by \( \mathcal{F}_{U_\alpha}(x) \) the connected component of \( \mathcal{F}(x) \cap U_\alpha \) containing \( x \). Then in coordinates on \( U_\alpha \) the local leaf \( \mathcal{F}_{U_\alpha}(x) \) is given by a set of equations of the form \( x^{k+1}_\alpha = \cdots = x^d_\alpha = \text{cst} \). If the local coordinates \( (x^1_\alpha, \ldots, x^d_\alpha) \) can be chosen uniformly \( C^r \) along the local leaves (i.e., to have uniformly \( C^r \) overlaps on the sets \( x^{k+1}_\alpha = \cdots = x^d_\alpha = \text{cst} \)) then we say that \( \mathcal{F} \) has \( C^r \)-leaves. If the \( (x^1_\alpha, \ldots, x^d_\alpha) \) can be chosen \( C^r \) on \( U_\alpha \) then \( \mathcal{F} \) is called a \( C^r \) foliation.

Note that the leaves of a foliation with \( C^r \) leaves are \( C^r \), injectively immersed submanifolds of \( M \).

**Lemma 8.** Let \( f \) be a \( C^{r+1} \) diffeomorphism of a closed Riemannian manifold \( M \). Let \( \mathcal{W} \) be an \( f \)-invariant foliation with uniformly \( C^r \)-leaves. For \( x \in M \), let \( \alpha_x := \| Df \|^{-1}_{TW(x)} \). Let \( E^1 \) and \( E^2 \) be continuous, \( f \)-invariant distributions on \( M \) such that the distribution \( E = E^1 \oplus E^2 \) is uniformly \( C^r \) along \( \mathcal{W} \) leaves and \( E^1 \oplus E^2 \) is a dominated splitting in the sense that for any \( x \in M \),

\[
\alpha_x^r := \max_{v \in E^2(x), \|v\|=1} \| Df(v) \| < 1.
\]

If \( \sup_{x \in M} \alpha_x^r \leq 1 \), then \( E^1 \) is uniformly \( C^r \) along the leaves of \( \mathcal{W} \). In particular if \( \alpha_x \leq 1 \) for all \( x \in M \) then \( E^1 \) is uniformly \( C^r \) along the leaves of \( \mathcal{W} \).

**Proof.** The proof of Lemma 8 is basically an application of \( C^r \)-section theorem in [39]; for a precise proof cf. Corollary 5.6 in [24] or [68].

Suppose \( \mathcal{F} \) is a foliation of a closed manifold \( M \) with \( C^1 \) leaves, and \( \mu \) is a Borel probability measure on \( M \). Let \( B \) be a foliation box, and let \( \mu_B \) be normalized Lebesgue measure on \( B \). There is a unique family of conditional measures \( \mu_x \) defined for \( \mu_B \)-almost every \( x \) in \( B \) with the following properties (see [73]). First, for almost every \( x \), the measure \( \mu_x \) is supported on the plaque \( \mathcal{F}(B)(x) \); second, for every \( \mu_B \)-integrable function \( \psi : B \to \mathbb{R} \), we have

\[
\int_B \psi(x) \, d\mu_B(x) = \int_B \int_{\mathcal{F}(B)(x)} \psi(y) \, d\mu_x(y) \, d\mu_B(x).
\]

We say \( \mu \) has **Lebesgue disintegration** along \( \mathcal{F} \) if for any foliation box \( B \) and \( \mu_B \)-almost every \( x \), the conditional measure of \( \mu_B \) on \( \mathcal{F}(B)(x) \) is equivalent to the Riemannian measure on \( \mathcal{F}(B)(x) \). The measure \( \mu \) has **atomic disintegration** (along \( \mathcal{F} \)) if there exists \( k \geq 1 \) such that for any foliation box \( B \) the conditional of \( \mu_B \) measure on \( \mathcal{F}(B)(x) \) is atomic, with at most \( k \) atoms, for \( \mu_B \)-almost every \( x \).

**Lemma 9.** Let \( \mathcal{F} \) be an orientable topological foliation of a closed manifold \( M \) such that all leaves are circles. Suppose that there exists a full volume set \( S \subset M \) and \( k \in \mathbb{N} \) such that \( S \) meets almost every leaf of \( \mathcal{F} \) in exactly \( k \) points. Denote by

\[
\mathcal{Z}^c := \{ g \in \text{Diff}_{\text{vol}}(M), g \text{ fixes all the leaves of } \mathcal{F} \text{ and preserves the orientation of } \mathcal{F} \}.
\]

then \( \mathcal{Z}^c \) is a finite cyclic group.
Proof. Since the action of $\mathcal{Z}^c$ fixes all the leaves of $\mathcal{F}$ and preserves the volume, on almost every leaf $\mathcal{F}(x)$, any element $g$ of $\mathcal{Z}^c$ maps atoms to atoms, which means that $g$ induces a permutation on $S \cap \mathcal{F}(x)$. Moreover since $g$ preserves the orientation of each circle leaf of $\mathcal{F}$ it induces a cyclic permutation (with respect to the circle ordering) of the atoms on almost every leaf.

Thus for every $x \in S$, the restriction of $g \in \mathcal{Z}^c$ to $\mathcal{F}(x)$ has rotation number $\frac{k'(g,x)}{k}$ (mod 1), for some $k \in \mathbb{Z}^+$ and $k' = k'(g,x) \in \mathbb{Z}/k\mathbb{Z}$, where $k$ is the number of atoms. Since the rotation number is a continuous function on diffeomorphisms, and $S$ is dense, $k'(g,x)$ is independent of $x$. Therefore on every center leaf, $g$ has rotation number $\frac{k'(g,x)}{k}$ (mod 1).

Moreover for any other $h \in \mathcal{Z}^c$ such that $k'(g) = k'(h)$, and every $x \in S$, $h$ induces the same permutation on $S \cap \mathcal{F}(x)$ as $g$, which implies that $g = h$, by the density of $S$. Therefore $k'$ induces an injective homomorphism from $\mathcal{Z}^c$ to $\mathbb{Z}/k\mathbb{Z}$. □

3.2. Lyapunov exponents and the Oseledec splitting. Suppose $M$ is a smooth manifold and $f \in \text{Diff}^1(M)$ is a diffeomorphism preserving a probability measure $\mu$ (for instance, volume). In analogy with the Birkhoff ergodic theorem, one can inquire about the asymptotic behavior of the composition of tangent maps of $f$

$$D_pf^n = D_{f^n(p)}f \circ \cdots \circ D_pf : T_pM \to T_{f^n(p)}M,$$

for $\mu$-a.e. $p \in M$. An answer is given by the Oseledec Multiplicative Ergodic theorem, which we describe here in the setting of continuous cocycles.

Suppose $\Omega$ is a compact metric space and $E \to \Omega$ is a (continuous) vector bundle. Let $T : \Omega \to \Omega$ be homeomorphism. A continuous cocycle over $T$ is a bundle map $F : E \to E$ covering $T$. On the fibers, $F$ is given by linear maps $F_\omega : E_\omega \to E_{T\omega}$ that vary continuously with $\omega$. For simplicity we assume that each $F_\omega$ is invertible, so that $F$ is a bundle isomorphism. Then we have

**Theorem 9** (Oseledec multiplicative theorem). Let $F : E \to E$ be a continuous, invertible cocycle over an ergodic probability measure preserving system $T : (\Omega, \mu) \to (\Omega, \mu)$. Assume that $E$ is equipped with a metric $\| \cdot \|_\omega$ on each fiber $E_\omega$ that depends continuously on $\omega$. Then there exist real numbers $\lambda_1 > \cdots > \lambda_k$ and measurable, $F$–invariant subbundles of $V$ defined for almost every $\omega \in \Omega$:

$$E = E^{\lambda_1} \oplus \cdots \oplus E^{\lambda_k},$$

such that for $v \in E_\omega \setminus \{0\}$,

$$v \in E^{\lambda_i}_\omega \iff \lim_{n \to \pm \infty} \frac{1}{n} \log \|F^n(v)\|_{T^n(\omega)} = \lambda_i.$$

The splitting $E = \oplus E^{\lambda_i}$ is called the Oseledec splitting. The numbers $\lambda_i$ are called Lyapunov exponents.

The following well-known result allows one to deduce uniform growth of cocycles from knowledge about exponents for every invariant measure.
Lemma 10. Let $f : X \to X$ be a continuous map of a compact metric space, and let $F : E \to E$ be a linear cocycle over $f$, where $p : E \to X$ is a continuous vector bundle over $X$.

(1) If for any $f$-invariant ergodic measure $\nu$, the top Lyapunov exponent $\lambda^+(F, \nu) \leq \lambda$, then for any $\epsilon > 0$, there exists $n \in \mathbb{Z}^+$ such that
$$\|F^n(x)\| \leq e^{n(\lambda + \epsilon)}, \forall x \in X.$$ 

(2) If for any $f$-invariant ergodic measure $\nu$, the bottom Lyapunov exponent $\lambda^-(F, \nu) \geq \lambda'$, then for any $\epsilon > 0$, there exists $n \in \mathbb{Z}^+$ such that
$$\|F^n(x)^{-1}\|^{-1} \geq e^{n(\lambda' - \epsilon)}, \forall x \in X.$$

Proof. Lemma 10 is a corollary of a classical result on subadditive sequences (cf. [78] or chapter 4 in [43]). □

3.3. Some useful properties of commuting maps. A basic principle in the study of abelian actions is the following: if $f$ and $g$ are commuting maps, and $Y$ is an $f$-invariant object, then $g^*(Y)$ is also $f$-invariant. For example, if $f(p) = p$, then $g(f(p)) = g(p)$. Thus $g(Fix(f)) \subset Fix(f)$; in other words, the set of $f$-periodic points of period $k$ is a $g$-invariant set. Similar results hold for invariant sets of commuting homeomorphisms, such as the limit set and non-wandering set.

In the measurable context, if $\mu$ is an $f$-invariant measure, then $g^*\mu$ is also $f$-invariant, and so $g^*$ preserves the set of $f$-invariant measures. When further assumptions are added, such as those in the present context, we get the following useful lemma.

Lemma 11. Let $M$ be a closed manifold, and suppose that $f, g \in \text{Diff}(M)$ satisfy $fg = gf$. If $f$ is volume preserving and topologically transitive (for example, if $f$ is ergodic with respect to volume), then $g$ is volume preserving as well.

Proof. The commutativity implies that $\text{vol}_M$ and $g^*(\text{vol}_M)$ are both $f$-invariant measures. Since $g$ is $C^1$, the induced Radon-Nikodym derivative $\frac{d(g^*\text{vol}_M)}{d\text{vol}_M}$ is an $f$-invariant continuous function. Then by transitivity of $f$ we obtain that this derivative is constant and equal to the degree of $g$, which is 1. Thus $g^*(\text{vol}_M) = \text{vol}_M$. □

If $f$ and $g$ are commuting diffeomorphisms, then their derivatives commute as well. It follows that if $f(p) = p$, then the derivative of $f$ at $p$ is conjugate to its derivative at $g(p)$, and so $D_pf$ and $D_{g(p)}f$ have the same eigenvalues. More generally:

Lemma 12. Let $M$ be a closed manifold, and suppose that $f, g \in \text{Diff}(M)$ satisfy $fg = gf$. If $\mu$ is an ergodic invariant measure for $f$, then the Lyapunov exponents of $\mu$ are the same as the Lyapunov exponents of $g^*\mu$.

Applying the same principle to the invariant subbundles in a dominated splitting, we obtain
Lemma 13. Let \( M \) be a closed manifold, and suppose that \( f, g \in \text{Diff}(M) \) satisfy \( fg = gf \). If \( f \) preserves a dominated splitting

\[
TM = E^1 \oplus \cdots \oplus E^\ell,
\]

then so does \( g \). Moreover if, for some \( i \), \( W^i \) is the unique \( f \)-invariant foliation tangent to the bundle \( E^i \) in this dominated splitting, then \( g(W^i) = W^i \).

Proof. For \( v \in TM \setminus \{0\} \), denote the limits

\[
\limsup_{n \to \pm \infty} \frac{1}{n} \log \|Df^n(v)\|, \quad \liminf_{n \to \pm \infty} \frac{1}{n} \log \|Df^n(v)\|
\]

by \( \bar{\lambda}^\pm(v) \), \( \underline{\lambda}^\pm(v) \) respectively. It is not hard to check that if \( fg = gf \) then \( \bar{\lambda}^\pm, \underline{\lambda}^\pm \) are \( Dg \)-invariant functions. Moreover, since \( E^1 \oplus \cdots \oplus E^\ell \) is a dominated splitting, there are \( \ell \) disjoint bands (closed intervals) \([a_i, b_i], 1 \leq i \leq \ell\) such that for any \( v \in TM \setminus \{0\} \),

\[
v \in E^i \iff \bar{\lambda}^\pm(v), \underline{\lambda}^\pm(v) \in [a_i, b_i].
\]

It follows that each \( E^i \) is \( Dg \)-invariant.

Clearly if \( W^i \) is \( f \)-invariant then so is \( g(W^i) \). By \( Dg \)-invariance of \( E^i \) we obtain that \( g(W^i) \) is an \( f \)-invariant foliation that is tangent to \( E^i \). The uniqueness assumption in the lemma then implies that \( g(W^i) = W^i \). \( \square \)

Lemma 14. Let \( \varphi_t \) be a \( C^1 \) Anosov flow, and suppose that \( \varphi_\alpha \) is transitive, for some \( \alpha \neq 0 \) (this will hold for every nonzero \( \alpha \) if \( \varphi_t \) is mixing). If \( g \) is a \( C^1 \) diffeomorphism that commutes with \( \varphi_\alpha \), then \( g \) commutes with \( \varphi_t \), for all \( t \).

Proof. By the Anosov property of \( \varphi_t \), the vector field \( \dot{\varphi} \) is the unique vector field up to scaling that is preserved by \( D\varphi_t \). The partial hyperbolicity of \( \varphi_\alpha \) implies that if \( X \) is a vector field that is invariant under \( D\varphi_\alpha \), then \( X(x) = c(x)\dot{\varphi}(x) \). The fact that \( \varphi_\alpha \) has a dense orbit implies that \( c(x) \) must be constant.

Now if \( gf = fg \), then the vector field \( g_\alpha \dot{\varphi} \) is invariant under \( Df \). By the previous considerations, \( g_\alpha \dot{\varphi} \) is a constant multiple of \( \dot{\varphi} \), and so \( g \) commutes with \( \varphi_t \). \( \square \)

Sufficiently high regularity of a map plus some hyperbolicity can force high regularity of its centralizer. A basic motivating example is a linear map on \( \mathbb{R} \). If \( f(x) = 2x \) and \( fg = gf \), then \( g(0) = 0 \), and the commutativity of \( f \) and \( g \) implies that for all \( x \neq 0 \) and \( n \):

\[
\frac{g(x)}{x} = \frac{f^n g f^{-n}(x)}{x} = \frac{2^n}{x^n} g \left( \frac{x}{2^n} \right) = g \left( \frac{x}{2^n} \right) - g(0) \frac{x}{2^n}.
\]

If \( g \) is differentiable at 0, then the right hand side converges as \( n \to \infty \) to \( g'(0) \). Thus \( g(x) = g'(0) x \) is linear.

A more sophisticated illustration of this principle is the following result of Jungreis and Hirsch.

Theorem 10 ([49]). Suppose \( f : M \to M \) is a homeomorphism that commutes with a volume-preserving Anosov flow \( \varphi_t \) \( (f \circ \varphi_t = \varphi_t \circ f) \) for all \( t \in \mathbb{R} \), and such that \( f \) sends every periodic orbit of \( \varphi \) to itself. Then \( f = \varphi_t \) for some \( t \in \mathbb{R} \).
In the setting of linear Anosov diffeomorphisms, there is a similar result, due to Adler and Palais:

**Lemma 15.** Suppose $A \in \text{SL}(k, \mathbb{Z})$ does not have a root of unity as an eigenvalue, and let $T_A$ be the induced automorphism of $\mathbb{T}^k$. Suppose $h : \mathbb{T}^k \to \mathbb{T}^k$ is a homeomorphism such that $T_A h = h T_A$. Then $h$ is affine and $h(0) \in \mathbb{Q}^k/\mathbb{Z}^k$.

For such toral automorphisms $T_A$, we thus have

$$Z_{\text{Homeo}(\mathbb{T}^k)}(T_A) \subseteq \{ x \mapsto T_L x + b : L \in Z_{\text{SL}(k, \mathbb{Z})}(A), \ b \in \mathbb{Q}^k/\mathbb{Z}^k \}.$$  

When $A$ is irreducible, the linear part of the right hand side can be computed using the following lemma.

**Lemma 16.** Let $A \in \text{SL}(k, \mathbb{Z})$ be a matrix with characteristic polynomial irreducible over $\mathbb{Z}$. Denote by $Z_{\text{SL}(k, \mathbb{Z})}(A)$ the centralizer of $A$ in $\text{SL}(k, \mathbb{Z})$. Then $Z_{\text{SL}(k, \mathbb{Z})}(A)$ is abelian, and in fact it is virtually $\mathbb{Z}^r + c - 1$ where $r$ is the number of real eigenvalues and $c$ is the number of pairs of complex eigenvalues, $r + 2c = k$.

**Proof.** This follows from the Dirichlet unit theorem, or cf. Proposition 3.7 in [49] \qed  

3.4. More on partial hyperbolicity. In this section we discuss fundamental concepts in the study of partially hyperbolic diffeomorphisms: normal hyperbolicity, leaf conjugacy, center bunching, and accessibility. Along the way, we state and prove in Proposition 20 a key criterion for invariance of center foliations under the centralizer. We also discuss some results of Avila–Viana–Wilkinson [2, 3] and Hammerlindl [37] that we use in this paper.

3.4.1. Normal hyperbolicity. Suppose $M$ is closed manifold, and let $f_1, f_2 \in \text{Diff}(M)$. Assume that $F_1, F_2$ are foliations of $M$ with $C^1$ leaves and that $f_1$ and $f_2$ respectively preserve $F_1$ and $F_2$.

**Definition 1.** A leaf conjugacy from $(f_1, F_1)$ to $(f_2, F_2)$ is a homeomorphism $h : M \to M$ sending $F_1$ leaves diffeomorphically onto $F_2$ leaves, equivariantly in the sense that

$$h(f_1(F_1(p))) = f_2(F_2(h(p))), \ \forall p \in M.$$  

**Definition 2.** Suppose $f \in \text{Diff}(M)$ and $F$ is an $f$–invariant foliation of $M$ with $C^1$ leaves. $F$ is normally hyperbolic if there exists a $Df$–invariant dominated splitting $TM = E^u \oplus E^c \oplus E^s$, with at least two of the bundles nontrivial, such that $Df$ uniformly expands $E^u$, uniformly contracts $E^s$, and such that $T F = E^c$.

Note that a diffeomorphism with a normally hyperbolic foliation is partially hyperbolic, with $E^c = T F$, but, as remarked above, the converse does not hold in general: the center bundle of a partially hyperbolic diffeomorphism is not necessarily tangent to a foliation, let alone an invariant foliation.

**Definition 3.** A partially hyperbolic diffeomorphism $f$ is dynamically coherent if there exist $f$–invariant center stable and center unstable foliations $W^{cs}$ and $W^{cu}$, tangent to the bundles $E^{cs}$ and $E^{cu}$, respectively; intersecting their leaves gives an invariant center foliation $W^c$. 


3.4.2. Foliations with compact leaves. An important family of examples are the partially hyperbolic systems with compact center foliations. We say that a foliation $\mathcal{F}$ of a closed manifold $M$ has compact leaves if every leaf of $\mathcal{F}$ is compact. Foliations with compact leaves can be rather complicated; for example, there exist closed manifolds foliated by circles whose length is unbounded [84, 30]. A foliation $\mathcal{F}$ has uniformly compact leaves if there exists a constant $C > 0$ such that the restricted Riemannian volume of every leaf $\mathcal{F}$ is bounded by $C$, with respect to some (any) Riemannian metric on $M$. The simplest foliations with uniformly compact leaves are fibrations, but there are many others.

In many of the cases of interest here, the center foliation $\mathcal{W}_c^c$ of a partially hyperbolic diffeomorphism $f$ has compact leaves, and the simplest cases, these leaves form a fibration, meaning that $f$ is a fibered dynamical system. We distinguish between several cases of such fibered systems.

**Definition 4.** Let $f$ be a partially hyperbolic diffeomorphism of a closed manifold $M$. Assume that there exists an $f$–invariant compact center foliation $\mathcal{W}_c^c$.

- If $\mathcal{W}_c^c$ is a topological fibration of $M$, i.e. the quotient space $M/\mathcal{W}_c^c$ is a topological manifold, then $f$ is called a fibered partially hyperbolic system, and the map $\bar{f} : M/\mathcal{W}_c^c \to M/\mathcal{W}_c^c$ canonically induced by $f$ is called the base map.
- A fibered partially hyperbolic system $f$ is smoothly fibered (or $C^r$–fibered, for $r \in \mathbb{R}^+$) if $\mathcal{W}_c^c$ is a $C^\infty$ (respectively $C^r$) foliation, and $f$ is $C^\infty$ (resp. $C^r$).
- A fibered partially hyperbolic system $f$ is isometrically fibered if there is a continuous Riemannian metric on $E^c$ such that $Df|_{E^c}$ is an isometry.
- An isometrically fibered partially hyperbolic system $f$ is an isometric extension (or smoothly isometrically fibered) if $f$ is smoothly fibered.

3.4.3. Plaque expansiveness and structural stability. A central result in [39] concerns perturbations of normally hyperbolic systems. It provides techniques to study integrability of the central distribution and robustness of the central foliation for partially hyperbolic systems. To state this result, we need to define some preliminary concepts.

**Definition 5.** A plaquation of a $d$–dimensional foliation $\mathcal{F}$ of an $n$–dimensional compact manifold $M$ is determined by a choice of finitely many foliation boxes with uniform size: $B_d(0,1) \times B_{n-d}(0,1)$ such that the corresponding half size foliation boxes $B_d(0,\frac{1}{2}) \times B_{n-d}(0,\frac{1}{2})$ cover $M$ as well. The plaquation $\mathcal{P}$ consists of the unit size plaques (local leaves) $(B_d(0,1) \times \{y\})$. They cover the leaves of $\mathcal{F}$ in a uniform way.

Recall that a $\delta$-pseudo orbit of $f$ is a bi-infinite sequence of points $(x_n)$ such that for each $n \in \mathbb{Z}$, $d(f(x_n), x_{n+1}) < \delta$. It respects the plaquation $\mathcal{P}$ if $f(x_n)$ and $x_{n+1}$ always belong to a common plaque in $\mathcal{P}$.

An $f$–invariant foliation $\mathcal{F}$ is plaque expansive if there exist a plaquation $\mathcal{P}$ of $\mathcal{F}$ and a $\delta > 0$ such that any two $\delta$-pseudo orbits of $f$ that respect $\mathcal{P}$ and $\delta$–shadow each other necessarily belong to the same plaques of $\mathcal{P}$. The property of plaque expansiveness is independent of the choice of plaquation and Riemannian metric (see the discussion in [39], Chapter 7).

---

6 Or, equivalently, if $\mathcal{W}_c^c$ has trivial holonomy; see [30].
To study the precise smoothness of the leaves of a normally hyperbolic foliation, we refine the definition of normal hyperbolicity. For \( r \geq 1 \) we say that \((f, F)\) is \( r \)-normally hyperbolic if there exists \( k \geq 1 \) such that
\[
\sup_p \|D_p f^k|_{E^s}\| \cdot \|(D_p f^k|_{T F})^{-1}\| < 1, \quad \text{and} \quad \sup_p \|(D_p f^k|_{E^u})^{-1}\| \cdot \|D_p f^k|_{T F}\| < 1.
\]
Note that 1-normally hyperbolic = normally hyperbolic, and \( r \)-normal hyperbolicity is a \( C^1 \)-open condition.

**Theorem 11** (Foliation Stability and Hölder continuity of the leaf conjugacy). Let \( M \) be a closed manifold, and let \((f, F)\) be an \( r \)-normally hyperbolic foliation of \( M \), for some \( r \geq 1 \), with \( Df \)-invariant splitting \( E^u \oplus (T F = E^c) \oplus E^s \). Then the leaves of \( F \) are uniformly \( C^r \), and we have the following.

1. Suppose that one of the following holds:
   - (a) the restriction \( Df|_{T F} \) is an isometry, or
   - (b) the bundles \( E^{cu} \) and \( E^{cs} \) are \( C^1 \), or
   - (c) \( F \) is uniformly compact.
   Then \( f \) is dynamically coherent, and plaque expansive and \( r \)-normally hyperbolic with respect to the foliations \( W^{cu}, W^{cs} \) and \( F = W^{cu} \cap W^{cs} \).

2. If \((f, F)\) is plaque expansive then it is structurally stable in the following sense. For each diffeomorphism \( g \) that \( C^1 \)-approximates \( f \), there exists a unique \( g \)-invariant foliation \( \mathcal{F}_g \) (with \( C^1 \)-leaves) near \( F \). The foliation \( \mathcal{F}_g \) is normally hyperbolic, plaque expansive, and \((f, F)\) is leaf conjugate to \((g, \mathcal{F}_g)\) by a homeomorphism \( h^c: M \to M \) close to the identity.

3. If in addition, condition (b) or (c) holds in item (1), then the leaf conjugacy \( h^c \) in (2) is Hölder continuous.

4. If in addition, condition (a), (b) or (c) holds in item (1), then \( g \) is dynamically coherent.

**Proof.** (1): Under assumptions (a) or (b), this is proved in [39, Theorems 7.5 and 7.6] (see also Remark 4 on p. 117). Under assumption (c) this is proved in [6, Theorem 1.26]. See the discussion in [68, Section 3].

(2): The main stability result is proved in [39, Theorem 7.1].

(3): Hölder continuity of \( h^c \) is proved in [68, Theorems A and B].

(4): The local center-stable and unstable plaques are dynamically characterized and preserved under leaf conjugacy; they intersect in the center plaques ([39, Chapter 7]). By item (1), the hypotheses of (4) imply that these plaques determine unique foliations that are also plaque expansive. Applying (2) we obtain center-stable and center-unstable foliations for \( g \) intersecting in the unique \( g \)-invariant foliation for near \( F \), which is \( \mathcal{F}_g \). This implies dynamical coherence of \( g \).

**3.4.4. Criteria for \( Z(f) \)-invariance of \( f \)-invariant foliations.** An obvious step in analyzing the centralizer of a partially hyperbolic diffeomorphism \( f \) is to examine its action on the \( f \)-invariant foliations. Lemma [13] gives a criterion for when a diffeomorphisms \( g \) commuting
with \( f \) preserves an \( f \)-invariant foliation: the foliation \( F \) should be the unique \( f \)-invariant foliation tangent to an invariant bundle in a dominated splitting. Since the stable and unstable bundles are uniquely integrable, this gives

**Lemma 17.** Let \( f \) be a partially hyperbolic diffeomorphism. If \( g \) is a diffeomorphism commuting with \( f \), then \( gW_f^u = W_f^u \), and \( gW_f^s = W_f^s \).

When \( f \) is dynamically coherent, it is natural to try to apply Lemma 13 to the \( f \)-invariant, normally hyperbolic foliations \( W_f^{cu} \), \( W_f^{cs} \), and \( W_f^c \) as well. The problem is verifying the uniqueness hypotheses. There is no known criterion for unique integrability of a normally hyperbolic foliation other than smoothness, and smoothness is not stable under perturbations. Even adding \( f \)-invariance does not help: it is a priori possible that the tangent bundle to normally hyperbolic foliation might be tangent to two invariant foliations. Thus we seek additional criteria to obtain invariance of \( W_f^{cu} \), \( W_f^{cs} \), and \( W_f^c \) under the centralizer of \( f \).

Our first lemma shows that it suffices to prove invariance of \( W_f^c \) under the centralizer of \( f \).

**Lemma 18.** Let \( f \) be a partially hyperbolic, dynamically coherent diffeomorphism of a closed manifold \( M \). If \( g \in \text{Diff}(M) \) preserves both \( W_f^c \) and \( W_f^u \), then \( gW_f^{cu} = W_f^{cu} \).

**Proof.** Fix \( x, y \) in the same local \( W^{cu} \) leaf. We need only show that \( g(x) \) and \( g(y) \) lie in the same \( W^{cu} \) leaf. By Proposition 2.4 of [16], we have that \( W^c \) and \( W^u \) subfoliate \( W^{cu} \), and so \( W^u(x, \text{loc}) \) intersects \( W^c(y, \text{loc}) \) in a unique point \( z \). Now \( g \)-invariance of \( W^c \) and \( W^u \) implies that \( g(y) \in W^u(g(z), \text{loc}) = W^u(W^c(x, \text{loc}), \text{loc}) \). Since \( W^c \) and \( W^u \) subfoliate \( W^{cu} \), we conclude that \( g(y) \in W^{cu}(g(x)) \). \( \square \)

We now develop a new criterion for invariance of a normally hyperbolic foliation under the centralizer, which we will eventually apply to \( W^c \).

**Definition 6.** Let \( (f, F) \) be a normally hyperbolic foliation of a closed manifold \( M \). We say that \( f \) is pathwise center Lyapunov stable if for any \( x, y \) in the same \( F \) leaf, there is a \( C^1 \) path \( \gamma \) connecting \( x \) and \( y \) such that

\[
\sup_{n \in \mathbb{Z}} |f^n(\gamma)| < +\infty.
\]

We say that \( f \) is stably pathwise center Lyapunov stable if it is plaque expansive, and for any diffeomorphism \( g \) that \( C^1 \)-approximates \( f \), the associated unique \( g \)-invariant foliation \( F_g \) defined by Theorem 11 is pathwise center Lyapunov stable.

**Proposition 19.** Let \( (f, F) \) be a 1-dimensional normally hyperbolic foliation of a closed manifold \( M \). Then \( f \) is stably center pathwise Lyapunov stable if \( f \) satisfies one of the following conditions:

(a) \( F \) is uniformly compact, or
(b) \( f = \psi_{t_0} \) is the time-\( t_0 \) map of an Anosov flow \( \psi_t \) and the lift \( \tilde{\psi}_t \) of \( \psi_t \) to the universal cover \( \tilde{M} \) has no closed orbits.
Proof. Suppose that (a) holds. Since $\mathcal{F}$ is uniformly compact and 1–dimensional, the leaves of $\mathcal{F}$ have uniformly bounded length, which easily implies that $f$ is center pathwise Lyapunov stable. By Theorem 11 any $g \in \text{Diff}(M)$ that $C^1$ approximates $f$ is partially hyperbolic with a 1–dimensional invariant uniformly compact center foliation $W_c^c$, and therefore $g$ is center pathwise Lyapunov stable as well which implies Proposition 10.

Now consider the case that (b) holds: $g$ is a $C^1$–perturbation of $f = \psi_{t_0}$ on $M$. Since $Df|_{T\mathcal{F}}$ is an isometry, Theorem 11 implies that $g$ is dynamically coherent; hence $g$ is uniformly close to a translation by $t_0$. Consider the lifts $\hat{\psi}_t, \hat{g}$ of $\psi_t, g$ respectively to $\hat{M}$, where $\hat{g}$ is uniformly $C^1$–close to $\hat{\psi}_t$, and $\hat{g}$ preserves the lift $\hat{\mathcal{F}}$ of $\mathcal{F}_g$. On each $\hat{\mathcal{F}}_g$–leaf, the action of $\hat{g}$ is uniformly close to a translation by $t_0$ on $\mathbb{R}$. Therefore it is topologically conjugate to a translation.

For $x, y$ in the same $\mathcal{F}_g$ leaf, fix a $C^1$–embedded path $\gamma : [0, 1] \to \mathcal{F}_g(x)$ connecting $x$ and $y$, and consider an arbitrary lift $\hat{\gamma}$ of $\gamma$ to $\hat{M}$. Since the action of $\hat{g}$ on $\hat{\mathcal{F}}_g(x)$ is topologically conjugate to a translation, there exists $j \in \mathbb{Z}$ such that $\hat{\gamma}(1)$ must lie on some oriented $\hat{\mathcal{F}}_g$–arc $[\hat{g}^j(\hat{\gamma}(0)), \hat{g}^{j+1}(\hat{\gamma}(0))]$. As a consequence, for any $n$, $|\hat{g}^n(\hat{\gamma})|$ (hence $|g^n(\gamma)|$) can not be greater than

$$
\begin{aligned}
\max_{k \in \mathbb{Z}} \text{the central arc } [\hat{g}^k(\hat{\gamma}(0)), \hat{g}^{k+j}(\hat{\gamma}(0))], & \text{ if } j \geq 0 \\
\max_{k \in \mathbb{Z}} \text{the central arc } [\hat{g}^k(\hat{\gamma}(0)), \hat{g}^{k+j}(\hat{\gamma}(0))], & \text{ if } j < 0
\end{aligned}
$$

Since $\hat{g}$ is uniformly $C^1$–close to $\hat{\psi}_{t_0}$, the terms in (1) are finite. Therefore $g$ is center pathwise Lyapunov stable.

Recall that the distance between nonempty subsets $F_1, F_2$ of a metric space $(X, d)$ is defined by

$$
d_X(F_1, F_2) = \inf_{x \in F_1, y \in F_2} d_X(x, y).
$$

Definition 7. Suppose $(f, \mathcal{F})$ is a normally hyperbolic foliation of a compact manifold $M$. We say $f$ is globally center expansive if some (and hence, any) lift $(\hat{f}, \hat{\mathcal{F}})$ of $(f, \mathcal{F})$ to the universal cover $\hat{M}$ satisfies the following property:

- For any $x, y \in \hat{M}$, $\limsup_{n \to \pm \infty} d_M(\hat{\mathcal{F}}(\hat{f}^n(x))), \hat{\mathcal{F}}(\hat{f}^n(y))) < \infty$ (if and only if $\hat{F}(x) = \hat{F}(y)$).

We say that $f$ is stably globally center expansive if it is plaque expansive and any diffeomorphism $g$ that $C^1$ approximates $f$ is globally center expansive with respect to the unique continuation $\mathcal{F}_g$ defined by Theorem 11.

Proposition 20. Let $(f, \mathcal{F})$ be a normally hyperbolic foliation of a closed manifold $M$. If $f$ is globally center expansive and pathwise center Lyapunov stable, then for any $g \in \mathcal{Z}_{\text{Diff}(M)}(f)$, we have $g\mathcal{F} = \mathcal{F}$.

Proof. Fix $x$ and $y$ in the same $\mathcal{F}$–leaf. We want to show that $g(x)$ and $g(y)$ lie in the same $F$–leaf. Pathwise center Lyapunov stability implies there exists a $C^1$ path $\gamma_0 : I \to M$.
connecting \( x \) and \( y \) such that the lengths \( |f^n(\gamma_0)|, n \in \mathbb{Z} \) are uniformly bounded. We show that \( \gamma := g \circ \gamma_0, \gamma(0), \gamma(1) \) are in the same leaf of \( \mathcal{F} \). First observe that the lengths
\[
|f^n(\gamma)| = |f^n g(\gamma_0)| = |g(f^n(\gamma_0))| \leq \|g\|_{\text{C}^1} \cdot |f^n(\gamma_0)|, \forall n \in \mathbb{Z}
\]
are also uniformly bounded.

Now consider the universal cover \( \tilde{M} \) of \( M \) and (arbitrary) lifts \( \tilde{f}, \tilde{\gamma}, \tilde{\mathcal{F}} \) of \( f, \gamma, \mathcal{F} \) to \( \tilde{M} \) respectively. By (2), the lengths \( |\tilde{f}^n(\tilde{\gamma})| \) are uniformly bounded for \( n \in \mathbb{Z} \). As a consequence, \( \limsup_{n \to \pm \infty} d_{\tilde{M}}(\tilde{f}(\tilde{f}^n(\tilde{\gamma}(0))), \tilde{\mathcal{F}}(\tilde{f}^n(\tilde{\gamma}(1)))) < \infty \). Since \( f \) is globally center expansive, \( \tilde{\gamma}(0), \tilde{\gamma}(1) \) must lie on the same \( \tilde{\mathcal{F}} \)–leaf, which implies the desired conclusion.

An immediate consequence is

**Corollary 21.** Let \( (f_0, \mathcal{F}_0) \) be a normally hyperbolic foliation of a closed manifold \( M \). Assume that \( f_0 \) is stably globally center expansive and stably pathwise center Lyapunov stable. Suppose that \( f \) that \( C^1 \)–approximates \( f_0 \), and denote by \( \mathcal{F}_f \) the center foliation of \( f \) provided by Theorem \([11] \). Then for any \( g \in \mathcal{Z}_{\text{Diff}(M)}(f) \), we have \( g\mathcal{F}_f = \mathcal{F}_f \).

We have the following criterion for stable global center expansiveness.

**Proposition 22.** Let \( (f, \mathcal{F}) \) be a normally hyperbolic foliation of a closed manifold \( M \). Then \( f \) is stably globally center expansive if and only if \( f \) is globally center expansive and plaque expansive.

**Proof.** For any \( g \) that \( C^1 \) approximates \( f \), Theorem \([11] \) gives a \( g \)–invariant normally hyperbolic foliation \( \mathcal{F}_g \) of \( M \) and a leaf conjugacy \( h^c : M \to M \) between \( (f, \mathcal{F}) \) and \( (g, \mathcal{F}_g) \) that is uniformly close to \( \text{id}_M \). Let \( \tilde{f}, \tilde{\mathcal{F}}, \tilde{g}, \tilde{\mathcal{F}}_g, \tilde{h}^c \) be lifts of \( f, \mathcal{F}, g, \mathcal{F}_g, h^c \), respectively, to the universal cover \( \tilde{M} \) such that \( \tilde{h}^c, \tilde{h}^{c-1} \) are \( \epsilon \)–close to \( \text{id}_{\tilde{M}} \) for some \( \epsilon > 0 \), and \( \tilde{h}^c \) is a leaf conjugacy between \( (\tilde{f}, \tilde{\mathcal{F}}) \) and \( (\tilde{g}, \tilde{\mathcal{F}}_g) \).

Suppose \( x, y \in \tilde{M} \) satisfy \( \limsup_{n \to \pm \infty} d_{\tilde{M}}(\tilde{F}_g(\tilde{g}^n(x)), \tilde{F}_g(\tilde{g}^n(y))) < \infty \). Then there exists \( D > 0 \) and points \( x_n, y_n \) for \( n \in \mathbb{Z} \) such that for all \( n \in \mathbb{Z} \) we have
\[
(3) \quad x_n \in \tilde{F}_g(\tilde{g}^n(x)), \quad y_n \in \tilde{F}_g(\tilde{g}^n(y)), \quad \text{and}
(4) \quad d_{\tilde{M}}(x_n, y_n) < D.
\]
Since \( \tilde{h}^c, \tilde{h}^{c-1} \) are \( \epsilon \)–close to \( \text{id}_{\tilde{M}} \), equation (4) implies that for any \( n \in \mathbb{Z} \),
\[
d_{\tilde{M}}(\tilde{h}^{c-1}(x_n), \tilde{h}^{c-1}(y_n)) < D + 2\epsilon,
\]
which implies that
\[
(5) \quad d_{\tilde{M}}(\tilde{F}(\tilde{h}^{c-1}(x_n)), \tilde{F}(\tilde{h}^{c-1}(y_n))) < D + 2\epsilon.
\]
Since \( \tilde{h}^c \) is a leaf conjugacy from \( (\tilde{f}, \tilde{\mathcal{F}}) \) to \( (\tilde{g}, \tilde{\mathcal{F}}_g) \), by (3) we have that \( \tilde{h}^{c-1}(x_n) \in \tilde{F}(\tilde{f}^{n}(\tilde{h}^{c-1}(x))), \tilde{h}^{c-1}(y_n) \in \tilde{F}(\tilde{f}^{n}(\tilde{h}^{c-1}(y))). \) Combining this with (5), we obtain that
\[
d_{\tilde{M}}(\tilde{F}(\tilde{f}^{n}(\tilde{h}^{c-1}(x))), \tilde{F}(\tilde{f}^{n}(\tilde{h}^{c-1}(y)))) < D + 2\epsilon.
\]
Global center expansiveness of $f$ then implies that $\tilde{h}^{-1}(x), \tilde{h}^{-1}(y)$ must lie in the same $\tilde{F}$–leaf, and hence $x, y$ must lie in the same $\tilde{F}_g$–leaf since $\tilde{h}$ is a leaf conjugacy. □

**Proposition 23.** Let $f : M \to M$ satisfy one of the following conditions.

(a) $M = \mathbb{T}^d$, and $f$ is a $C^1$–small perturbation of an isometric extension of an Anosov diffeomorphism of $\mathbb{T}^{d-1}$;

(b) $M = T^1X$, where $X$ is a closed, negatively curved locally symmetric space, and $f$ is a $C^1$–small perturbation of the discretized geodesic flow $\psi_{t_0}$, for some $t_0 \neq 0$.

(c) $M$ is a 3-manifold, and $f$ is a partially hyperbolic diffeomorphism with an $f$–invariant compact center foliation $W^c_f$.

Then $f$ is dynamically coherent, and for any $g \in Z_{\text{Diff}(M)}(f)$ we have $gW^*_f = W^*_f$, for $* \in \{c, s, u, cs, cu\}$.

**Proof.** Note that in all three cases (a)–(c) dynamical coherence of $f$ follows from item (4) of Theorem 11.

Assume that $f$ satisfies the conditions of (a) or (b). Let $f_0$ be the isometric extension $C^1$-close to $f$, in case (a), and let $f_0 = \psi_{t_0}$ in case (b).

Lemma 17 implies that $gW^u_f = W^u_f$ and $gW^s_f = W^s_f$. By Corollary 21, to prove $gW^c_f = W^c_f$ we need only prove that $f_0$ is stably globally center expansive and stably pathwise center Lyapunov stable. The stable global center expansiveness of $f_0$ is an easy corollary of Proposition 22. Proposition 19 gives that $f_0$ is stably pathwise center Lyapunov stable. Then Lemma 18 implies that $gW^{cs}_f = W^{cs}_f$ and $gW^{cu}_f = W^{cu}_f$.

Now suppose that $f$ satisfies the conditions in (c). As above, we have that $gW^u_f = W^u_f$ and $gW^s_f = W^s_f$. By Lemma 13, we only need to prove that each of $W^*_f, * \in \{c, cs, cu\}$ is the unique $f$–invariant foliation tangent to corresponding distribution. By Proposition 24, there is a finite cover $\tilde{M}$ of $M$ such that the lift $f$ of $f$ on $\tilde{M}$ is a fibered partially hyperbolic system, and $M$ is a circle bundle over $\mathbb{T}^2$.

If the circle bundle is trivial then $\tilde{M}$ is a smooth torus (notice that there is no exotic smooth structure in dimension $\leq 3$). Theorem 1.1 in [11] implies that the lift of $E^c, E^{cs}, E^{cu}$ to $\tilde{M}$ are uniquely integrable, and hence for arbitrary $* \in \{c, cs, cu\}$ the lifted foliation $\tilde{W}^*$ of $W^*$ is the unique $f$–invariant foliation tangent to the lift of the distribution $E^*$, which implies that $W^*$ is the unique $f$–invariant foliation tangent to $E^*$.

If the circle bundle is non-trivial then by [38] Theorem 1.2, for arbitrary $* \in \{c, cs, cu\}$ the lifted foliation $\tilde{W}^*$ of $W^*$ is the unique $f$–invariant foliation tangent to the lift of the distribution $E^*$, which implies that $W^*$ is the unique $f$–invariant foliation tangent to $E^*$. □

3.4.5. Center bunching conditions. For $r \geq 1$, we say that a partially hyperbolic diffeomorphism $f$ is $r$–bunched if there exist $k \geq 1$ and a Riemannian metric on $M$ such
\[
\sup_p \left\{ \|D_p f^k|_{E^s}\| \cdot \|(D_p f^k|_{E^c})^{-1}\| \cdot \|D_p f^k|_{E^c}\| \right\} < 1,
\]

\[
\sup_p \|D_p f^k|_{E^s}\| \cdot \|(D_p f^k|_{E^c})^{-1}\| \cdot \|D_p f^k|_{E^c}\| < 1, \quad \text{and}
\]

\[
\sup_p \|(D_p f^k|_{E^s})^{-1}\| \cdot \|D_p f^k|_{E^c}\| \cdot \|(D_p f^k|_{E^c})^{-1}\| < 1.
\]

When \( f \) is \( C^r \) and dynamically coherent, the first of these three inequalities is \( r \)-normal hyperbolicity and implies that the leaves \( W_{cs}, W_{cu} \) are \( C^r \). If \( f \) is \( C^{r+1} \) and dynamically coherent they also imply the stable and unstable holonomy and \( E^s, E^u \) are \( C^r \) along \( W^c \), cf. \([68, 87]\). We say that \( f \) is center bunched if it is \( 1-bunched \). If \( E^c \) is 1-dimensional, then \( f \) is automatically center bunched. All systems we consider here have 1-dimensional center and thus are center bunched.

### 3.4.6. Accessibility

A partially hyperbolic diffeomorphism \( f : M \to M \) is accessible if any two points \( x, y \) in \( M \) can be joined by an \( su \)-path, that is, a piecewise \( C^1 \) path such that every piece is contained in a single leaf of \( W^u_f \) or a single leaf of \( W^s_f \). It is easy to see that if \( E^u_f \) and \( E^s_f \) are jointly integrable (i.e. there exists an \( f \)-invariant foliation \( W \) with \( C^1 \) leaves everywhere tangent to the bundle \( E^u \oplus E^s \)) then \( f \) fails to be accessible.

Pugh and Shub conjectured that if \( f \in \text{Diff}^{2}_{\text{vol}}(M) \) is partially hyperbolic and accessible, then \( f \) is ergodic. This was proved for center bunched \( f \) by Burns–Wilkinson \([14]\). In particular, accessibility implies ergodicity for systems with 1-dimensional center bundle, and stable accessibility — i.e., accessibility that persists under \( C^1 \)-small perturbations — implies stable ergodicity.

Pugh and Shub also conjectured that stable accessibility is a dense property among \( C^r \) partially hyperbolic diffeomorphisms, volume-preserving or not. Dolgopyat–Wilkinson \([27]\) proved \( C^1 \) density of stable accessibility among all \( C^r \) partially hyperbolic diffeomorphisms, and Hertz-Hertz-Ures \([72]\) proved \( C^r \) density (for any \( r \)) among the systems with 1-dimensional center foliation.

### 3.4.7. Rigidity and Lebesque disintegration

The following rigidity result for volume preserving partially hyperbolic diffeomorphisms on 3–manifolds is proved by Avila–Viana–Wilkinson in \([3]\). We will use it later in the proof of Theorems 7 and 8.

**Theorem 12.** Let \( M \) be a closed 3-manifold and let \( f \in \text{Diff}^{\infty}_{\text{vol}}(M) \) be partially hyperbolic and ergodic. Assume that there exists a foliation \( W^c \), invariant under \( f \), whose leaves are all circles.

If \( \text{vol}_M \) has Lebesgue disintegration along \( W^c \), then up to a 4–fold covering, \( f \) is \( C^\infty \) conjugate to an isometric extension of a volume preserving Anosov diffeomorphism on \( \mathbb{T}^2 \). In particular, \( W^c \) is \( C^\infty \).
3.4.8. \textit{AB-systems}. The partially hyperbolic diffeomorphisms in Theorems 2, 4, 6 and 7 all belong to a class of diffeomorphisms called \textit{AB-systems}, a notion introduced by Hammerlindl in [37].

To define an \textit{AB-system}, we start with automorphisms $A$ and $B$ of a compact nilmanifold $N$ such that $A$ is hyperbolic, and $AB = BA$. Then $A$ and $B$ define a diffeomorphism $f_{AB} : M_B \to M_B, (v,t) \mapsto (Av,t)$ on the manifold $M_B = N \times \mathbb{R}/(v,t) \sim (Bv,t - 1)$. Such $f_{AB}$ is called an \textit{AB-prototype}. It is partially hyperbolic with 1-dimensional integrable center bundle spanned by the vector field $\partial/\partial t$.

A partially hyperbolic diffeomorphism $f : M_B \to M_B$ is an \textit{AB-system} if it has a 1-dimensional, orientable center foliation and is leaf conjugate to an \textit{AB-prototype}, via a leaf conjugacy that is orientation preserving on center leaves. Fibered partially hyperbolic systems with trivial circle center foliations (over a hyperbolic automorphism of a nilmanifold) (see Section 3.4.2) correspond to \textit{AB-systems} for which $B = \text{id}$. The suspensions of Anosov diffeomorphisms correspond to the case $A = B$. And these are not the only cases. In [37], Hammerlindl proved the following useful result for \textit{AB-systems}:

\textbf{Theorem 13.} [Theorem 2.2 in [37]] Suppose $B$ is an automorphism of a closed nilmanifold $N$, and $M_B$ is equipped with a smooth volume $\text{vol}$. Let $f \in \text{Diff}^2_{\text{vol}}(M_B)$ be an \textit{AB-system}. Then one of the following holds

1. $f$ is accessible and stably ergodic.
2. $E^u$ and $E^s$ are jointly integrable and $f$ is topologically conjugate to the map $(v,t) \mapsto (Av,t + \theta)$ on $M_B$, for some $\theta \in \mathbb{R}$. Moreover $f$ is ergodic if and only if $\theta$ is irrational.
3. There exist $n \geq 1$, a $C^1$ surjection $p : M_B \to S^1$, and a non-empty open set $U \subset S^1$ such that
   - for every connected component $I$ of $U$, $p^{-1}(I)$ is an $f^n$-invariant subset homeomorphic to $N \times I$, and the restriction of $f^n$ to $p^{-1}(I)$ is accessible and ergodic, and
   - for every $t \in S^1 \setminus U$, $p^{-1}(t)$ is an $f^n$-invariant submanifold tangent to $E^u \oplus E^s$ and homeomorphic to $N$.

It is natural to ask when a fibered partially hyperbolic system is an \textit{AB-system}. Hammerlindl answered this question in [37].

\textbf{Theorem 14.} [Proposition 3.2, Theorem 3.3, Corollary 3.4 in [37]] Suppose $f$ is a fibered partially hyperbolic diffeomorphism of a closed manifold $M$ where the base map $\bar{f}$ is topologically conjugate to a hyperbolic automorphism of a nilmanifold. If $E^c_f$ is 1-dimensional and orientable, then:

1. If $f$ preserves the orientation on $E^c$, then $f$ is an \textit{AB-system} if and only if the fiber bundle $M \to M/W^c_f$ is trivial. In this case, if $f$ is a volume preserving and $C^2$ then $f$ satisfies one of the three cases in Theorem 13.
2. If $f$ is not accessible, then the fiber bundle $M \to M/W^c_f$ is trivial.
The following proposition on the fibration structure and disintegration of volume for the partially hyperbolic diffeomorphisms on 3-manifolds is a key step in the proofs of Theorems 7, 8.

**Proposition 24.** Let $M$ be a 3-manifold, and let $f : M \to M$ be a partially hyperbolic diffeomorphism. Assume that there exists a foliation $W^c$, invariant under $f$, whose leaves are all circles. Then:

1. There is a finite cover (at most 4-fold) $\hat{M}$ of $M$ such that the lifts of $E^u, E^c$ to $\hat{M}$ are orientable and any $g \in \text{Diff}(M)$ lifts to $\hat{g} \in \text{Diff}(\hat{M})$.

2. There is an equivariant fibration $\pi : \hat{M} \to T^2$ such that $\pi \circ \hat{f} = T_{A_f} \circ \pi$, where $A_f \in \text{SL}(2, \mathbb{Z})$ is hyperbolic. The fibers of $\pi$ are the leaves of the foliation $\hat{W}^c$, which is the lift of $W^c$. In particular, the lift $\hat{f}$ on $\hat{M}$ is a fibered partially hyperbolic system.

3. If $f \in \text{Diff}^2_{\text{vol}}(M)$ and the disintegration of volume along $W^c$ leaves is not Lebesgue, then one or both of the following holds:

   • there exists a full volume set $S \subset \hat{M}$ and $k \in \mathbb{N}$ such that $S$ meets every leaf of $\hat{W}^c$ in exactly $k$ points, i.e. $\text{vol}_M$ has atomic disintegration along $\hat{W}^c$; or

   • $\hat{f}$ is ergodic and topologically conjugate to $\tau : \mathbb{T}^{d-1} \times \mathbb{T} \to \mathbb{T}^{d-1} \times \mathbb{T}$, for some $\theta \notin \mathbb{Q}$.

**Proof.**

1. If $E^c$ and $E^u$ are both orientable we can take $\hat{M} = M$; otherwise the construction of $\hat{M}$ is similar to the orientation covering of any non-orientable manifold. And it is easy to see that any $g \in \text{Diff}(M)$ can be lifted to $\hat{M}$.

2. is proved in Theorem A of [6].

3. If $\hat{f}$ is accessible, then by [2, 3] $\text{vol}_M$ has atomic disintegration along $W^c$.

If $\hat{f}$ is not accessible then by part (2) of Theorem 14 the fiber bundle $\hat{M} \to \hat{M}/\hat{W}^c$ is trivial, where $\hat{W}^c$ is the lift of $W^c$ to $\hat{M}$. But by (2) we know the base map induced by $\hat{f} : \hat{M} \to \hat{M}$ is conjugate to a hyperbolic automorphism of $\mathbb{T}^2$. Therefore by item (1) of Theorem 14 $\hat{f}$ is an AB-system and one of the three cases in Theorem 13 holds.

   • Since we assume $f$ is not accessible, the first case in Theorem 13 does not hold.

   • If the second case holds then $\hat{f}$ is topologically conjugate to $\tau : \mathbb{T}^{d-1} \times \mathbb{T} \to \mathbb{T}^{d-1} \times \mathbb{T}$. We claim that $\theta \notin \mathbb{Q}$. Otherwise we could easily construct an $\hat{f}$-invariant set $S \subset \hat{M}$ with arbitrary small positive volume. Denote by $p : \hat{M} \to M$ the covering map. Then we could pick $S$ such that $\text{vol}(p(S)) \neq 0, 1$, which contradicts the ergodicity of $f$. In particular by Theorem 13 $\hat{f}$ is ergodic as well.

   • If the third case in Theorem 13 holds, then there exists an open accessibility class $U$ in $M$ and a finite iterate of $f$ that fixes $U$ and is ergodic on $U$. In particular the disintegration of $\text{vol}_U$ along $\hat{W}^c$ must be atomic by [2, 3]. Therefore the disintegration of $\text{vol}$ along $W^c$ contains atoms, since the property of the disintegration of volume along a foliation containing atoms is local and pulls back through covering
maps. By ergodicity of \( f \), the volume \( \text{vol}_M \) must have pure atomic disintegration along \( \mathcal{W}^c \), which implies that \( \text{vol}_{\hat{M}} \) has pure atomic disintegration along \( \hat{\mathcal{W}}^c \). Therefore the first case in (3) of Proposition 24 holds. \[\Box\]

In the higher dimensional setting, we have similar result, which we will use in the proofs of in Theorems 4 and 6.

**Lemma 25.** Let \( f_0 \in \text{Diff}_\text{vol}^\infty(\mathbb{T}^d) \) satisfy the hypotheses of Theorem 4, and let \( f \in \text{Diff}_\text{vol}^\infty(\mathbb{T}^d) \) be a \( C^1 \)-small, ergodic perturbation of \( f_0 \). Then

1. \( f \) is a fibered partially hyperbolic system. There is an equivariant fibration \( \pi : \mathbb{T}^d \to \mathbb{T}^{d-1} \) such that \( \pi \circ f = T_{A_f} \circ \pi \), where \( A_f \in \text{SL}(d-1,\mathbb{Z}) \) is hyperbolic. The fibers of \( \pi \) are the leaves of the center foliation \( \mathcal{W}^c \) of \( M \) by circles, where \( \mathcal{W}^c \) can be constructed from Theorem 11.
2. If the disintegration of volume along \( \mathcal{W}^c \) leaves is not Lebesgue, then one or both of the following holds:
   - either there exists a full volume set \( S \subset \mathbb{T}^d \) and \( k \in \mathbb{N} \) such that \( S \) meets every leaf of \( \mathcal{W}^c \) in exactly \( k \) points, i.e. volume has atomic disintegration along \( \mathcal{W}^c \);
   - or \( f \) is topologically conjugate to \( T_{A_f} \times R_\theta \), for some \( \theta \notin \mathbb{Q} \).

**Proof.** By Theorem 11 (cf. [39, 68]) \( f \) is leaf conjugate to \( f_0 \), hence (1) holds and \( f \) is a (trivial) \( AB \)-system. Then we can again apply Theorem 13. The proof of (2) is similar to that of (4) of Proposition 24 (even easier since here \( \hat{M} = M \)) so we skip it. \[\Box\]

3.4.9. **Estimate of the Hölder exponents of leaf conjugacies in the presence dominated splittings.** As in Theorems 4 and 6 let \( f_0 \in \text{Diff}_\text{vol}^\infty(\mathbb{T}^d) \) be an isometric extension of an automorphism \( T_{A_f} \) on \( \mathbb{T}^{d-1} \), where \( A_f \in \text{SL}(d-1,\mathbb{Z}) \) is hyperbolic (we do not need irreducibility here). We denote by \( P : \mathbb{T}^d \to \mathbb{T}^{d-1} \) the projection along the \( \mathcal{W}^c_{f_0} \) leaves, which is just projection onto the first factor in \( \mathbb{T}^{d-1} \times \mathbb{T} \). Under the identification \( T\mathbb{T}^{d-1} \cong \mathbb{T}^{d-1} \times \mathbb{R}^{d-1} \), the action of \( DT_{A_f} \) is just \( T_{A_f} \times A_f \), and \( T\mathbb{T}^{d-1} = \mathbb{T}^{d-1} \times (\oplus V^i) \) is the \( T_{A_f} \)-invariant dominated splitting, where \( \mathbb{R}^{d-1} = \oplus V^i \) is the decomposition into generalized eigenspaces of \( A_f \).

There is a \( Df_0 \)-invariant dominated splitting \( TM = \oplus E^i_{f_0} \) projecting to the dominated splitting for \( T_{A_f} \), so that \( DpE^i_{f_0} = \{p(\xi)\} \times V^i \), for each \( i \). Moreover the Lyapunov exponent of \( Df_0|_{E^i_{f_0}} \) is equal to Lyapunov exponent of \( A_f|_{V^i} \).

\[\text{In fact, for completeness we need to prove that the number of atoms on the center leaf is a constant on almost every leaf, since } f \text{ could be non-ergodic. Suppose that the disintegration of } \text{vol}_M \text{ along } \mathcal{W}^c \text{ leaves has } k \text{ atoms for some positive } k, \text{ we claim that the same holds for } \hat{M}. \text{ The reason is that by Theorem A of } \[\text{and construction of } M, \text{ the restriction of covering map } p : \hat{M} \to M \text{ on each } \hat{\mathcal{W}}^c \text{ leaf is diffeomorphism except at most } 4 \text{ “bad” leaves. In particular, for any good leaf } \hat{\mathcal{W}}^c(\hat{x}) \text{, the covering map restricted to a small neighborhood of } \hat{\mathcal{W}}^c(\hat{x}) \text{ which is saturated by } \hat{\mathcal{W}}^c \text{ is a local diffeomorphism. As a corollary, if } \mathcal{W}^c(x) \text{ is a typical leaf in } M \text{ with } k \text{ atoms then } \hat{\mathcal{W}}^c(\hat{x}) \text{ also contains } k \text{ atoms.} \]
As in Theorem 6, we now assume that \( f \in \text{Diff}_{\text{vol}}^2(T^d) \) is a \( C^1 \)-small perturbation of \( f_0 \). Then \( Df \) also preserves a dominated splitting \( TM = \oplus E^i_f \). By Theorem 11, \( f \) is a fibered partially hyperbolic system, and \( (f_0; \mathcal{W}^c_{f_0}) \) is leaf conjugate to \( (f; \mathcal{W}^c_f) \) by a bi-Hölder continuous homeomorphism \( h^c : M \to M \). The leaf conjugacy \( h^c \) is canonical in the sense that
\[
\pi \circ h^c = P,
\]
where \( \pi \) is defined in Lemma 25. For the estimate of the bi-Hölder exponents of \( h^c \), cf. 68. In this context we can give a concrete description of how \( h^c \) is constructed. Fixing a smooth normal bundle \( \mathcal{N} \) to \( E^c \), the map \( h^c = h^c_{\mathcal{N}} \) is defined by
\[
h^c(x) = \pi^{-1}(P(x)) \cap D(x),
\]
where \( \{D(x) : x \in M\} \) is the smooth family of embedded disks defined by
\[
D(x) = \exp_x(\{tv : t \in [0, \epsilon), v \in \mathcal{N}(x)\}).
\]
If \( f \) is sufficiently \( C^1 \) close to \( f_0 \) and \( \epsilon > 0 \) is sufficiently small, then \( h^c \) is a well-defined homeomorphism that is smooth along the leaves of \( \mathcal{W}^c_{f_0} \) (as smooth as the leaves of \( \mathcal{W}^c_{f_0} \)).

It is easy to see that \( E^i_{f_0}, E^i_{f_0} \oplus E^c_{f_0} \) are integrable and we denote by \( \mathcal{W}^i_{f_0}, \mathcal{W}^c_{f_0} \) their integral manifolds respectively. In general, \( E^i_{f_0} \) and \( E^i_{f_0} \oplus E^c_{f_0} \) might not be integrable.

**Lemma 26.** Suppose \( E^i_f, E^i_f \oplus E^c_f \) are integrable and their integral manifolds are denoted by \( \mathcal{W}^i_f, \mathcal{W}^c_f \) respectively. If, for some normal bundle \( \mathcal{N} \), the map \( h^c = h^c_{\mathcal{N}} \) sends \( \mathcal{W}^c_{f_0} \) to \( \mathcal{W}^c_f \), for each \( i \), then the Hölder exponents of \( h^c \) and \( (h^c)^{-1} \) may be chosen close to 1.

**Proof.** Fix \( i \) and consider the foliation \( \mathcal{W}^c_i \). Its leaves are jointly foliated by \( \mathcal{W}^i_f \) and the uniformly compact foliation \( \mathcal{W}^i_f \). By taking \( f^{-1} \) if necessary, we may assume that the leaves of \( \mathcal{W}^i_f \) are uniformly contracted by the dynamics. Let \( \lambda_i < 0 \) be the corresponding Lyapunov exponent for \( A_f|_{\mathcal{L}^i} \). Since \( f_0 \) is an isometric extension of a linear map, for any \( \epsilon > 0 \) we may choose a continuous adapted metric on \( T^d \) such that for all \( f \) sufficiently \( C^1 \)-close to \( f_0 \), for all \( p \in T^d \), and all \( v \in E^i_f(p) \):
\[
e^{\lambda_i-\epsilon}\|v\| \leq \|D_p f(v)\| \leq e^{\lambda_i+\epsilon}\|v\|.
\]
Let \( \mu_i = e^{\lambda_i-\epsilon} \) and \( \nu_i = e^{\lambda_i+\epsilon} \). If \( f \) is sufficiently \( C^1 \)-close to \( f_0 \), then for any \( w \in M \) and \( w' \in \mathcal{W}^i_f(w, \text{loc}) \), if \( f^{-j}(w') \in \mathcal{W}^i_f(f^{-j}(w'), \text{loc}) \) for \( j = 0, \ldots, n \), then
\[
\nu_i^{-n}d(w, w') \leq d_{\mathcal{W}^i_f}(f^{-n}(w), f^{-n}(w')) \leq \mu_i^{-n}d(w, w').
\]
(This is easily proved by induction on \( n \).

Consider the restriction of \( h^c \) to \( \bigcup \mathcal{W}^c_{f_0} \), whose image is \( \bigcup \mathcal{W}^c_f \), sending \( \mathcal{W}^c_{f_0} \) leaves smoothly to \( \mathcal{W}^c_f \) leaves. Now \( h^c \) does not necessarily send \( \mathcal{W}^c_{f_0} \) leaves to \( \mathcal{W}^c_f \), but we can estimate the Hölder exponent of \( h^c \) restricted to \( \mathcal{W}^c_{f_0} \) leaves via a standard argument, which we now describe.

Fix \( \eta > 0 \) such that for all \( w, w' \in \bigcup \mathcal{W}^c_{f_0} \), with \( d(w, w') < \eta \), then for any \( z \in \mathcal{W}^i_f(w) \), there is a unique point \( z' \) in \( \mathcal{W}^i_f(z, \text{loc}) \cap \mathcal{W}^i_f(w') \) and the distance between \( z \) and \( z' \) is
uniformly comparable to the distance between $z$ and $z'$ as measured along $\mathcal{W}^s_i(z, \text{loc})$. This is possible because the foliation $\mathcal{W}^s_i$ has uniformly compact leaves. Next fix a small constant $\delta > 0$ such that $d(w, w') < \delta$ implies $d(h^c(w), h^c(w')) < \eta$.

Now let $x \in M$ and $x' \in W^s_{f_0}(x)$. Let $y = h^c(x)$ and $y' = h^c(x')$. We want to estimate $d(y, y')$ in terms of $d(x, x')$. Let $z = W^s_j(y) \cap W^s_i(y')$. By the construction of $h^c$ using the smooth normal bundle $\mathcal{N}$, we have that $d(y', z) = O(d(y, z))$, so it suffices to estimate $d(y, z)$ in terms of $d(x, x')$.

We may assume that $d(x, x') < \delta$. Fix $n \geq 0$ such that
\[ d(x, x') \in [\delta \mu_i^{n+1}, \delta \mu_i^n). \]
Then, since $x' \in W^s_{f_0}(x, \text{loc})$, we have $d(f_0^{-n}(x), f_0^{-n}(x')) < \mu_i^{-n}d(x, x') < \delta$. By our choice of $\delta$, we have that $d(h^c(f_0^{-n}(x)), h^c(f_0^{-n}(x'))) < \eta$. Since $h^c$ is a leaf conjugacy, $h^c(f_0^{-n}(x)) \in W^s_j(f_0^{-n}(y))$ and $h^c(f_0^{-n}(x')) \in W^s_j(f_0^{-n}(y')) = W^s_j(f_0^{-n}(z))$. Since $f_0^{-n}(y) \in W^s(f_0^{-n}(z), \text{loc})$, our choice of $\eta$ implies that $d(f_0^{-n}y, f_0^{-n}z)$ is comparable to the distance measured along $\mathcal{W}^s_j$, which is at least $\nu_i^{-n}d(y, z)$. Thus $d(y, z) = O(\nu_i^{-n}) = O(\mu_i^{-n} \beta) = O(d(x, x')^\beta)$, where
\[ \beta = \frac{\log \mu_i}{\log \nu_i} = \frac{\lambda_i + \epsilon}{\lambda_i - \epsilon}. \]
Since we may choose $\epsilon > 0$ arbitrarily small by setting $d_{C^1}(f_0, f)$ small enough, this shows that we may choose $\beta$ arbitrarily close to 1.

This shows that $h^c$ is uniformly $\beta$-Hölder continuous along $\mathcal{W}^s_{f_0}$-leaves, for all $i$. It is thus $\beta$-Hölder continuous.

A similar argument (reversing the roles of $f_0$ and $f_0^{-1}$) shows that $(h^c)^{-1}$ is $\beta$-Hölder continuous.

3.5. Pesin theory and Lyapunov charts. We will also use the following well-known corollaries of Pesin theory. Let $f$ be a $C^r(r > 1)$ diffeomorphism of a closed $d$–manifold $M$, let $\nu$ be an $f$–invariant ergodic probability measure, and let $\lambda^+ = \lambda_1 \geq \cdots \geq \lambda_d \geq \lambda^-$ be the Lyapunov exponents of $Df$ with respect to $\nu$. For $x \in M$, $\delta > 0$, and $\lambda < 0$, we define the local stable set
\[ \mathcal{W}^s(x, \lambda, \delta) := \{ y \in M : d(f^n(x), f^n(y)) \leq \delta \exp(\lambda n), \forall n \geq 0 \}. \]
The set of regular points for $(f, \nu)$ in $M$ (also called the Lyapunov–Perron regular points, cf. [34]) have full $\nu$-measure in $M$ and the following important property.

**Proposition 27** (Stable manifold theorem). Fix $\lambda < 0$ such that $\lambda_{k+1} < \lambda < \lambda_k$ holds for some $k$. Then for any regular point $x$, the local stable set $\mathcal{W}^s(x, \lambda, \delta)$ is a $C^r$ embedded disk in $M$ for small enough $\delta$. The dimension of the disk is $d - k$.

We call the set $\mathcal{W}^s(x, \lambda, \delta)$ defined by the Proposition the local Pesin stable manifold, and we denote it by $\mathcal{W}^s(x, \lambda, \text{loc})$ (cf. [34] for a concrete estimate on $\delta$). Suppose $x$
Suppose there exists Lemma 29. Proposition 28. For \( \lambda \) the argument here. Suppose that the asymptotic behavior described by Lyapunov exponents is realized immediately under iteration. The nonuniformity of the derivative cocycle is thus uniformized at the expense of the uniformity of the charts.

**Proposition 28** (cf. [77] and the references therein). Let \( \mathcal{F} \) be an \( f \)-invariant, \( n \)-dimensional foliation of \( M \) with \( C^2 \) leaves, and let \( E = TF \). Let \( \lambda^\pm \) be the largest and smallest Lyapunov exponents for the cocycle \( (Df|_E, \nu) \).

Then for \( \epsilon > 0 \) sufficiently small, there exists an \( f \)-invariant set \( \Lambda_0 \subset M \) of full measure with the following properties.

- There exists a measurable function \( r : \Lambda_0 \to (0, 1] \) and a collection of embeddings \( \Psi(x) : B(0, r(x)) \to \mathcal{F}(x, \text{loc}) \) such that \( \Psi(x)(0) = x \) and
  \[
  \exp(-\epsilon) < \frac{r(f(x))}{r(x)} < \exp(\epsilon).
  \]

- \( \text{If } F_x := \Psi^{-1}(f(x)) \circ f|_{\mathcal{F}(x, \text{loc})} \circ \Psi(x) : B(0, r(x)) \to \mathbb{R}^n, \text{ then } D_0 F_x \text{ satisfies}
  \[
  \exp(\lambda^- - \epsilon) \leq \|D_0 F_x^{-1}\|^{-1} \leq \|D_0 F_x\| \leq \exp(\lambda^+ + \epsilon).
  \]
  

We write \( F^n_x := F_{f^n(x)} \circ \cdots \circ F_x \) for later use.
- \( d_{C^1}(F_x, D_0 F_x) < \epsilon \) in \( B(0, r(x)) \).
- There exist \( K > 0 \) and a measurable function \( A : \Lambda_0 \to \mathbb{R} \) such that for \( y, z \in B(0, r(x)) \),
  \[
  K^{-1}d_{\mathcal{F}}(\Psi(x)(y), \Psi(x)(z)) \leq ||y - z|| \leq A(x)d_{\mathcal{F}}(\Psi(x)(y), \Psi(x)(z)),
  \]
  with \( \exp(-\epsilon) < \frac{A(F(x))}{A(x)} \leq \exp(\epsilon). \)

For \( x \in \Lambda_0 \), the map \( \Psi(x) : B(0, r(x)) \to \mathcal{F}(x, \text{loc}) \) is called a Lyapunov chart at \( x \) (for \( \mathcal{F}(x, \text{loc}) \)).

A corollary of Proposition 28 is the following lemma. Here \( f, E, \mathcal{F} \) are defined as in Proposition 28.

**Lemma 29.** Suppose there exists \( \lambda_0 < 0 \) such that for any point \( x \in M \) there exists \( C_x > 0 \) and \( y \in \mathcal{F}(x, \text{loc}) \) arbitrary close to \( x \) such that

\[
  d_{\mathcal{F}}(h^n(x), h^n(y)) \leq C_x e^{n\lambda_0} \cdot d_{\mathcal{F}}(x, y), \quad \forall n \geq 0.
  \]

Then for any \( f \)-invariant ergodic measure \( \nu \), we have \( \lambda^-(Df|_E, \nu) \leq \lambda_0 \).

**Proof.** The proof is a classical application of Lyapunov charts; for completeness we give the argument here. Suppose that \( \lambda^-(Df|_E, \nu) > \lambda_0 \). We pick \( \epsilon > 0 \) small enough that

\[
  \lambda^-(Df|_E, \nu) - \lambda_0 \gg \epsilon, \quad e^{\lambda^-(Df|_E, \nu)} \gg \epsilon,
  \]

is a regular point and \( W^u(x, \lambda, \text{loc}) \) is defined as above. The **global Pesin manifold** (of \( W^u(x, \lambda, \text{loc}) \)) is defined by

\[
  W^u(x, \lambda) = \bigcup_{n=0}^{\infty} f^{-n}(W^u(f^n(x), \lambda, \text{loc})).
  \]
and such that there exists a \( \nu \)-full measure set \( \Lambda_0 \), Lyapunov charts \( \Psi(x) : B(0, r(x)) \to \mathcal{F}(x, \text{loc}) \), function \( r : \Lambda_0 \to (0, 1] \), constant \( K > 0 \) and function \( A : \Lambda_0 \to \mathbb{R} \) satisfying all of the conclusions of Proposition 28.

Now we pick \( x \in \Lambda_0 \) and \( y \in \mathcal{F}(x, \text{loc}) \) sufficiently close to \( x \) satisfying the condition in Lemma 29. We denote \( d_{\mathcal{F}}(x, y) \) by \( \delta \). Then \( d_{\mathcal{F}}(h^n(x), h^n(y)) \leq C_x \cdot e^{n\lambda_0} \). By our conditions on \( r \) and \( A \), we have \( r(f^n(x)) \geq r(x) \cdot e^{-n\epsilon} \) and \( A(f^n(x)) \leq A(x) \cdot e^{n\epsilon} \). If \( \delta \) is small enough such that

\[
(7) \quad \delta \cdot C_x A(x) \cdot e^{n(\lambda_0 + \epsilon)} \leq r(x) \cdot e^{-n\epsilon} \leq r(f^n(x))
\]

(If \( \epsilon \ll |\lambda_0| \) and (7) holds for \( n = 0 \) then it holds for any \( n \geq 0 \), then \( f^n(y) \) is in the Lyapunov chart of \( f^n(x) \) for each \( n \). Therefore by the estimates in Proposition 28 we have

\[
(8) \quad \|\Psi^{-1}(f^n(x)) \cdot (f^n(y))\| \leq \delta \cdot A(x) C_x e^{n(\lambda_0 + \epsilon)}, \ \forall n \geq 0.
\]

On the other hand, by the estimates in Proposition 28 we have that for any \( n \geq 0 \)

\[
\|\Psi^{-1}(f^n(x)) \cdot (f^n(y))\| = \|\Psi^{-1}(f^n(x)) \cdot (f^n(y)) - \Psi^{-1}(f^n(x)) \cdot (f^n(x))\| \\
= \|F^n_x \cdot \Psi^{-1}(y) - F^n_x \cdot \Psi^{-1}(x)\| \\
\geq \inf_{z \in B(0, r(x))} \|DF^n_x\|^{-1} \cdot K^{-1} \delta \\
\geq (e^{\lambda - \epsilon} - \epsilon)^n \cdot K^{-1} \delta \quad \text{since} \quad f^n(y) \text{ lies in the Lyapunov chart} \\
\geq e^n(\lambda_0 + 2\epsilon) \cdot K^{-1} \delta, \quad \text{if } \epsilon \text{ is small enough},
\]

which contradicts (8).

By a similar proof, the following higher dimensional generalization of Lemma 29 holds. Here \( f, \mathcal{F} \) are defined as in Lemma 29.

**Lemma 30.** ([66]) Suppose there exists \( \lambda_0 < 0 \) such that for any point \( x \in M \), there exists \( C_x > 0 \) and a topological \( d \)-dimensional disc \( F(x) \subset \mathcal{F}(x, \text{loc}) \) containing \( x \) such that for any \( y \in F(x) \),

\[
d_{\mathcal{F}}(h^n(x), h^n(y)) \leq C_x e^{n\lambda_0} \cdot d_{\mathcal{F}}(x, y), \quad \forall n \geq 0.
\]

Then for any \( f \)-invariant ergodic measure \( \nu \), the associated \( d \) smallest Lyapunov exponents for the cocycle \( (Df|_E, \nu) \) are at most \( \lambda_0 \).

### 3.6. Normal forms for uniformly contracting foliations.

Classical normal form theory for local contractions has origins which go back to Poincaré and was fully developed by Sternberg [82] and Chen [18]. It gives conditions under which a local diffeomorphism can be (smoothly) linearized in a neighborhood of a periodic orbit. In this setting, smooth linearization fails in the presence of resonances between eigenvalues of the derivative. By replacing a linear normal form with a polynomial, Takens [85] extended this normal form theory to hold in the presence of resonances.

An approach that turns out to have numerous applications is the study of normal forms near an invariant manifold at a fixed point; extensive study in this direction was done in [10]. Subsequently non-stationary linearizations and the theory of non-stationary normal forms was developed on contracting invariant foliations for smooth diffeomorphisms ([38],
and much more recently in [46, 47] and [58] in more general settings. The theory of non-stationary normal forms theory has proved to be an extremely useful tool in applications, primarily to a range of problems treating smooth dynamical systems and group actions with some hyperbolicity. This tool was crucial in the proof in [54] of local rigidity for large classes of Anosov actions (see [46, 47] for a more complete list of references).

Here we will use a state-of-the art version of this theory, Theorem 15 below, recently developed by Kalinin. This gives normal forms for the restriction of a $C^r$ diffeomorphism to a uniformly contracted foliation that simultaneously normalize its smooth centralizer.

Let $f$ be a diffeomorphism of a closed manifold $M$. Let $\mathcal{W}$ be an $f$–invariant foliation of $M$ with uniformly $C^1$ leaves. We assume that $f$ uniformly contracts the leaves $\mathcal{W}$. Let $E = TW$ be the tangent bundle to $\mathcal{W}$. We denote by $F: E \to E$ the bundle automorphism induced by the derivative of $f$: $F_x = Df|_{T_xW}: E_x \to E_{fx}$. Then $F$ induces a bounded linear operator $F^*$ on the space of continuous sections of $E$ by $F^*v(x) = F(v(f^{-1}x))$. The spectrum of the complexification of $F^*$ is called the Mather spectrum of $F$. If the non-periodic points of $f$ are dense in $M$, then the Mather spectrum consists of finitely many closed annuli centered at 0, see [57, 64].

**Definition 8.** We say that the bundle automorphism $F$ has narrow band spectrum if its Mather spectrum is contained in a finite union of closed annuli $A_i$, $i = 1, \ldots, \ell$, bounded by circles of radii $e^{\lambda_i}$ and $e^{\mu_i}$, where the numbers

$$\lambda_1 \leq \mu_1 < \lambda_2 \leq \mu_2 < \cdots < \lambda_\ell \leq \mu_\ell < 0$$

satisfy $\mu_i + \mu_\ell < \lambda_i$, for $i = 1, \ldots, \ell$.

For the given spectral intervals $\{(\lambda_i, \mu_i)\}$, the bundle $E$ splits into a direct sum

$$E = E^1 \oplus \cdots \oplus E^\ell$$

of continuous, $F$–invariant sub-bundles such that Mather spectrum of $F|_{E^i}$ is contained in the annulus $A_i$. A sub-resonance relation for $(\lambda, \mu) = (\lambda_1, \ldots, \lambda_\ell, \mu_1, \ldots, \mu_\ell)$ with $\lambda_1 \leq \mu_1 < \cdots < \lambda_\ell \leq \mu_\ell < 0$ is a relation of the form $\lambda_i \leq \sum s_j \mu_j$, where $s_1, \ldots, s_\ell$ are non-negative integers.

For vector spaces $E$ and $\bar{E}$ we say that a map $P: E \to \bar{E}$ is polynomial if for some bases of $E$ and $\bar{E}$, each component of $P$ is a polynomial. A polynomial map $P$ is homogeneous of degree $n$ if $P(av) = a^n P(v)$ for all $v \in E$ and $a \in \mathbb{R}$. More generally, for a given splitting $E = E^1 \oplus \cdots \oplus E^\ell$ we say that $P: E \to \bar{E}$ has homogeneous type $s = (s_1, \ldots, s_\ell)$ if for any real numbers $a_1, \ldots, a_\ell$ and vectors $t_j \in E^j$, $j = 1, \ldots, \ell$, we have

$$P(a_1 t_1 + \cdots + a_\ell t_\ell) = a_1^{s_1} \cdots a_\ell^{s_\ell} P(t_1 + \cdots + t_\ell).$$

Suppose $E = E^1 \oplus \cdots \oplus E^\ell$, $\bar{E} = \bar{E}^1 \oplus \cdots \oplus \bar{E}^\ell$ and $P: E \to \bar{E}$ is a polynomial map. Split $P$ into components $P_j: E_j \to \bar{E}_j$ and write $P = (P_1, \ldots, P_\ell)$. Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \ldots, \mu_\ell)$ with $\lambda_1 \leq \mu_1 < \cdots < \lambda_\ell \leq \mu_\ell$. We say that $P$ is of $(\lambda, \mu)$ sub-resonance type if each component $P_j$ has only terms of homogeneous types $s = (s_1, \ldots, s_\ell)$ and they satisfy the sub-resonance relations $\lambda_i \leq \sum s_j \mu_j$. 


Now we can state the main results in this section. Let $f$ be a $C^r$ diffeomorphism of a closed manifold $M$, and let $W$ be an $f$–invariant topological foliation of $M$ with uniformly $C^r$ leaves. Suppose that the leaves of $W$ are contracted by $f$ and that $Df|_{TW}$ has narrow band spectrum (see Definition 8). More precisely, we assume that the Mather spectrum of $Df|_{TW}$ is contained in a finite union of closed annuli $A_i$, $i = 1, \ldots, \ell$, bounded by circles of radii $e^{\lambda_i}$ and $e^{\mu_i}$, where $\lambda_1 \leq \mu_1 < \cdots < \lambda_\ell \leq \mu_\ell < 0$ satisfy $\mu_i + \mu_\ell < \lambda_i$, $1 \leq i \leq \ell$. For $Df^{-1}|_{E^s}$, the Mather spectrum is contained in a finite union of closed annuli $A_i$, $i = 1, \ldots, \ell$, bounded by circles of radii $e^{\lambda_i}$ and $e^{\mu_i}$ that satisfy similar inequalities.

**Theorem 15.** Suppose that $r > \lambda_1/\mu_\ell$, and let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \ldots, \mu_\ell)$. Then there exists a family $\{H_x \}_{x \in M}$ of $C^r$ diffeomorphisms $H_x : W_x \to E_x = T_x W$ such that

1. $P_x = H_{fx} \circ f \circ H_x^{-1} : E_x \to E_{fx}$ is a polynomial map of $\lambda, \mu$ sub-resonance type for each $x \in M$;
2. $H_x(x) = 0$ and $DxH_x$ is the identity map for each $x \in M$;
3. $H_x$ depends continuously on $x \in M$ in the $C^r$ topology and is jointly $C^r$ in $x$ and $y \in W_x$ along the leaves of $W$;
4. $H_y \circ H_x^{-1} : E_x \to E_y$ is a polynomial map of $(\lambda, \mu)$ sub-resonance type for each $x \in M$ and each $y \in W_x$; and
5. if $g$ be a homeomorphism of $M$ which commutes with $f$, preserves $W$, and is $C^s$, with $s > \lambda_1/\mu_\ell$, along the leaves of $W$, then the maps $H_x$ bring $g$ to a normal form as well, i.e. the map $Q_x = H_{fx} \circ g \circ H_x^{-1}$ is a polynomial of $(\lambda, \mu)$ sub-resonance type for each $x \in M$.

We will use normal form theory to upgrade the regularity of certain homeomorphisms in the centralizer of the partially hyperbolic systems under consideration, as detailed in the following proposition. Theorem 15 plays an important role in its proof.

**Proposition 31.** Let $f$ be a $C^\infty$ partially hyperbolic diffeomorphism of a closed manifold $M$. Assume that $f$ has a 1–dimensional center foliation $W^c$ with $C^\infty$ leaves. Suppose that $\varphi = \varphi_t : M \times \mathbb{R} \to M$ is a flow generated by a continuous vector field $X$ that commutes with $f$ for any $t$. Suppose that $f$, $\varphi_t$, and $X$ satisfy the following conditions.

1. The vector field $X$ is tangent to $E^c_f$ and uniformly smooth along the leaves of $W^c_f$.
2. The Mather spectrums of $Df|_{E^s_f}$ and $Df^{-1}|_{E^u_f}$ satisfy the narrow band conditions above\(^8\)\(^9\).
3. There exists a dense set $D \subset \mathbb{R}$ such that for any $t \in D$, $\varphi_t \in \text{Diff}^r(M)$, where $r > \max(\lambda_\ell/\mu_\ell, \lambda_\ell/\mu_\ell)$.

Then $\varphi_t$ is a $C^\infty$ flow.

---

\(^8\)More precisely, the Mather spectrum of $Df|_{E^s}$ is contained in a finite union of closed annuli $A_i$, $i = 1, \ldots, \ell$, bounded by circles of radii $e^{\lambda_i}$ and $e^{\mu_i}$, where $\lambda_1 \leq \mu_1 < \cdots < \lambda_\ell \leq \mu_\ell < 0$ satisfy $\mu_i + \mu_\ell < \lambda_i$, $1 \leq i \leq \ell$. For $Df^{-1}|_{E^u}$, the Mather spectrum is contained in a finite union of closed annuli $A_i$, $i = 1, \ldots, \ell$, bounded by circles of radii $e^{\lambda_i}$ and $e^{\mu_i}$ that satisfy similar inequalities.

\(^9\)Here $\lambda_i, \mu_i$ (resp. $\lambda_i, \mu_i$) are the outer and inner radii of the of Mather spectrum of $Df|_{E^s}$ (resp. $Df^{-1}|_{E^u}$).
Let $E$ be a set and $\epsilon > 0$. In particular, for any $g \in Z^{\text{diff}}(M)(f)$, the map $H_g \circ g \circ H_x^{-1}$ is a sub-resonance polynomial (with fixed type) as well.

Thus for $t \in D$, $\{H_x\}$ is also a normalization for $\varphi_t$ on $W^s_f$. Now consider the homeomorphism $\varphi_t$ for an arbitrary fixed $t \in \mathbb{R}$. Pick $t_k, k = 1, 2, \ldots$ in $D$ such that $\lim_{k \to \infty} t_k = t$. Then the sequence

$$H_{\varphi_{t_k}} \circ \varphi_{t_k} \circ H_x^{-1} : E^s_f(x) \to E^s_f(\varphi_{t_k}(x))$$

uniformly converges to $H_{\varphi_f} \circ \varphi_t \circ H_x : E^s_f(x) \to E^s_f(\varphi_t(x))$.

But each of $H_{\varphi_{t_k}} \circ \varphi_{t_k} \circ H_x^{-1}$ is a sub-resonance polynomial (with fixed type), so their $C^0$-limit is a sub-resonance polynomial as well. Thus $H_{\varphi_f} \circ \varphi_t \circ H_x^{-1}$ is uniformly smooth along $E^s_f$, which means $\varphi_t$ is uniformly smooth along $W^s_f$.

The same arguments applied to $f^{-1}|_{W^{-}}$ give that $\varphi_t$ is uniformly smooth along $W^s_f$. Item (1) of Proposition 31 implies that $\varphi_t$ is uniformly smooth along $W^c$, and the evaluation map $t \mapsto \varphi_t(x), x \in M$ is smooth, uniformly in $x$. Applying Journé’s Lemma as in [2], we obtain that $\{\varphi_t\}$ is a smooth flow and $D = \mathbb{R}$. ∎

3.7. Thermodynamic formalism. In this section we review some classical concepts from thermodynamic formalism: pressure, equilibrium states, and the variational principle. For more details, see [48, Chapter 20]. We will use this thermodynamic formalism in Sections 6.8 and 6.10 to characterize volume as the unique measure of maximal entropy for certain diffeomorphisms with large centralizer.

**Definition 9.** Let $f$ be a homeomorphism of a compact metric space $X$, and let $\varphi : X \to \mathbb{R}$ be a continuous function (called a potential). For any $f$–invariant Borel probability measure $\mu$ on $X$, the pressure of $\mu$ with respect to $\varphi$ is defined by

$$P_\mu(\varphi) = h_\mu(f) + \int \varphi d\mu,$$

where $h_\mu(f)$ is the measure-theoretic entropy of $f$ with respect to $\mu$.

**Definition 10.** Let $f$ be a homeomorphism of a compact metric space $X$. For $x \in X$, $\epsilon > 0$, and $n \geq 0$, the Bowen ball $B_f(x, \epsilon, n)$ is defined by

$$B_f(x, \epsilon, n) := \{y \in X : d(f^k(x), f^k(y)) < \epsilon, \text{ for } 0 \leq k \leq n - 1\}.$$

A set $E \subset X$ is called $(n, \epsilon)$–separated if for any $x, y \in E$

$$\max_{0 \leq k \leq n - 1} d(f^k(x), f^k(y)) > \epsilon.$$

Let

$$N(f, \varphi, \epsilon, n) := \sup \left\{ \sum_{x \in E} e^{S_n(\varphi, x)} : E \subset X \text{ is } (n, \epsilon) – \text{ separated} \right\},$$
where $S_n(\varphi, x) := \varphi(x) + \cdots + \varphi(f^{n-1}(x))$. The topological pressure $P(\varphi)$ of $f$ is defined by

$$P(\varphi) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(f, \varphi, \epsilon, n),$$

and we have the associated variational principle (cf. [48, Theorem 20.2.4])

$$P(\varphi) = \sup_{\mu \text{ is } f^{-\text{inv}}} P_{\mu}(\varphi). \quad (10)$$

**Definition 11.** Let $f$ be a homeomorphism of a compact metric space $X$ and let $\varphi \in C^0(X, \mathbb{R})$ be a potential function. An $f$-invariant measure $\mu$ is called an equilibrium state of $\varphi$ if $P_{\mu}(\varphi) = P(\varphi)$. 

**3.8. Partially hyperbolic higher rank abelian actions.** A detailed ground treatment of Anosov and partially hyperbolic abelian higher rank actions, including a variety of techniques and examples, can be found in [51]. For a detailed treatment of smooth ergodic theory of general abelian actions, see [12].

Let $\alpha : \mathbb{Z}^k \to \text{Diff}^2(M)$ be an action by diffeomorphisms of a closed manifold $M$. We say $\alpha$ is partially hyperbolic if it contains a partially hyperbolic diffeomorphism $\alpha(a)$, for some $a \in \mathbb{Z}^k$, and Anosov if it contains at least one Anosov diffeomorphism. Some basic questions and difficulties related to partially hyperbolic abelian actions are described in [21], [22]. For basics and background on partially hyperbolic abelian actions with compact center foliation we refer to [25] and the references therein.

Corresponding to Oseledec’s theorem (Section 3.2) for a cocycle over a single ergodic measure preserving transformation, there is a “higher-rank Oseledec theorem” ([12, Theorem 2.4]) for abelian group actions. Let $E \to M$ be a continuous vector bundle and let $\mathcal{A}$ be a linear $\mathbb{Z}^k$-cocycle on $E$ over an ergodic, $\mu$-preserving abelian action $\alpha$ of $M$: that is, $\mathcal{A} : \mathbb{Z}^k \to \text{Aut}(E)$ is a $\mathbb{Z}^k$-action by bundle isomorphisms projecting to the action of $\alpha$ on $M$.

The higher-rank Oseledec theorem implies existence of finitely many linear functionals $\chi : \mathbb{R}^k \to \mathbb{R}$ and an $\mathcal{A}$-invariant measurable splitting $\oplus E_\chi$ of the bundle $E$ on a full $\mu$-measure set, such that for any $a \in \mathbb{Z}^k$ and $v \in E_\chi(x)$:

$$\lim_{a \to \infty} \frac{\log \| \mathcal{A}(a, x)(v) \| - \chi(a)}{\|a\|} = 0.$$

The splitting $\oplus E_\chi$ is called the Oseledec decomposition for $\mathcal{A}$ (with respect to $\mu$), and the linear functionals $\chi$ are called the Lyapunov functionals for $\mathcal{A}$ (with respect to $\mu$). The hyperplanes $\ker \chi \subset \mathbb{R}^k$ are called Weyl chamber walls, and the connected components of $\mathbb{R}^k - \bigcup \ker \chi$ are called the Weyl chambers for $\mathcal{A}$ (with respect to $\mu$). Even though elements of the Weyl chambers are vectors in $\mathbb{R}^k$, we will often say that the diffeomorphism $\alpha(a)$ is in the Weyl chamber $C$ if $a \in C$.

Two nonzero Lyapunov functionals $\chi_i$ and $\chi_j$ are coarsely equivalent if they are positively proportional, meaning there exists $c > 0$ such that $\chi_i = c \cdot \chi_j$. This is an equivalence relation on the set of Lyapunov functionals, and a coarse Lyapunov functional is
an equivalence class under this relation. Given a fixed ordering of non-zero coarse Lyapunov functionals \((\chi_1, \ldots, \chi_r)\), each Weyl chamber \(C\) can be labelled by its signature \((\text{sgn} \chi_1(a), \ldots, \text{sgn} \chi_r(a))\), where \(a\) is any element in \(C\). The Weyl chambers of \(A\) in \(\mathbb{R}^k\) together with their assigned signatures we call the Weyl chamber picture of \(A\) over \(\alpha\) (with respect to the ergodic invariant measure \(\mu\)).

If, for two \(\mathbb{Z}^k\) cocycles (over possibly two distinct \(\mathbb{Z}^k\) actions), the Weyl chamber walls in \(\mathbb{R}^k\) coincide and the signatures of each Weyl chamber also coincide, we will say that the two cocycles have the same Weyl chamber picture. If for two Lyapunov functionals \(\chi^1, \chi^2\), we have \(\ker \chi^1 = \ker \chi^2\) and \(\chi^1(a)\chi^2(a) > 0\) for some \(a\), then \(\chi^1, \chi^2\) are positively proportional. This implies the following:

**Lemma 32.** Suppose that the Lyapunov functionals \(\{\chi^i\}, \{\chi'^i\}\) of two ergodic cocycles \(A\) and \(A'\) have the same Weyl chambers, and suppose that for any \(i\), there is an element \(a \in \mathbb{Z}^k\) such that \(\chi^i(a)\chi'^i(a) > 0\). Then \(A\) and \(A'\) have the same the Weyl chamber picture.

The higher-rank Oseledec theorem is typically (especially for Anosov actions) applied to the derivative cocycle \(D\alpha\) on the tangent bundle \(TM\). The “Weyl chambers for \(D\alpha\)” are simply called Weyl chambers. In this more classical setting, the Weyl chamber picture in \(\mathbb{R}^k\) depends only on \(\alpha\) and on the invariant measure. In the presence of sufficiently many Anosov elements of the action (for example, one Anosov element in each Weyl chamber), and an ergodic measure of full support, even the dependence on the measure can be removed. Moreover, in this case the coarse Lyapunov distributions are intersections of stable distributions for finitely many elements of the action, they are well defined everywhere, Hölder continuous, and tangent to foliations with smooth leaves. (For more details cf. Section 2.2 in [45] and the references therein). The same holds for actions that have many elements normally hyperbolic to a common center foliation [22].

For actions containing a partially hyperbolic element, one is often most interested in the Lyapunov exponents transverse to the center distribution of that element. Adapted to this situation is a different collection of Weyl chamber, called the hyperbolic Weyl chambers of the action. Suppose \(\alpha : \mathbb{Z}^k \to \text{Diff}^\omega_{\text{vol}}(M)\) is a \(C^2\), volume preserving ergodic abelian action on a compact manifold \(M\). We assume that there exists at least one \(a \in \mathbb{Z}^k\) such that \(\alpha(a)\) is a partially hyperbolic diffeomorphism. By the discussion in Section 3.3 the sum of the stable and unstable distributions of a single partially hyperbolic element \(\alpha(a)\) is \(\alpha\)–invariant. Denote it by \(E^H := E^u_a \oplus E^s_a\). The Weyl chamber walls for the cocycle \(D\alpha|_{E^H}\), are called hyperbolic Weyl chamber walls for \(\alpha\), the chambers the hyperbolic Weyl chambers for \(\alpha\), and the Weyl chamber picture the hyperbolic Weyl chamber picture for \(\alpha\).

For the abelian actions we shall consider in this paper there is a priori just one partially hyperbolic element (and its powers). In particular, a priori the hyperbolic Weyl chamber picture may depend on the choice of the invariant measure. This is one of the main difficulties in our problem. Thus one of the main challenges is: given a partially hyperbolic action, show existence of more partially hyperbolic elements in distinct Weyl chambers. A significant portion of the work in subsequent sections is dedicated to resolving this problem.
The (hyperbolic) Weyl chamber picture for an Abelian action reflects the “incompatibility” of the hyperbolicity of various action elements. This incompatibility is what causes in many cases rigidity of differentiable or measurable structures for the action (see [21], [22] and references therein). The general paradigm is: the more rich the Weyl chamber picture for an abelian action, the more reason to expect that the action is rigid. We list here some properties of abelian actions which are described via the Weyl chamber structure, and which we will make use of in subsequent sections.

- \( \alpha \) is **maximal** if there are exactly \( k + 1 \) coarse Lyapunov exponents corresponding to \( k + 1 \) distinct Lyapunov hyperplanes, and if the Lyapunov hyperspaces are in general position (namely, if no Lyapunov hyperspace contains a non-trivial intersection of two other Lyapunov hyperspaces).

- \( \alpha \) is **totally non-symplectic** (TNS) if there are no negatively proportional Lyapunov exponents. In particular, if \( \alpha \) is maximal, then \( \alpha \) is TNS.

Maximality implies a special property of Weyl chambers: there is any combination of signs of Lyapunov functionals among the Weyl chambers, except all positive, and all negative. Classical examples of maximal Anosov actions are certain actions by toral automorphisms. In particular we have

**Lemma 33.** [cf. [49]] Suppose \( A \in \text{SL}(k, \mathbb{Z}) \) is a hyperbolic irreducible matrix. Then \( Z_{\text{SL}(k, \mathbb{Z})}(A) \) induces a maximal abelian Anosov action on \( \mathbb{T}^k \).

An action \( \alpha \) is called **Cartan** if all coarse Lyapunov distributions are one-dimensional. The maximal \( Z^{k-1} \) Anosov actions on the torus \( \mathbb{T}^k \) by toral automorphisms are prime examples of Cartan actions; the centralizer \( Z_{\text{SL}(k, \mathbb{Z})}(A) \) of any hyperbolic \( k \times k \) matrix \( A \) with distinct real eigenvalues is virtually \( Z^{k-1} \).

The simplest such example is obtained by choosing two commuting hyperbolic automorphisms of \( \mathbb{T}^3 \) (particular examples of commuting matrices defining such actions can be found in [51]). Such a picture is depicted in Figure 1. In this case the Weyl chamber picture in the acting group \( Z^2 \) is simple to observe, and yet it is as rich as possible, i.e. it is maximal Cartan.

We remark that for the **partially hyperbolic** action obtained as a product of such a maximal Cartan action on \( \mathbb{T}^3 \) and the identity action on another manifold, the **hyperbolic** Weyl chamber picture would look exactly the same as the picture above.

### 4. Proofs of Theorems 3 and 5

We begin with a general discussion of perturbations of discretized geodesic flows in negative curvature. Let \( X \) be a closed, negatively curved Riemannian manifold of any dimension, and let \( \psi_t: T^1X \to T^1X \) be the geodesic flow on the unit tangent bundle \( T^1X \).

The centralizer of the flow \( \psi_t \) (and hence any element of the flow) contains the flow itself. If \( X \) admits an isometry \( h \), then the derivative \( Dh \) preserves the unit tangent bundle \( T^1X \) and commutes with the flow. While the flow fixes its own orbits, the derivative of a nontrivial isometry permutes the orbits nontrivially.
Suppose \( g: T^1X \to T^1X \) is an arbitrary continuous map, and let \( g_*: \pi_1(T^1X) \to \pi_1(T^1X) \) be the induced map on the fundamental group. We claim that \( g \) induces a homomorphism \( g_*: \pi_1(X) \to \pi_1(X) \) such that \( g_*p_* = p_*g_* \), where \( p: T^1X \to X \) is the canonical projection. When \( \dim(X) \geq 3 \), this is immediate, because the fibers of \( T^1X \) are simply connected. When \( X \) is a surface, this follows from the fact that \( \pi_1(T^1X) \) is a central extension of the simple group \( \pi_1(X) \).

Note that since \( \psi_t \) is isotopic to the identity, it induces a trivial map on \( \pi_1(X) \), whereas the derivative of a nontrivial isometry \( h \) induces a nontrivial automorphism \( \overline{Dh}_* \) of \( \pi_1(X) \), namely \( h_* \) itself. The latter automorphism \( h_* \) induces a nontrivial outer automorphism; that is, it is not induced by a conjugacy on \( \pi_1(X) \). This is because, as we shall see, homeomorphisms of \( T^1X \) that leave invariant the orbit foliation of \( \varphi_t \) and that induce inner automorphisms of \( \pi_1(X) \) must fix the leaves of the orbit foliation.

**Proposition 34.** Let \( X \) be a closed, negatively curved manifold, and suppose that \( g: T^1X \to T^1X \) is a homeomorphism that leaves invariant the orbit foliation of the geodesic flow \( \psi_t \).

The following are equivalent:

1. there exists \( \hat{\gamma} \in \pi_1(X) \) such that \( \tilde{g}_*\gamma = \hat{\gamma}\gamma\hat{\gamma}^{-1} \), for every \( \gamma \in \pi_1(X) \).
2. \( g \) leaves invariant each orbit of \( \psi_t \).

**Proof.** (1): Since \( g \) preserves the orbits of the geodesic flow, the map \( \tilde{g}_* \) has a simple description: given \( \gamma \in \pi_1(X) \), represent \( \gamma \) by a closed, unit-speed geodesic \( c_\gamma \) in \( X \) (here we
are using free homotopy equivalence): this representation is unique up to reparametrization, because $X$ is negatively curved. The lift $c'_\gamma$ to $T^1X$ is a closed orbit of $\varphi_\gamma$ and is taken to a closed orbit $c'\gamma$ by $g$; the projection of this orbit to $X$ is a closed geodesic $\hat{c} = c_{\hat{g}_s(\gamma)}$ representing the class $\bar{g}_s(\gamma)$.

Now suppose that there exists $\hat{\gamma} \in \pi_1(X)$ such that for every $\gamma \in \pi_1(X)$, $\hat{g}_s(\gamma) = \hat{\gamma}\gamma\hat{\gamma}^{-1}$. The group $\Gamma = \pi_1(X)$ acts freely on the universal cover $\tilde{X}$ on the left by isometries. Since $X$ is closed and negatively curved, each $\gamma \in \Gamma$ has a unique axis $\alpha_\gamma$, which is a geodesic in $\tilde{X}$, invariant under $\gamma$ and on which $\gamma$ acts by translations.

Denote by $\pi: \tilde{X} \to X$ the covering projection. It is easy to see that

$$\pi^{-1}(c_\gamma) = \bigcup_{\eta \in \Gamma} \eta\alpha_\gamma = \bigcup_{\eta \in \Gamma} \alpha_{\eta\gamma\eta^{-1}}.$$

Denote by $\hat{g}$ the action of $g$ on lifted geodesics in $\tilde{X}$, which is well-defined up to deck transformations. Then

$$\hat{g} \left( \pi^{-1}(c_\gamma) \right) = \pi^{-1}(c_{\hat{g}_s(\gamma)}) = \bigcup_{\eta \in \Gamma} \alpha_{\eta\gamma(\eta\hat{\gamma})^{-1}} = \pi^{-1}(c_\gamma).$$

Thus $g(c'_\gamma(\mathbb{R})) = c'_\gamma(\mathbb{R})$, for every closed $\psi_\tau$-orbit $c'_\gamma(\mathbb{R})$. Since $X$ is closed and negatively curved, $\psi_\tau$-periodic orbits are dense in $T^1X$, and so $g$ fixes all $\psi_\tau$-orbits.

(2) If $g$ fixes all $\psi_\tau$-orbits, then by the argument for (1), we obtain that $\hat{g}_s$ preserves the conjugacy classes in $\pi_1(X)$ and thus must act by conjugation. □

Suppose that $f \in \text{Diff}^r(T^1X), r \geq 1$ a $C^1$-small perturbation of $\psi_{t_0}$. By Theorem 11, $f$ is plaque expansive and dynamically coherent, and $(f, W^c)$ is leaf conjugate to $(\psi_{t_0}, \tilde{W}^c_{\psi_{t_0}})$. Proposition 23 implies that for any $g \in Z_{\text{Diff}^r(T^1X)}(f), g(W^*) = W^*$, for $* \in \{u, c, s, cu, cs\}.$

Let $Z^+_{\text{Diff}^r(T^1X)}(f)$ be the subgroup of $Z_{\text{Diff}^r(T^1X)}(f)$ consisting of the elements that preserve the orientation of $W^c$. Clearly $Z^+_{\text{Diff}^r(T^1X)}(f)$ has finite index in $Z_{\text{Diff}^r(T^1X)}(f)$. We denote by $Z^c_{\text{Diff}^r(T^1X)}(f)$ the set of $g \in Z^+_{\text{Diff}^r(T^1X)}(f)$ fixing the leaves of $W^c(f)$. Observe that $Z^c_{\text{Diff}^r(T^1X)}(f)$ is a normal subgroup of $Z^+_{\text{Diff}^r(T^1X)}(f)$.

**Proposition 35.** Let $\psi_{t_0}$ be the discretized geodesic flow over a closed, negatively curved manifold $X$. There exists $\epsilon > 0$ such that for any $r \geq 1$, if $f \in \text{Diff}^r(T^1X)$, and $d_{C^1}(f, \psi_{t_0}) < \epsilon$, then $Z^+_{\text{Diff}^r(T^1X)}(f)/Z^c_{\text{Diff}^r(T^1X)}(f)$ is isomorphic to a subgroup of the outer automorphism group $\text{Out}(\pi_1(X)).$

**Proof.** Consider the map that sends $g \in Z^+_{\text{Diff}^r(T^1X)}(f)$ to $[\hat{g}_s] \in \text{Out}(\pi_1(X))$. It suffices to prove that the kernel of this map is $Z^c_{\text{Diff}^r(T^1X)}(f)$. Suppose then that $g$ lies in the kernel, i.e. that there exists $\hat{\gamma} \in \pi_1(X)$ such that $\hat{g}_s(\gamma) = \hat{\gamma}\gamma\hat{\gamma}^{-1}$, for all $\gamma \in \pi_1(X)$.

Let $h: T^1X \to T^1X$ be the leaf conjugacy between $(W^c_f, \psi_f)$ and $(W^c_{\psi_{t_0}}, \psi_{t_0})$, satisfying

$$h \left( W^c_{\psi_{t_0}}(v) \right) = W^c_{\psi_{t_0}}(h(v)).$$
for all \( v \in T^1 X \), and let \( g_1 = h \circ g \circ h^{-1} \), which is a homeomorphism preserving the orbit foliation of \( \psi_t \). Since \( h \) is homotopic to the identity, the induced maps \( \tilde{g}_1 \) and \( \tilde{g}_1^* \) are the same (i.e., conjugacy by \( \hat{\gamma} \)). Proposition 34 implies that \( g_1 \) fixes the \( \psi_t \) orbits, and so \( g \) fixes the leaves of \( W^s \), i.e. \( g \in Z_{\operatorname{Diff}^r(T^1 X)}^c(f) \). Similarly, if \( g \in Z_{\operatorname{Diff}^r(T^1 X)}^c(f) \), then \( g \) lies in the kernel. □

**Corollary 1.** Let \( X \) be a closed, negatively curved locally symmetric manifold of dimension at least 3. There exists \( \epsilon > 0 \) such that for any \( r \geq 1 \), if \( f \in \operatorname{Diff}^r(T^1 X) \), and \( d_{C^1}(f, \psi_{t_0}) < \epsilon \), then \( Z_{\operatorname{Diff}^r(T^1 X)}^+(f)/Z_{\operatorname{Diff}^r(T^1 X)}^c(f) \) is finite.

**Proof.** Mostow rigidity implies that the outer automorphism group \( \operatorname{Out}(\pi_1(X, p(v))) \) is finite, isomorphic to the isometry group of \( X \), and so this follows immediately from Proposition 35. □

We remark that Proposition 35 and the discussion above also imply that for \( X \) negatively curved and locally symmetric, of dimension at least 3,

\[
Z_{\operatorname{Diff}^r(T^1 X)}^+(\psi_{t_0})/Z_{\operatorname{Diff}^r(T^1 X)}^c(\psi_{t_0}) \cong \operatorname{Out}(\pi_1(X)),
\]

since every outer automorphism is represented by a unique isometry. With a little more work (see, e.g., [42]), one can show that for any \( t_0 \neq 0 \) the centralizer of \( \psi_{t_0} \) in \( \operatorname{Diff}^1(T^1 X) \) is precisely the group generated by the flow itself and the isometry group of \( X \). Details are left to the reader.

**Proof of Theorem 5.** Let \( f \) be a diffeomorphism satisfying all the hypotheses in Theorem 5. Then we have

**Lemma 36.** If \( \operatorname{vol}_{T^1 X} \) has singular disintegration along \( W^s_f \), then there exists \( k \geq 1 \) and a full volume set \( S \subset T^1 X \) that intersects every center leaf in exactly \( k \) orbits of \( f \).

**Proof.** The proof is essentially the same as that of [2]. We remark that a \( C^2 \) regularity hypothesis in the discussion in Section 6 in [2] is sufficient. □

By [50] and [72], \( \psi_{t_0} \) in Theorem 5 is stably accessible and hence stably ergodic (by, e.g. [14]), and so we may assume that \( f \) is accessible and ergodic. Therefore by Lemma 11 \( Z_{\operatorname{Diff}(T^1 X)}(f) \subset Z_{\operatorname{vol}}(M) \). Corollary 1 then implies that \( Z_{\operatorname{Diff}(T^1 X)}^+(f) \), and hence \( Z_{\operatorname{Diff}(T^1 X)}^+(f) \), is virtually \( Z_{\operatorname{Diff}(T^1 X)}^c(f) \).

We now show that \( Z_{\operatorname{Diff}(T^1 X)}^c(f) \) is virtually \(< f^n >\), which completes the proof of Theorem 5. First, since \( (f, W^s_f) \) is leaf conjugate to \( \psi_{t_0} \), almost every (all but countably many) \( W^s_f \)-leaves are noncompact. For any noncompact \( W^s_f \)-leaf, we consider the total order “\( < \)” induced by the canonical orientation on \( W^s_f \). The action of \( f \) on every non-compact \( W^s_f \)-leaf is uniformly close to a translation by \( t_0 \) on \( \mathbb{R} \) and, therefore, is topologically conjugate to a translation.
Lemma 36 then implies that there is a full volume set $S \subset T^1X$ and $k \in \mathbb{Z}^+$ such that for almost every $v \in T^1X$, $W^c_f(v)$ is non-compact,

$$\text{(11)} \quad S \cap W^c_f(v) = \{x_{i,j}(v), i \in \mathbb{Z}, 1 \leq j \leq k, \}$$

and

$$\text{(12)} \quad f^i(v) \leq x_{i,1}(v) < x_{i,2}(v) < \cdots < x_{i,k}(v) < f^{i+1}(v), \quad f(x_{i,j}(v)) = x_{i+1,j}(v).$$

Fix an arbitrary $g \in Z^c_{\text{Diff}(T^1X)}(f)$. Lemma 11 implies that $g$ is volume preserving, which implies that modulo a zero set, $gS = S$. As a consequence, there is an $f$–invariant full volume set $\Omega \subset T^1X$ such that for any $v \in \Omega$,

- $W^c_f(v)$ is noncompact;
- $S$ meets $W^c_f(v)$ in exactly $k$ orbits and (11), (12) hold, i.e. we can define $x_{i,j}(v)$ associated to $v$;
- $g(S \cap W^c_f(v)) = S \cap W^c_f(v)$; and
- $f(S \cap W^c_f(v)) = S \cap W^c_f(v)$.

Since $g$ preserves the orientation on $W^c_f$–leaves, for any $v \in \Omega$, the restriction of $g$ to $W^c_f(v) \cap S(= \{x_{i,j}(v), i \in \mathbb{Z}, 1 \leq j \leq k\})$ is an order preserving transformation. By (12), for any $v \in \Omega$, both $g|_{W^c_f(v) \cap S}$, $f|_{W^c_f(v) \cap S}$ are conjugate to a translation on $\mathbb{Z}$.

In particular, for any $v \in \Omega$, there exists $k'(g,v) \in \mathbb{Z}$ such that on $W^c_f(v) \cap S$, we have $g^k = f^{k'(g,v)}$. Moreover by the construction of $x_{i,j}$, the fact that $fg = gf$ implies $k'(g,v)$ is an $f$–invariant function on $v$. Ergodicity of $f$ then implies that $k'(g,v)$ is almost everywhere a constant $k'(g)$, and on a full measure subset of $S$, $g^k = f^{k'(g)}$. But any full measure subset of $S$ is dense in $T^1X$, and hence $g^k = f^{k'(g)}$ on all of $T^1X$. In addition, any $g_1, g_2 \in Z^c_{\text{Diff}(T^1X)}(f)$ satisfying $k'(g_1) = k'(g_2)$ must induce the same transformation on $S \cap W^c_f(v)$ for almost every $v \in T^1X$, which implies that $g_1 = g_2$. Therefore $k'$ induces a group embedding

$$k' : Z^c_{\text{Diff}(T^1X)}(f) \rightarrow \mathbb{Z},$$

and $k'(<f^n>) = k\mathbb{Z}$. Then $Z^c_{\text{Diff}(T^1X)}(f)$ is virtually $<f^n>$. \hfill \Box

**Proof of Theorem 3** Returning to the proof of Theorem 3, if the volume has singular disintegration along $W^c_f$, then Theorem 3 is just a corollary of Theorem 5.

Suppose now the volume has Lebesgue disintegration along $W^c_f$. As in the proof of Theorem 5 we may assume that $f$ is accessible and ergodic. In particular by the arguments in [2] one can show that the disintegration of vol along $W^c_f$ has continuous density function, moreover as in [2] we can even construct a continuous vector field $Y$ tangent to $W^c_f$ such that the continuous flow (a priori it might not be smooth) $\varphi_t$ generated by $Y$ satisfies the following:

- $Y$, and hence $\varphi_t$, is uniformly smooth along the leaves of $W^c_f$, (proved by [2]), and
- $\varphi_1 = f$. 

Fix $r > r_0$, where
\[
    r_0 = \begin{cases} 
        1 & \text{if } X \text{ is real hyperbolic} \\
        2 & \text{otherwise},
    \end{cases}
\]
and consider $h \in Z^c_{\text{Diff}^r(T^1X)}(f)$. By ergodicity of $f$, $h$ preserves the disintegration of volume along $W^s_f$. Therefore $h = \varphi_t$ for some $t \in \mathbb{R}$. If follows that
\[
    Z^c_{\text{Diff}^r(T^1X)}(f) = \{ \varphi_t, t \in D \}, \quad \text{where } D := \{ t \in \mathbb{R} : \varphi_t \in \text{Diff}^r(T^1X) \}.
\]

Since $f = \varphi_1$ is $C^\infty$, it follows that $D$ is a non-empty subgroup of $\mathbb{R}$, and by Lemma ??, $Z^c_{\text{Diff}^r(T^1X)}(f)$ contains $\{ \varphi_t : t \in D \}$ as a finite index subgroup.

**Case 1:** $D$ is discrete. Then, since $f = \varphi_1$, it follows that $< f >$ has finite index in $\{ \varphi_t : t \in D \}$, and hence in $Z^c_{\text{Diff}^r(T^1X)}(f)$. Thus $f$ has virtually trivial centralizer in $\text{Diff}^r(T^1X)$.

**Case 2:** $D$ is dense in $\mathbb{R}$. We use the normal form theory from Section 3.6 to show that the $C^\infty$ smoothness of the $\varphi_t$ with $t \in D$ extends to all $t \in \mathbb{R}$. Here we use the fact that the unperturbed system $\psi_{t_0}$ is the geodesic flow on a locally symmetric space, and so has constant expansion and contraction factors on invariant subbundles, either conformal or exactly 1/2-pinched on the unstable bundle, depending on whether $X$ is real hyperbolic or not. In particular, the Mather spectrum of $D\varphi_{t_0}|_{E^s}$ and $D\varphi_{t_0}^{-1}|_{E^u}$ has either one or two bands, and in the notation of Definition 8, either
\[
    \lambda^s_1 = \mu^u_1 = -1 = \lambda^u_1 = \mu^s_1
\]
in the case where $X$ is real hyperbolic, or
\[
    \lambda^s_1 = \mu^u_1 = -2 = \lambda^u_1 = \mu^s_1; \quad \lambda^s_2 = \mu^u_2 = -1 = \lambda^u_2 = \mu^s_2,
\]
otherwise.

Since $D\varphi_{t_0}|_{E^s}$ and $D\varphi_{t_0}^{-1}|_{E^u}$ have point Mather spectrums (i.e., $\lambda_i = \mu_i$ in Definition 8) on $W^s_{\psi_{t_0}}$, it follows that if $f$ is sufficiently $C^1$—close to $\psi_{t_0}$, then $Df|_{E^s}, Df^{-1}|_{E^u}$ also satisfy the narrow band conditions on $W^s_f$ and moreover
\[
    r > \max(\lambda^s_i(f), \frac{\lambda^u_i(f)}{\mu^s_i(f)}).
\]

Therefore, the triple $(f, \varphi_t, Y)$ satisfies all the conditions in Proposition 31. As a result, by Proposition 31 we know that $D = \mathbb{R}$, $Y$ is a $C^\infty$ vector field and $\varphi_t$ is a $C^\infty$ flow.

As a consequence, by (13) for any $s \geq r$ we have
\[
    Z^c_{\text{Diff}^s(T^1X)}(f) = \{ \varphi_t, t \in \mathbb{R} \} \subset Z^c_{\text{Diff}^r(T^1X)}(f) \subset Z_{\text{Diff}^r(T^1X)}(f),
\]
which implies $Z^c_{\text{Diff}^s(T^1X)}(f) = \{ \varphi_t, t \in \mathbb{R} \}$. Thus by Corollary 1 for any $s \geq r$, $Z^c_{\text{Diff}^s(T^1X)}(f)$ hence $Z^c_{\text{Diff}^s(T^1X)}(f) = Z^c_{\text{Diff}^s_{\text{vol}}(T^1X)}(f)$ is virtually $\{ \varphi_t : t \in \mathbb{R} \} \cong \mathbb{R}$. \qed
\section{Proofs of Theorems \textbf{7} and \textbf{8}}

\subsection{Proof of Theorem \textbf{8}}
Let $f \in \text{Diff}_{\text{vol}}^\infty(M)$ partially hyperbolic and ergodic, preserving a foliation $\mathcal{W}^c$ by circles.

We consider the finite cover $\tilde{M}$ of $M$ such that item (2) of Proposition \textbf{24} holds. By Theorem A of \cite{6}, $f$ has infinitely many periodic center leaves and for any period $T < \infty$, there are only finitely many center leaves $\mathcal{W}^c_{f,0}(x_i)$ such that $f^T(\mathcal{W}^c_{f,0}(x_i)) = \mathcal{W}^c_{f,0}(x_i)$. Moreover we can pick a $T$-periodic center leaf $\mathcal{W}^c_{f,0}(x_0)$ of $f$ such that the covering map $p : \tilde{M} \to M$ restricted to each component of the lift of $\mathcal{W}^c_{f,0}(x_0)$ is a diffeomorphism (in fact this holds for all but at most 4 center leaves, cf. \cite{6} for more details). By Proposition \textbf{23}, for any $h \in Z_{\text{Diff}(M)}(f)$, $h\mathcal{W}^c_f = \mathcal{W}^c_f$, and so $Z_{\text{Diff}(M)}(f)$ is virtually $Z_0$, where

$$Z_0 := \{h \in Z_{\text{Diff}(M)}(f) \mid h(\mathcal{W}^c_{f,0}(x_0)) = \mathcal{W}^c_{f,0}(x_0)\},$$

and $f^T \in Z_0$. Without loss of generality we may assume $T = 1$; for general $T$ the proof is the same.

We pick an arbitrary lift $\hat{x}_0$ of $x_0$ and $\mathcal{W}^c_{f,0}(\hat{x}_0)$ of $\mathcal{W}^c_{f,0}(x_0)$. For any $h \in Z_0$ there is unique lift $\hat{h}$ of $h$ such that $\hat{h}\mathcal{W}^c_{f,0}(\hat{x}_0) = \mathcal{W}^c_{f,0}(\hat{x}_0)$. Let

$$\hat{Z}_0 := \{\hat{h} : h \in Z_0\}$$

It is easy to see that $\hat{h}$ commutes with $\hat{f}$ for all $\hat{h} \in \hat{Z}_0$\footnote{since the commutator $[\hat{h}, \hat{f}]$ is the unique lift on $\tilde{M}$ of $\text{id}\mid_M$ preserving $\mathcal{W}^c_f(\hat{x}_0)$}, and $\hat{Z}_0$ is a group isomorphic to a finite index subgroup of $Z_0$. To prove Theorem \textbf{8} we only need to prove that either $\hat{Z}_0$ is virtually $\mathbb{Z}$ or the volume $\text{vol}_M$ has Lebesgue disintegration along $\mathcal{W}^c$.

By (3) of Proposition \textbf{24} there is an equivariant fibration $\pi : \tilde{M} \to \mathbb{T}^2$ such that $\pi \circ \hat{f} = T_{A_f} \circ \pi$, the fibers of $\pi$ are the the leaves of $\mathcal{W}^c$ and $A_f \in \text{SL}(2, \mathbb{Z})$ is hyperbolic. Then by Lemma \textbf{15} for any $\hat{h} \in \hat{Z}_0$ there exists $A_{\hat{h}} \in \text{SL}(2, \mathbb{Z})$ such that $\pi \circ \hat{h} = T_{A_{\hat{h}}} \circ \pi$.

If $\text{vol}_{\hat{M}}$ has Lebesgue disintegration along $\mathcal{W}^c$, then $\text{vol}_M$ must have Lebesgue disintegration along $\mathcal{W}^c$ as well. If the disintegration of $\text{vol}_M$ along $\mathcal{W}^c$ is not Lebesgue, then by item (4) of Proposition \textbf{24} either

\textbf{Case 1}: the $\text{vol}_{\hat{M}}$ has atomic disintegration along $\mathcal{W}^c$, or

\textbf{Case 2}: $\hat{f}$ is conjugate to $T_{A_f} \times R_\theta$, where $\theta \notin \mathbb{Q}$

holds.

We consider the group $\hat{Z}^c$ generated by all the $\mathcal{W}^c$-fixing elements in $\hat{Z}_0$, and we set $H = \{A_{\hat{h}} : h \in \hat{Z}_0\}$. Then $\hat{Z}_0$ is a group extension of $H$ by $\hat{Z}^c$.

In Case 1, by Lemma \textbf{9} $\hat{Z}^c$ is finite. But by Lemma \textbf{16} $H$ is abelian and virtually $\mathbb{Z}$. This implies that $\hat{Z}_0$ is virtually $\mathbb{Z}$, which is the desired conclusion.

In Case 2, $\hat{f}$ is conjugate to $T_{A_f} \times R_\theta$, for some $\theta \notin \mathbb{Q}$. But this implies that $E^u$ and $E^s$ are jointly integrable. Denote by $\mathcal{W}^H$ the horizontal foliation that is tangent to the lift of
$E^u \oplus E^s$ on $\hat{M}$ and by $\varphi$ the conjugacy map satisfying $\varphi \circ \hat{f} \circ \varphi^{-1} = T_{A_f} \times R_\theta$. We claim that for any $\hat{h} \in \hat{Z}_c$, there exists $\rho = \rho_{\hat{h}} \in \mathbb{T}$ such that

$$\hat{h} = \varphi \circ (\text{id} \times R_\rho) \circ \varphi^{-1}.$$

By the definition of $\hat{Z}_c$ and commutativity, $\varphi^{-1} \circ \hat{h} \circ \varphi$ takes the form

$$\varphi^{-1} \circ \hat{h} \circ \varphi = (x, y) \mapsto (x, R_\rho(x,y)(y)),$$

where $\rho$ is a continuous function satisfying

$$\rho(x,y) = \rho(T_{A_f}(x), R_\theta(y)) = \rho((T_{A_f} \times R_\theta)(x,y)).$$

Transitivity of $T_{A_f} \times R_\theta$ implies that $\rho = \rho(x,y)$ is a constant function. Let

$$D := \{ \rho \in \mathbb{T} : \varphi^{-1} \circ (\text{id} \times R_\rho) \circ \varphi \in \hat{Z}_c \}.$$

If $D$ is discrete, then $\hat{Z}_c$ is finite. Hence $\hat{Z}_0$ is virtually $\mathbb{Z}$, which completes the proof.

If $D$ is dense, we will prove that in this case $\text{vol}_{\hat{M}}$ has Lebesgue disintegration along $\hat{W}^c$, which contradicts our assumption above. By density of $D$, the only invariant measure on $\mathbb{T}$ under $\{ R_\rho : \rho \in D \}$ is the Lebesgue measure $\text{vol}_\mathbb{T}$. Therefore any $\hat{Z}_0$–invariant measure $\mu$ has the form $\varphi_* (\nu \times \text{vol}_\mathbb{T})$, where $\nu$ is some probability measure on $\mathbb{T}^2$. Notice that by Theorem 13, $\hat{f}$ is ergodic with respect to $\text{vol}_{\hat{M}}$, and therefore by Lemma 11, $\text{vol}_{\hat{M}}$ is $\hat{Z}_c$–invariant and hence has the form $\varphi_* (\nu \times \text{vol}_\mathbb{T})$.

We denote by $\text{Pr}^c$ the projection from $\hat{M}$ to $\hat{W}^c(\hat{x}_0)$ along $\hat{W}^H$; and $\text{Pr}^H$ the canonical projection from $\hat{M}$ to $\hat{M}/\hat{W}^c$. It follows that any $\hat{Z}_c$–invariant measure $\mu$ is the product of $\text{Pr}^c_\ast (\mu)$ with $\text{Pr}^H_\ast (\mu)$. In particular, for almost every $x$, the conditional measure $m^c_x$ on $\hat{W}^c(x)$ of $\text{vol}_{\hat{M}}$ has the following form

$$m^c_x = \text{Pr}^c_\ast |_{\hat{W}^c(x)} (\text{Pr}^c_\ast (\text{vol}_{\hat{M}}));$$

that is, $m_x$ is the pullback of $\text{Pr}^c_\ast (\text{vol}_{\hat{M}})$ on $\hat{W}^c(x)$ by $\text{Pr}^c_\ast |_{\hat{W}^c(x)}$.

We now use the key fact that $\hat{W}^H$ is a $C^1$ foliation. Since $\hat{W}^c$ is one dimensional, $f$ is center bunched, and so by [67], the stable and unstable holonomies between $\hat{W}^c$ leaves are uniformly $C^1$. The holonomy maps along $\hat{W}^H$ between $\hat{W}^c$ leaves are the composition of stable and unstable holonomies. The $C^1$–ness of stable and unstable holonomies imply that the holonomy map along $\hat{W}^H$–leaves between center leaves is $C^1$ as well. Since the lift of $E^u \oplus E^s$ is integrable and tangent to $\hat{W}^H$, therefore $\hat{W}^H$ has uniformly $C^1$ leaves. Then by [67] Theorem 6.1], $\hat{W}^H$ is $C^1$.

It follows that $\text{Pr}^c$ is $C^1$, and so $\text{Pr}^c_\ast |_{\hat{W}^c(x)} (\text{Pr}^c_\ast (\text{vol}_{\hat{M}}))$ has continuous density function for any $x$. This implies that $\text{vol}_{\hat{M}}$ has Lebesgue disintegration along $\hat{W}^c$ leaves. Hence $\text{vol}_{\hat{M}}$ has Lebesgue disintegration along $\hat{W}^c$ leaves, completing the proof of Theorem 8.
5.2. Proof of Theorem 7. Theorem 7 is a corollary of Theorem 8 and Theorem 12. Suppose \( f \) satisfies all the hypotheses in Theorem 7. If the disintegration of volume along \( W^c \) is not Lebesgue, then by Theorem 8, \( f \) has virtually trivial centralizer in \( \text{Diff}(M) \).

If the disintegration of volume along \( W^c \) is Lebesgue, then Theorem 12 implies that, up to a 4-fold covering, \( f \) is \( C^\infty \)-conjugate to a rotation extension of a volume preserving Anosov diffeomorphism of \( T^2 \). Moreover, similar to the proof of Proposition 24 we can easily prove that this must be an irrational rotation extension, since \( f \) is ergodic.

Proposition 23 and Lemma 15 imply that the centralizer in \( \text{Diff}_r^r(M) \), \( r \geq 1 \) of an irrational rotation extension of a volume preserving Anosov diffeomorphism is virtually \( \{ \text{id} \times R_\theta : \theta \in \mathbb{T} \} \). Therefore, up to a 4-fold covering, \( \mathcal{Z}_{\text{Diff}_r^r(M)}(f) \) is virtually \( \mathbb{T} \) for any \( r \geq 1 \).

6. Proof of Theorem 6: the Cartan case

In this section we prove Theorem 6 in the case that all the eigenvalues of \( A_f \) are real:

**Proposition 37.** Suppose that \( f_0 : M \to M \) satisfies the hypotheses of Theorem 6 and that all eigenvalues of \( A_{f_0} \) are real. Let \( f \in \text{Diff}_\text{vol}^r(T^d) \) be a \( C^1 \)-small ergodic perturbation of \( f_0 \). Then either the vol\(_{\mathbb{T}^d}\) has Lebesgue disintegration along \( W^c \) or \( \mathcal{Z}_{\text{Diff}_r^r(M)}(f) \) is virtually \( \mathbb{Z}^l \), for some \( l < \ell(A_f) \).

As mentioned in the introduction, a key idea in the proof of Proposition 37 is to construct genuinely new partially hyperbolic elements commuting with \( f \), an argument that we now detail.

6.1. The groups \( G \) and \( G_0 \). Two central players in the proof of Proposition 37 are groups \( G \) and \( G_0 \), which we define in this subsection. We start with an easy observation.

**Lemma 38.** Let \( f \) be as in Proposition 37. Then there is a \( Df \)-invariant dominated splitting
\[
T_{\mathbb{T}^d} = \bigoplus_i E^i \oplus E^c
\]
such that for any \( i \), \( \dim E^i = 1 \).

Consider an arbitrary element \( g \in \mathcal{Z}_{\text{Diff}_r^r(T^d)}(f) \). Proposition 23 implies that \( gW^c = W^c \). Thus \( g \) and \( f \) induce homeomorphisms \( \bar{f}, \bar{g} \) on the topological manifold \( \mathbb{T}^d/W^c \) such that \( \bar{f} \bar{g} = \bar{g} \bar{f} \). Moreover \( \bar{f} \) is Hölder conjugate to the hyperbolic automorphism \( T_{A_f} \) on \( \mathbb{T}^{d-1} \).

By Lemma 15, \( \bar{g} \) is conjugate to an affine map by the same conjugacy. We denote the linear part of this affine map by \( T_{A_g} \), where \( A_g \in \text{SL}(d-1, \mathbb{Z}) \).

Let \( \pi : \mathbb{T}^d \to \mathbb{T}^{d-1} \) be the fibration given by Lemma 25 which satisfies \( \pi \circ f = T_{A_f} \circ \pi \). Then the center leaf \( \pi^{-1}(0) \) is invariant under \( f \); denote it by \( W^c_f(x_0) \). We use this leaf to define \( G \) and \( G_0 \).
Definition 12. Let $G_0$ be the group of all the elements $g \in Z_{\text{Diff}^2(T^d)}(f)$ such that $g$ fixes $W_f^c(x_0)$ and preserves the orientation of $W^c$ and $T^d/W^c$. Let $G < \text{SL}(d-1, \mathbb{Z})$ be the group generated by $\{A_g : g \in G_0\}$.

Lemma 39. Suppose $f$ satisfies the hypotheses of Theorem 6. Then

1. $Z_{\text{Diff}^2(T^d)}(f)$ is virtually $G_0$.
2. If the disintegration of vol along $W^c$ is not Lebesgue, then $G_0, G$ are finitely generated abelian groups.
3. If the disintegration of vol along $W^c$ is not Lebesgue, then at least one of the following cases holds:
   I. $G$ is virtually $\mathbb{Z}^\ell$ for some $\ell < \ell(A_f)$.
   II. $G$ is a finite index subgroup of $Z_{\text{SL}(d-1, \mathbb{Z})}(A_f)$. In particular, $G$ induces a maximal Cartan Anosov action on $T^{d-1}$ if $\text{rank}(Z_{\text{SL}(d-1, \mathbb{Z})}(A_f)) > 1$.

Proof. (1) Let $Z^+$ be the group of all the elements $g \in Z_{\text{Diff}^2(T^d)}(f)$ such that $g$ preserves the orientation of $W^c$ and $T^d/W^c$. Clearly $Z^+$ has finite index in $Z_{\text{Diff}^2(T^d)}(f)$. Denote by $Z^c$ the set of center-fixing elements of $Z^+$.

Consider the map from $Z^+$ to $Z_{\text{Homeo}^+}(T^{d-1})(T_{A_f})$, sending $g$ to $\pi_*g$. The kernel is $Z^c$, and so $Z^+/Z^c$ is isomorphic to a subgroup of $Z_{\text{Homeo}^+}(T^{d-1})(T_{A_f})$. By Lemmas 15 and 16, the group $Z_{\text{Homeo}^+}(T^{d-1})(T_{A_f})$ is virtually abelian, and hence so is $Z^+/Z^c$.

Note that since there are finitely many center leaves fixed by $f$, and each element of $Z^+$ permutes the fixed center leaves, there exists $k \geq 1$ such that for every element $g \in Z^+/Z^c$, we have $g^k \in G_0/Z^c$. Thus the virtually abelian quotient

$$\frac{Z^+/Z^c}{G_0/Z^c} \cong Z^+/G_0$$

has the property that every element has order at most $k$ and is therefore finite. This proves that $G_0$ has finite index in $Z^+$, as claimed.

(2) Since $A_f$ is irreducible, Lemma 16 implies that $Z_{\text{SL}(d-1, \mathbb{Z})}(A_f)$ (hence $G$) is a finitely generated abelian group.

Now we prove that $G_0$ is a finitely generated abelian group if the disintegration of vol$_{T^d}$ along $W_f^c$ is not Lebesgue. By Lemma 25, there are two possibilities:

**Case 1:** the volume vol$_{T^d}$ has atomic disintegration along $W^c$. Observe that the map $h \mapsto A_h$ is a group homomorphism from $G_0$ to $G$. The elements in the kernel are center-fixing. Lemma 9 implies that the center-fixing elements form a cyclic group, and hence $G$ is solvable. For arbitrary $g_1, g_2 \in G_0$, we consider the commutator $h = [g_1, g_2] := g_1g_2g_1^{-1}g_2^{-1} \in G_0$. We claim that $h = \text{id}$. Otherwise, $h$ is center-fixing, since $G$ is abelian. Mimicking the proof of Lemma 9, there exist $k \in \mathbb{Z}^+$ and $k' \in \mathbb{Z}/k\mathbb{Z}$ such that on every center leaf, $h$ has rotation number $k'/k$.

In particular, if $h \neq \text{id}$ then $h|_{W_f^c(x_0)}$ has non-zero rotation number, where $W_f^c(x_0)$ is the $G_0$-fixed center leaf we defined in Section 6.1. On the other hand, $G_0|_{W_f^c(x_0)}$ is a
solvable, orientation-preserving action on a circle. It is known (cf. [59]) that rotation number induces a group homomorphism from any solvable subgroup of Homeo$^+(S^1)$ to $T^1$. Thus the commutator $h|_{Wc_f(x_0)}$ must have rotation number 0, which is a contradiction. We conclude that $h := [g_1, g_2]$ is the identity, and so $G_0$ is abelian.

**Case 2:** the diffeomorphism $f$ is topologically conjugate to $T_{A_f} \times R_\theta : T^{d-1} \times \mathbb{T} \to T^{d-1} \times \mathbb{T}$, for some $\theta \not\in \mathbb{Q}$. In this case, we again consider $h := [g_1, g_2] \in G_0$ for arbitrary $g_1, g_2 \in G_0$. Suppose $h \neq \text{id}$; then as in the proof of Theorem 8, by the same conjugacy as in the proof of Theorem 8, vol$_{T^d}$ must have Lebesgue disintegration along $W^c_f$, which contradicts our assumption.

If $\rho \not\in \mathbb{Q}/\mathbb{Z}$, then by the same argument as in the atomic case, $h|_{Wc_f(x_0)}$ can only have 0 rotation number, which implies $h = \text{id}$, contradicting our assumption on $h$. In conclusion, $G_0$ is abelian.

To prove $G_0$ is finitely generated, again we consider the homomorphism $h \mapsto A_h$, from $G_0$ to $G$ whose kernel consists of the center fixing elements of $G_0$. A discussed above, the group of center fixing elements is cyclic and finite. We thus obtain that $G_0$ is finitely generated.

(3) This is a corollary of Lemma 16. For more details cf. [71].

Returning to the proof of Proposition 37, we consider separately the two cases in item (3) of Lemma 39:

I. $G$ is virtually $\mathbb{Z}^\ell$ for some $\ell < \ell(A_f)$.

II. $G$ is a finite index subgroup of $\mathcal{Z}^{SL(d-1,\mathbb{Z})}(A_f)$. In particular, $G$ induces a maximal Cartan Anosov action on $T^{d-1}$ if rank($\mathcal{Z}^{SL(d-1,\mathbb{Z})}(A_f)$) > 1.

**Case I.** As in the proof of item (1) of Lemma 39, we denote by $\mathcal{Z}^c$ the set of center-fixing elements in $\mathcal{Z}^+$. Then $G_0$ is a group extension of $G$ by $\mathcal{Z}^c$. As in the proof of Theorem 8 we conclude that

(1) either vol$_{T^d}$ has Lebesgue disintegration along $W^c_f$,

(2) or vol$_{T^d}$ has singular disintegration along $W^c_f$ and $\mathcal{Z}^c$ is finite.

For the second case, by item (2) of Lemma 39 $G$ is a finitely generated abelian group. As a consequence there is a subgroup $G_1$ of $G_0$ isomorphic to $G$, which is virtually $\mathbb{Z}^\ell$ for some $\ell < \ell(A_f)$. By finiteness of $\mathcal{Z}^c$, $G_0$ is virtually $G_1$, therefore $G_0$ is virtually $\mathbb{Z}^\ell$. Thus Proposition 37 follows from (1) of Lemma 39.

**Case II.** Proposition 37 is a corollary of the following key proposition.
**Proposition 40.** Suppose \( f \) is as in Theorem 6 such that all the eigenvalues of \( A_f \) are real, and \( G_0, G \) are as in Definition 12. If \( G \) induces a maximal Cartan Anosov action on \( T^{d-1} \), then \( \text{vol}_T \) has Lebesgue disintegration along \( W^c_f \).

The rest of the section is devoted to the proof of Proposition 40, and henceforth we assume that \( G \) induces a maximal Cartan Anosov action on \( T^{d-1} \). Without loss of generality we may assume that \( G \) and \( G_0 \) are finitely generated abelian groups (otherwise Proposition 40 follows from Lemma 39). Then as in Case I, there is a subgroup \( G_1 \) of \( G_0 \) isomorphic to \( G \), through the map \( g \mapsto A_g \). Replacing \( G_0 \) with \( G_1 \), we may thus assume that \( G_0 \) is isomorphic to \( G \) through the map \( g \mapsto A_g \). Moreover we may assume \( G, G_0 \) are torsion free (otherwise we consider the free parts of them).

6.2. The projection \( \pi \). As above, let \( \pi \) be given by Lemma 25. Observe that for any \( h \in G_0 \), we have

\[
\pi \circ h = A_h \circ \pi.
\]

We will use the following classical Hölder estimate later. Notice that for any \( x \in M \), \( \pi : W^u_f(x) \to W^u_{A_f}(\pi(x)) \) is a homeomorphism.

**Lemma 41.** There exist \( C, \delta > 0 \) such that for any \( x \in T^d \) and \( y \in W^u_f(x, \text{loc}) \),

\[
d_{T^{d-1}}(\pi(x), \pi(y)) \leq C \cdot d_{T^d}(x, y)^\delta, \quad d_{T^d}(x, y) \leq C \cdot d_{T^{d-1}}(\pi(x), \pi(y))^\delta.
\]

6.3. Uniform hyperbolicity on the horizontal distribution. The following proposition is a significant step in the proof of Proposition 40. Recall that by linearity of the action of \( G \), we can define the Lyapunov functionals and associated (hyperbolic) Weyl chamber picture as in Section 3.8, independently of the invariant measure. In Lemma 38 we define the \( G_0 \)-invariant dominated splitting \( \oplus_i E^i \oplus E^c \).

**Proposition 42.** Assume that \( G \) induces a maximal Cartan Anosov action on \( T^{d-1} \). For any \( i \) and any element \( h \in G_0 \) such that \( A_h \) is not in any Weyl chamber wall of the action of \( G \), \( D_h \) uniformly contracts (or expands) \( E^i \).

6.4. \( G, G_0 \) have the same hyperbolic Weyl chamber picture.

**Lemma 43.** Any \( G_0 \)-invariant ergodic measure \( \nu \) has the same hyperbolic Weyl chamber picture as \( G \).

**Proof.** First we prove that the action of \((G_0, \nu)\) has the same Weyl chamber walls as \( G \). Lemma 17 implies that the foliations \( W^u_f \) and \( W^s_f \) are \( G_0 \)-invariant. Moreover \( \pi(W^*_f) = W^*_A_f \), for \( * \in \{u, s\} \). Therefore to analyze the hyperbolic Weyl chamber walls of \( G_0 \) we need only consider the action of \( G_0 \) on \( W^u, W^s \) separately. We show this for \( W^u \); the proof for \( W^s \) is analogous.
Recall that by Lemma 30 to prove Lemma 43, we only need to establish the following claim: for any \( A_h \in G \) that is not in any Weyl chamber wall, if \( A_h \) has \( d_u^{-}, d_u^{+} \)-dimensional stable and unstable distributions respectively within \( W_u^A_f \), then \( h \) has \( d_u^{-}, d_u^{+} \)-dimensional stable and unstable topological foliations (with exponential contracting or expanding speed) respectively within \( W_u^h \).

In fact if the claim holds, then Lemma 30 implies that for typical \( h \in G \), the map \( h \) and the matrix \( A_h \) have the same number of positive (resp. negative) Lyapunov exponents with respect to any \( G_0 \)-invariant ergodic measure \( \nu \). From this it follows that \( G \) and \( (G_0, \nu) \) have the same Weyl chamber walls.

To prove the claim, note that the restriction \( \pi : W^u_f(x) \to W^u_{T_Af}(\pi(x)) \) is a uniformly bi-Hölder homeomorphism for any \( x \in T^d \). Thus \( \pi \) restricted to each \( W^u_f \) leaf is a bi-Hölder conjugacy between \( h|_{W^u} \) and \( T_Ah|_{W^u_{T_Af}} \). This bi-Hölderness implies that the hyperbolicity of \( T_Ah|_{W^u_{T_Af}} \) lifts under \( \pi \) to uniform hyperbolicity of \( h|_{W^u} \). As a result, \( G \) and \( (G_0, \nu) \) have the same Weyl chamber walls.

Next consider the Lyapunov functionals \( \{\lambda^{u,i}_{G_0}(\cdot, \nu)\}, i = 1, \ldots, \dim E^u_f \} \) associated to the action of \( G_0 \) on \( W^u_f \) with respect to an ergodic measure \( \nu \), and the Lyapunov functionals \( \{\lambda^{u,i}_G(\cdot), i = 1, \ldots, \dim E^u_{T_Af} \} \) associated to the action of \( G \) on \( W^u_{T_Af} \). By our discussion above, without loss of generality we may assume that Weyl chamber wall \( \ker \lambda^{u,i}_{G_0}(\cdot, \nu) \) coincides with that of \( \lambda^{u,i}_G(\cdot) \). Moreover

\[
\lambda^{u,i}_{G_0}(f, \nu) > 0, \quad \text{and} \quad \lambda^{u,i}_G(T_Af) > 0.
\]

then by Lemma 32 (identifying \( G, G_0 \) with \( Z^k \) in the obvious way), the Weyl chamber picture of the action of \( G_0 \) on \( W^u_f \) with respect to \( \nu \) is the same as that of \( G \) on \( W^u_{T_Af} \). The same argument applied to the action of \( G_0 \) on \( W^s_f \) gives that the Weyl chamber picture of the action of \( G_0 \) on \( W^s_{T_Af} \) with respect to \( \mu \) is the same as that of \( G \) on \( W^u_{T_Af} \). In conclusion, \( (G_0, \nu) \) has the same hyperbolic Weyl chamber picture as \( G \).

6.5. Estimates for elements in the same Weyl chamber.

**Lemma 44.** Suppose \( h \in G_0 \) has the property that \( A_h \) and \( A_f \) lie in the same Weyl chamber. Then

1. there exists \( c > 0 \) such that for any \( h \)-invariant ergodic measure \( \nu \),
\[
\lambda^+(Dh|_{E^+_f}, \nu) < -c, \quad \lambda^-(Dh|_{E^-_f}, \nu) > c;
\]

2. for every \( i \), \( Dh \) either uniformly contracts or uniformly expands \( E^i \).

**Proof.** (1): The proof is basically the same as that of Lemma 43. If \( A_h \) is in the same Weyl chamber as \( A_f \), then as in Lemma 43 we have that \( T_Ah \) uniformly contracts \( \pi(W^s_f) \)
and uniformly expands $\pi(\mathcal{W}^u_f)$.

By Lemma 41, for any $x \in \mathbb{T}^d$ and $y \in \mathcal{W}^u_f(x, \text{loc})$,

$$\limsup_{n \to \infty} \frac{1}{n} \log d_{\mathcal{W}^u_f}(h^n(x), h^n(y)) \leq \delta \cdot \lambda(T_{Ah} | \pi(\mathcal{W}^u_f)) < 0,$$

where $\delta$ is the H"older exponent of $\pi$ in Lemma 41. Then for any $h$-invariant ergodic measure $\nu$, Lemma 30 and (15) together imply that for $\nu$—almost every $x \in \mathbb{T}^{n+1}$, there the Pesin stable manifold passing through $x$ is $\mathcal{W}^u_f(x, \text{loc})$, and

$$\lambda(Dh|_{E^s_f}, \nu) \leq \delta \lambda(T_{Ah} | \pi(\mathcal{W}^u_f)) < 0.$$

Similarly, for any $h$-invariant ergodic measure $\nu$, we have

$$\lambda^-(Dh|_{E^f}, \nu) \geq \delta \lambda^-(T_{Ah} | \pi(\mathcal{W}^u_f)) > 0.$$

Setting $c := \min(|\delta \lambda^+(T_{Ah} | \pi(\mathcal{W}^u_f))|, |\delta \lambda^-(T_{Ah} | \pi(\mathcal{W}^u_f))|)$ completes the proof of (1).

(2): Since $Dh|_{E^f} \cap Dh|_{E^s}$ are continuous, item (1) of Lemma 10 implies that $Dh|_{E^f}$ and $Dh|_{E^s}$ satisfy the conditions of Lemma 10. Thus $Dh|_{E^f}$ and $Dh^{-1}|_{E^s}$ have uniform exponential growth, which implies (2).

□

As a corollary, Proposition 42 holds in one case:

$$h \text{ and } f \text{ lie in the same hyperbolic Weyl chamber}$$

$$\iff A_h \text{ and } A_f \text{ lie in the same Weyl chamber}.$$

6.6. Proof of Proposition 42. In this section we will prove Proposition 42 for those $h$ for which $A_h$ and $A_f$ lie in different Weyl chambers.

**Step 1.** First we assume $A_h$ is in the Weyl chamber adjacent to that of $A_f$. Since we are assuming that Case II in item (3) of Lemma 39 holds, we have that $G$ induces a maximal Cartan Anosov action on $\mathbb{T}^{d-1}$, and the signs of all the exponents of $A_f$ and $A_h$ are the same except for exactly one exponent.

In particular, without loss of generality we may assume that the affine foliations $\mathcal{W}^s_{T_{Ah}}$, $\mathcal{W}^u_{T_{Ah}}$, $\mathcal{W}^s_{T_{Af}}$, $\mathcal{W}^u_{T_{Af}}$ on $\mathbb{T}^{d-1}$ (for $* \in \{u, s\}$) satisfy

$$\mathcal{W}^u_{T_{Ah}} \supset \mathcal{W}^u_{T_{Af}} \cap \mathcal{W}^s_{T_{Ah}} \cap \mathcal{W}^s_{T_{Af}} = \mathcal{W}^u_{T_{Ah}} \oplus (\mathcal{W}^s_{T_{Ah}} \cap \mathcal{W}^s_{T_{Af}}),$$

and $\mathcal{W}^s_{T_{Ah}} \cap \mathcal{W}^u_{T_{Af}}$ is 1-dimensional.

We consider the topological foliation $\mathcal{W}^\#$ defined to be the lift of $\mathcal{W}^s_{T_{Ah}} \cap \mathcal{W}^u_{T_{Af}}$ by $\pi^{-1}$ on $\mathcal{W}^u_f$—leaves. It is easy to see $\mathcal{W}^\#$ is a 1-dimensional $G_0$—invariant topological foliation. Since $\pi$ is bi-Hölder restricted to stable and unstable leaves of $f$, we know $h$ contracts $\mathcal{W}^\#$ exponentially fast, i.e. for any $x \in \mathbb{T}^d$, $y \in \mathcal{W}^\#(x, \text{loc})$,

$$\limsup_{n \to \infty} \frac{1}{n} \log d_{\mathcal{W}^\#}(h^n(x), h^n(y)) \leq \lambda_0 < 0,$$

where $\mathcal{W}^\#(x, \text{loc})$ is defined to be the lift of $\mathcal{W}^s_{T_{Ah}} \cap \mathcal{W}^u_{T_{Af}}(\pi(x), \text{loc})$. 

PATHOLOGY AND ASYMMETRY 51
Lemma 30 implies that for any $G_0$–invariant ergodic measure $\nu$, the leaf $W^s(x)$ coincides with the (global) Pesin stable manifold $W^s_{h|W^u_f}(x, gl)$ of $x$ (for the restricted dynamics $h|W^u_f$), for $\nu$—almost every $x$ (since globally $h$ contracts $W^u$ exponentially fast). Therefore $W^s(x, loc)$ is tangent to $E^u_h \cap E^u_f(x)$ at $x$ for $\nu$—almost every $x$, where $E^u_h$ is the Oseledec stable space of $(Dh, \nu)$, and $E^u_f$ is the unstable distribution of $f$.

In addition if $y \in W^s(x)$ is a regular point in the sense of Pesin theory, then $W^s(x)$ is the Pesin stable manifold of $y$ for the restricted dynamics $h|W^u_f$ as well (we will use this fact later).

Recall that there is a $Df$—invariant dominated splitting $T\mathbb{T}^d = \oplus E^u \oplus E^c$ for the restricted dynamics $h|W^u_f$. The bundles in this splitting control the Oseledec decompositions of $f$–invariant measures:

**Lemma 45.** For any $G_0$—invariant ergodic measure $\nu$, there exists $i$ such that the measurable distribution $(E^s_h \cap E^u_f, \nu)$ defined above coincides with $E^i \nu$—almost everywhere.

**Proof.** Evidently $(E^s_h \cap E^u_f, \nu)$ is a $\nu$—almost everywhere defined 1—dimensional $Df$—invariant distribution within $E^u_f$. By existence of $Df$—invariant finest dominated splitting in $E^u_f$ we know that (a priori) up to a $\nu$—negligible set, $\mathbb{T}^d$ can be decomposed as a finite union of measurable sets $\cup_i X_i$ such that for all $i$, $f(X_i) = X_i$, and for all $x \in X_i$:

$$E^s_h \cap E^u_f(x) = E^i(x).$$

But this splitting is also $G_0$—invariant, by Lemma 13. Therefore by $G_0$—ergodicity of $\nu$ we have that one of the $X_i$ has full $\nu$—measure (here we use the fact that $(E^s_h \cap E^u_f, \nu)$ is $G_0$—invariant as well). \(\square\)

**Step 2.** Now we consider the case $\nu = \text{vol}_{\mathbb{T}^d}$. By Lemma 15 there exists $i$ such that $E^s_h \cap E^u_f = E^i$, $\text{vol}$—almost everywhere on $\mathbb{T}^d$. Let $E := E^i$. We first claim that for $\text{vol}$—almost every $x \in \mathbb{T}^d$, $W^u(x)$ is a $C^1$ submanifold tangent to $E$ everywhere. The absolute continuity of the unstable foliation $W^u_f$ implies that any set of full volume meets almost every leaf of $W^u_f$ in a set of full leaf volume. Hence there is a full volume, invariant set $P \subset M$ of Pesin regular points for $(f, \text{vol})$ in $M$ such that for every $p \in P$, the leaf $W^u_f(p)$ meets $P$ in a set of full leafwise volume.

Let $N := \bigcup_{P \in M} W^u_f(p)$ be the disjoint union of unstable manifolds: it is a non-compact $C^\infty$ Riemannian manifold. The maps induced by $f$ and $h$ on $N$ are $C^\infty$, with uniform bounds on the derivatives. Applying the arguments in 66 to the (Pesin regular) points in $P_N := \bigcup_{p \in P} P \cap W^u_f(p)$, we obtain that the Pesin local stable manifolds $W^{Pe}_{h|W^u_f}(x, loc)$ of $h|N$ form an absolutely continuous family of disks. In particular, for every $p \in P$, a set $B \subset W^u_f(p)$ has $\text{vol}_{W^u_f(p)}$—measure $0$ in $W^u_f(p)$ if and only if it has $\text{vol}_{W^{Pe}_{h|W^u_f}(p, loc)}$—measure $0$ in $W^{Pe}_{h|W^u_f}(z, loc)$, for almost every $z \in W^u_f(p)$. 

This implies in particular that for vol—almost every $x \in \mathbb{T}^d$, there is a dense subset of $y \in W^{PE}_{h|W^y_j}(x, loc)$ such that $y$ belongs to $P$. In particular, for such $y$, the smooth disk $W^{PE}_{h|W^y_j}(x, loc)$ is tangent to $E(y)$.

Consequently for vol—almost every $x \in \mathbb{T}^d$, the submanifold $W^{PE}_{h|W^y_j}(x, loc)$ is tangent to $E$ on a dense subset, and hence by continuity of $E$, $W^{PE}_{h|W^y_j}(x, loc)$ is tangent to $E$ everywhere and is therefore a $C^1$ submanifold. By Pesin theory, the size of $W^{PE}_{h|W^y_j}(x, loc)$ is at least $r_0 \cdot e^{-ck}$, where $\epsilon$ is a small constant (compared to Lyapunov exponents), and $k$ only depends on the Pesin block $\Lambda_k$ containing $x$. We pick a regular $x$ in some $\Lambda_k$ such that $f^{-n}(x), n \geq 0$ intersects $\Lambda_k$ infinitely many times (this property holds for $\nu$—almost every $x$, by Poincaré recurrence). Then the submanifold $f^n(W^{PE}_{h|W^y_j}(f^{-n}(x), loc))$

- is contained in $W^\#(x)$
- is tangent to $E$ everywhere.
- has internal diameter at least $O(e^{\lambda n} \cdot r_0 e^{-ck})$ for some $\lambda > 0$, if $f^{-n}(x) \in \Lambda_k$.

Letting $n$ tend to infinity, we obtain that $W^\#(x) = \cup_{n \geq 0} f^n(W^{PE}_{h|W^y_j}(f^{-n}x, loc))$ is a $C^1$ submanifold tangent to $E$ everywhere. In conclusion, for vol—almost every $x$, $W^\#(x)$ is a $C^1$ submanifold and tangent to $E$. We denote the set of such $x$ by $K$. Then $K$ is dense since $K$ has full volume.

**Step 3.** We now consider a cone field $C$, defined as follows. Let $E$ be defined as in Step 2, and let $E' := \oplus_{E^j \neq E, E^j \subset E^j} E^j$. Then any $v \in E^j_l$ may be written in the form

$$v = (v_1, v_2), \quad v_1 \in E, v_2 \in E'. $$

For $x \in \mathbb{T}^d$, define the cone $C(x)$ at $x$ by

$$C(x) = C(x, \epsilon_1, \epsilon_2) := \exp_{W^y_j}(\{ (v_1, v_2) \in E^j_l(x) : \|v_2\| \leq \epsilon_2 \|v_1\|, \|v_1\| \leq \epsilon_1 \}).$$

Here $\exp_{W^y_j}$ is the exponential map associated to the foliation $W^y_j$. For $\epsilon_1, \epsilon_2 > 0$, consider the family of topological disks $G = \{ G(x, \epsilon_1, \epsilon_2) : x \in \mathbb{T}^d \}$ defined by

$$G(x) = \{ \exp_{W^y_j}(\gamma) : \gamma \text{ is a graph of a } \epsilon_2 - \text{Lipschitz function from } B(0, \epsilon_1) \subset E \text{ to } E' \}. $$

By definition, the $\exp_{W^y_j}(\gamma$-preimage of any curve in $G(x, \epsilon_1, \epsilon_2)$ is contained in $C(x, \epsilon_1, \epsilon_2)$. And by continuity of $E$, if $\epsilon_1, \epsilon_2$ are chosen small enough (the smallness of $\epsilon_2$ depends on $\epsilon_1$), then for $x \in K$, the local manifold $W^\#(x, loc)$ passing through $x$ is an element in $G(x, \epsilon_1, \epsilon_2)$, where $K$ is defined in the end of Step 2. (Here we restrict the diameter of $W^\#(x, loc)$ if necessary).

The cone field $C$ has the important property that for any $C^1$ path $\sigma : I \rightarrow W^y_j(x)$ with $\sigma(0) = x$, if $\sigma'(0) \in E^j$ for some $E^j \neq E$, then $\sigma$ can not be tangent to $C(x)$.

We now consider an arbitrary $y \in \mathbb{T}^d$ and claim that if $W^\#(y)$ is differentiable at $y$, then $T_yW^\#$ cannot be any $E^j \neq E$. This claim holds if $y \in K$ by definition of $K$. If
y \notin K$, by density of $K$, we may choose a sequence $y_k \to y$, $k \to \infty$ such that $y_k \in K$. It is easy to see that the local manifold of $\mathcal{W}^\#(y_k)$ tends to that of $\mathcal{W}^\#(y)$. Morover by our discussion above, for each $k$, the local manifold of $\mathcal{W}^\#(y_k)$ is an element of $\mathcal{G}(y_k, \epsilon_1, \epsilon_2)$.

Since $\mathcal{W}^\#$ is a topological foliation, the local manifold $\mathcal{W}^\#(y, loc)$ is the uniform limit of the $\mathcal{W}^\#(y_k, loc)$. As the $\mathcal{W}^\#(y_k, loc)$ belong to $\mathcal{G}(y_k, \epsilon_1, \epsilon_2)$, so does $\mathcal{W}^\#(y, loc)$. It follows that $T_y \mathcal{W}^\#$ lies in the cone $C(y, \epsilon_1, \epsilon_2)$, and so $T_y \mathcal{W}^\#$ cannot be any $E^j \neq E$, if $\epsilon_1, \epsilon_2$ are sufficiently small.

**Step 4.** By Lemma 45, for any $G_0$–invariant ergodic measure $\nu$, $(E^s_h \cap E^f_j, \nu)$ coincides with some $E^j$, $\nu$–almost everywhere, and the Pesin stable manifold of $(E^s_h \cap E^f_j, \nu)$ for $\nu$–almost every $x$ is tangent to this $E^j$. But by our discussion in Step 3, this $E^j$ must be $E$, no matter which $\nu$ we chose. In summary, we have shown that for any $G_0$–invariant ergodic measure $\nu$, the one dimensional Oseledec stable distribution of $Dh$ within $E^s_0$ must be $E$, and the Pesin stable manifold of $(h|_{\mathcal{W}^\#}, \nu)$ coincides with $\mathcal{W}^\#$, $\nu$–almost everywhere.

But by (16), we know that $h$ contracts $\mathcal{W}^\#$ exponentially fast (with Lyapunov exponent smaller than $0 < \lambda_0$ defined in (16)). Then Lemma 29 gives the following important property:

- There exists $\lambda_0 < 0$ such that for any $G_0$–invariant ergodic measure $\nu$, the Lyapunov exponent of the cocycle $Dh|_E$ with respect to $\nu$ is less than $\lambda_0$.

We claim that this property holds for any $h$–invariant measure as well. Suppose there is an $h$–invariant measure $\nu_0$ such that the Lyapunov exponent of the cocycle $Dh|_E$ with respect to $\nu_0$ greater or equal to $\lambda_0$. Then for any $g \in G_0$, $g_* \nu_0$ is $h$–invariant and also satisfies this property. Therefore by an averaging argument we can construct a $G_0$–invariant measure $\nu_1$ such that the Lyapunov exponent of $Dh|_E$ with respect to $\nu_1$ is greater or equal to $\lambda_0$. Then by considering an ergodic component of $\nu_1$, we get a contradiction.

In summary, for any $h$–invariant measure $\nu$, the Lyapunov exponent of $Dh|_E$ with respect to $\nu$ is less than $\lambda_0$. Then by Lemma 10, $Dh$ uniformly contracts $E$.

**Step 5.** We complete the proof of Proposition 42 in the case where $A_h$ and $A_f$ lie in adjacent Weyl chambers. Since $Dh$ uniformly contracts $E$, we only need to prove that $Dh$ uniformly contracts $E^s_j$ and uniformly expands $E' = \oplus_{E^j \subset E^j_0 \cap E^\#_0} E^j$. By the same proof as Lemma 44, we obtain that $Dh$ uniformly contracts $E^s_j$. For $E'$, by considering $T_A$ and the map $\pi$, we obtain a $(\dim E^u_j - 1)$–dimensional topological foliation $\mathcal{W}'$ of $\mathcal{W}^\#$ such that $h$ expands $\mathcal{W}'$ exponentially fast. By Step 4, $h$ uniformly contracts $\mathcal{W}^\#$. Using an argument similar to the proof of Lemma 44, we obtain that for any $G_0$–invariant ergodic measure $\nu$, the top $(\dim E^u_j - 1)$ Lyapunov exponents are positive and uniformly bounded away from 0. Therefore, mimicking the argument in Step 4, using Lemma 10 we obtain that $h$ uniformly expands $E'$.

As a corollary, we obtain that both $E$ and $E'$ are integrable, and the integral manifolds are $\mathcal{W}^\#$ and $\mathcal{W}'$ (i.e. the lifts of $\mathcal{W}^\#_{T_A} \cap \mathcal{W}^\#_{T_f}$ and $\mathcal{W}'_{T_A}$ by $\pi^{-1}$). Recall that $\mathcal{W}^\#$ is
the lift of $\mathcal{W}_{T_{A_h}}^{i} \cap \mathcal{W}_{T_{A_h}}^{j}$ by $\pi^{-1}$. Therefore the $Df$-invariant bundles $E$ and $E \oplus E^c_f$ are both integrable.

**Step 6.** Notice that in Steps 1-5, there is no restriction on the bundle $E$. In fact by the following lemma we know for any $E^i$ there is a hyperbolic Weyl chamber adjacent to that of $f$ such that for any element $h(i)$ in the chamber, the signs of all the exponents of $A_f$ and $A_h(i)$ are the same except for exactly one Lyapunov exponent corresponding to $E^i$.

**Lemma 46.** Suppose $A \in \text{SL}(n, \mathbb{Z})$ is an irreducible hyperbolic matrix such that all eigenvalues of $A$ are real. Let $\mathbb{R}^n = \oplus V^i$ be the eigenspace decomposition of $A$. Then for any $i$ there exists $B \in \text{SL}(n, \mathbb{Z})$ such that $AB = BA$ and

- For any $j \neq i$, $A$ contracts (resp. expands) $V^j$ iff $B$ contracts (resp. expands) $V^j$.
- If $A$ contracts (resp. expands) $V^i$, then $B$ expands (resp. contracts) $V^i$.

**Proof.** It is a corollary of Lemma 33 and the definition of maximal structure. \qed

It follows that for every $Df$-invariant distribution $E^i$, both $E^i$ and $E^i \oplus E^c_f$ are integrable. We denote by $\mathcal{W}^i$ and $\mathcal{W}^{ic}$ the integral foliations.

The following lemma might not be necessary in the Cartan case but the proof will be useful in non-Cartan case. Recall that $f \in \text{Diff}^2_\text{vol}(\mathbb{T}^d)$ is a $C^1$—small perturbation of $f_0$, where $f_0$ is an isometric extension of $T_{A_f} = T_{A_f}$. As in Section 3.4.9, $Df_0$ preserves a dominated splitting $\mathbb{T}^d = \oplus E^i_{f_0} \oplus E^c_{f_0}$, and the distributions $E^i_{f_0}, E^c_{f_0}$ are integrable. Denote by $\mathcal{W}^i_{f_0}, \mathcal{W}^{ic}_{f_0}$ the integral manifolds of $E^i_{f_0}$ and $E^i_{f_0} \oplus E^c_{f_0}$ respectively. Without loss of generality we may assume $E^i_{f_0}$ and $E^c_{f_0}$ are ordered by increasing size of Lyapunov exponents. Then we have

**Lemma 47.** Any leaf conjugacy $h^c$ from $(f_0, \mathcal{W}^c_{f_0})$ to $(f, \mathcal{W}^c_f)$ maps each $\mathcal{W}^{ic}_{f_0}$ to $\mathcal{W}^{ic}_f$.

**Proof.** By continuity of $\pi$, there exist positive constants $\epsilon_0, C_0$ such that for any $i$ and any pair of points $x, y$ with $y \in \mathcal{W}^i_f(x)$, if $d_{\mathcal{W}^i_f}(x, y) \leq \epsilon_0$, then

$$d_{\mathcal{W}^i_f}(\pi(x), \pi(y)) \leq C_0\epsilon_0.$$  

We next show that $h^c$ maps $\mathcal{W}^{1c}_{f_0}$ to $\mathcal{W}^{1c}_f$. If not, there exists $i \neq 1$ such that $h(\mathcal{W}^{1c}_{f_0}) = \mathcal{W}^{1c}_f$. Then by (6) we have that

$$\pi(\mathcal{W}^{ic}_f) = P(\mathcal{W}^{1c}_{f_0}) = \mathcal{W}^{1c}_{T_{A_f}}$$  

Without loss of generality we may assume that $f$ expands $\mathcal{W}^j_f$ with rate less than $e^{\lambda_j(f_0)+\eta}$, for some $\eta \ll \lambda^j(f_0) - \lambda^i(f_0)$; here $\lambda^j(f_0) = \lambda^j(A_f)$ is the $j$th Lyapunov exponent for $Df_0$. Then $T_{A_f}$ expands (contracts) $\mathcal{W}^{ic}_{T_{A_f}}$ with rate $e^{\lambda^j(f_0)}$, since $f_0$ is an isometric extension of $T_{A_f}$.  


We pick two points \( x, y \in \mathbb{T}^d \) such that \( y \in \mathcal{W}^f_{loc}(x, loc) \) and \( d_{\mathcal{W}^f}(x, y) \leq \epsilon_0 \). Then by the discussion above, we have that for \( n \) large
\[
d_{\mathcal{W}^f}(f^n(x), f^n(y)) \leq \epsilon_0 e^{n(\lambda^i(\pi_0))},
\]
which implies that \( \pi(f^n(x)) \) and \( \pi(f^n(y)) \) can be connected by a \( \mathcal{W}^1_{T_{A_f}} \)-path of length \( O(C_0 \epsilon_0 e^{n(\lambda^i+n)}) \).

On the other hand, since \( \pi \circ f = T_{A_f} \circ \pi \) and \( T_{A_f} \) expands \( \mathcal{W}^1_{T_{A_f}} \) at the exact rate \( e^{\lambda^i(\pi_0)} \), the points \( \pi(f^n(x)) \) and \( \pi(f^n(y)) \) cannot be linked by a \( \mathcal{W}^1_{T_{A_f}} \)-path of length \( O(\epsilon_0 e^{n(\lambda^i(\pi_0)))}) \). This is a contradiction when \( n \) large. We thus have shown that \( h^c \) maps \( \mathcal{W}^1_{f_0} \) to \( \mathcal{W}^1_{f} \).

Applying the same argument inductively, we obtain that \( h^c \) maps \( \mathcal{W}^1_{f_0} \) to \( \mathcal{W}^1_{f} \) for all \( \mathcal{W}^1_{f_0} \) in \( \mathcal{W}^1_{f_0} \). To establish the same conclusion for the subfoliations of \( \mathcal{W}^1_{f_0} \), one considers \( f_0^{-1} \) instead of \( f_0 \).

By Theorem \ref{thm:leaf-conjugacy} Lemmas \ref{lem:holder-exponent} and \ref{lem:lyapunov-exponent} there exists a leaf conjugacy \( h^c : \mathbb{T}^d \to \mathbb{T}^d \) between \( \mathcal{W}^1_{f_0} \) and \( \mathcal{W}^1_{f} \) such that \( h^c \) and \( (h^c)^{-1} \) are Hölder continuous with exponents close to 1. Since \( \pi \circ h^c = P, \) and \( P \) and \( P^{-1} \) along \( \mathcal{W}^1_{f_0} \) are smooth, we immediately obtain the following improvement of Lemma \ref{lem:Holder-exponent}

**Lemma 48.** There exist \( C, \delta > 0 \), with \( \delta \) close to 1, such that for any \( x, y \in \mathcal{W}^f_{loc}(x, loc) \), we have

\[
(17) \quad d_{\mathbb{T}^d}(\pi(x), \pi(y)) \leq C d_{\mathbb{T}^d}(x, y)^\delta, \quad \text{and} \quad d_{\mathbb{T}^d}(x, y) \leq C d_{\mathbb{T}^d}(\pi(x), \pi(y))^{\delta}.
\]

**Step 7:** Fix a \( Df \)-invariant distribution \( E^i_f \), and pick an element \( h(i) \in G_0 \) as in Step 6 such that the signs of all the exponents of \( A_f \) and \( A_{h(i)} \) are the same except for exactly one Lyapunov exponent corresponding to \( E^i_f \). By Steps 1-5, we know that \( h(i) \) satisfies Proposition \ref{prop:hyperbolicity}.

While at this point it is tempting to prove Steps 1-5 for general \( h \in \mathcal{Z}_{\text{Diff}^2(f)} \) by an inductive argument — i.e., to prove Proposition \ref{prop:hyperbolicity} for elements in the Weyl chamber adjacent to that of \( h(i) \) and so on — more work is required to do this. The reason is that in Steps 1-5 we use the the partial hyperbolicity of \( f \) to produce the uniformly contracted and expanded foliations \( \mathcal{W}^1_f \) and \( \mathcal{W}^{sc}_f \) that play a key role in the arguments. To make an inductive argument work, we thus need to establish partial hyperbolicity of \( h(i) \), for \( A_{h(i)} \) in a chamber adjacent to \( T_f \). We will use the improved Hölder exponents in Lemma \ref{lem:Holder-exponent} to establish this.

---

\(^{11}\)Here we are using that the invariant bundles \( E^i_{T_{A_f}} \) are 1-dimensional, but more generally, if the exponents of \( T_{A_f} \) are constant on \( E^i_{T_{A_f}} \), this expansion can be made arbitrarily close to \( e^{\lambda^i(\pi_0)} \) by using an adapted metric, and this suffices for the argument.
Lemma 49. For any $\epsilon > 0$, there exists $n \in \mathbb{Z}^+$ such that for any $j$, $\|Dh(i)^n|_{E^j_i}\|$ lies in the interval

$$\left(\left(\delta^{-1}\lambda^j(A_{h(i)}) - \epsilon\right)\cdot n, \left(\delta\lambda^j(A_{h(i)}) + \epsilon\right)\cdot n\right), \text{ if } \lambda^j(A_{h(i)}) < 0;$$

$$\left(\left(\delta\lambda^j(A_{h(i)}) - \epsilon\right)\cdot n, \left(\delta^{-1}\lambda^j(A_{h(i)}) + \epsilon\right)\cdot n\right), \text{ if } \lambda^j(A_{h(i)}) > 0,$$

where $\lambda^j(A_{h(i)})$ is the $j$th Lyapunov exponent of $T_{A_{h(i)}}|_{\pi(h^j_i)}$ and $\delta \approx 1$ is the Hölder exponent given by Lemma 48.

Proof. Without loss of generality, assume that $\lambda^j(A_{h(i)}) < 0$. By Lemma 10 to prove Lemma 49, we need only show that for any $h$–invariant ergodic measure, the Lyapunov exponent of $Dh(i)|_{E^j_i}$ lies in the interval

$$(\delta^{-1}\lambda^j(A_{h(i)}) - \epsilon, \delta\lambda^j(A_{h(i)}) + \epsilon).$$

Since $\lambda^j(A_{h(i)})$ is the Lyapunov exponent of $T_{A_{h(i)}}|_{\pi(h^j_i)}$, and $\pi$ restricted to $W^j_i$ and its inverse are bi-Hölder conjugacies between $h(i)|_{W^j_i}$ and $A_{h(i)}|_{\pi(h^j_i)}$, with Hölder exponent $\delta$, the desired bounds follow.

Lemma 11 implies that $h(i)$ is volume preserving. Since $E^j_f, E^c_j$ are all 1–dimensional continuous distributions in $TT^d$, there exists $C_0 \geq 1$, depending only on the angles between $E^j_f, E^c_j$, such that for any $k \in \mathbb{Z}$,

$$C_0^{-1} \leq \prod_j \|Dh(i)^k|_{E^j_i}\| \cdot \|Dh(i)^k|_{E^c_i}\| \leq C_0;$$

since $A_h$ has determinant 1, we also have

$$\sum_j \lambda^j(A_{h(i)}) = 0.$$ 

Therefore by (18), (19), (20) we have that for $n$ large enough,

$$\|Dh(i)^n|_{E^j_i}\| \in [e^{-\gamma n}, e^{\gamma n}],$$

where $\gamma$ is small if $\delta$ is sufficiently close to 1 and $\epsilon$ in (18) is small.

Comparing (21) with (18), for $|\lambda^j(A_{h(i)})| \gg \gamma$ (which holds for any $f$ which is sufficiently $C^1$ close to $f_0$ and any $h$ that is not close to the Weyl chamber wall), we get $h(i)$ is in fact a partially hyperbolic diffeomorphism, with $E^s_{h(i)} \oplus E^u_{h(i)} = \oplus_j E^j_f$, and $E^c_{h(i)} = E^c_f$.

We now repeat Steps 1-5 for $h(i')$ with $A_{h(i')}$ in an adjacent Weyl chamber to $A_{h(i)}$. To obtain that $A_{h(i')}$ is partially hyperbolic, with the same center distribution and Weyl chamber picture as $A_{h(i)}$. Arguing inductively, we complete the proof Proposition 42 for those $h$ for which $A_h$ and $A_f$ lie in different Weyl chambers. Combined with the discussion in Section 6.5 this completes the proof of Proposition 42.

Moreover, we have proved:

**Proposition 50.** Suppose $f$ satisfies the hypotheses of Proposition 44. Then in each hyperbolic Weyl chamber of $G_0$, there exists a partially hyperbolic element $h$. 
6.7. Integrability of the horizontal distribution and topological rigidity. We continue to assume that $G$ induces a maximal Cartan Anosov action on $\mathbb{T}^{d-1}$. Consider the hyperbolic subbundle $E^H := \oplus_j E^j_f$, which is hyperbolic for the entire $G$-action. The next key proposition is

**Proposition 51.** $E^H$ is tangent to a $C^1$ foliation $W^H$.

**Proof.** By Proposition 50 and maximality of the $G$-action, for any $\ell$, there exists $h \in G_0$ such that $h$ is partially hyperbolic and

$$E^u_h = E^u_f, \text{ and } E^s_h = \oplus_{k \neq \ell} E^k_f.$$ 

It follows that $\oplus_{k \neq \ell} E^k_f$ is integrable, for any $\ell$. We denote by $W^\ell_f$ the $G_0$-invariant foliation that is tangent to $\oplus_{k \neq \ell} E^k_f$. Since $h$ is partially hyperbolic, the foliations $W^\ell_f$ and $W^\ell'_f$ have uniformly $C^2$ leaves. For any pair $(i, j)$,

$$E^i_f \oplus E^j_f = \cap_{\ell \neq i, j} \oplus_{k \neq \ell} E^k_f$$

is integrable and the integral manifold is $\cap_{\ell \neq i, j} W^\ell'_f$. We denote this intersection $\cap_{\ell \neq i, j} W^\ell'_f$ by $W^i_f$.

We have the following lemma about the regularity of the $E^j_f$:

**Lemma 52.** Each $E^j_f$ is a $C^1$ distribution.

**Proof.** Since $f$ is a $C^2$ partially hyperbolic diffeomorphism with 1-dimensional center, $f$ is center bunched. Therefore $E^j_f$ is uniformly $C^1$ along $W^c_f$, and $W^u_f$, $W^s_f$ have uniformly $C^2$ leaves. Similarly by considering partially hyperbolic elements in each Weyl chamber we get that each $E^\ell_f$ is uniformly $C^1$ along $W^c_f$ and $W^\ell_f$ has $C^2$ leaves. As a consequence each $E^\ell_f$ is $C^1$ along $W^\ell_f$.

Next we consider the regularity of $E^j_f$ along $W^j_f$, for $i \neq j$. We claim that for any $i \neq j$, there exists an $h \in G_0$ that uniformly expands $W^i_f$ and such that $W^j$ is the fast unstable foliation of $h$. Lemma 8 will then imply that $E^j$ is uniformly $C^1$ along $W^i_f$. To do this, it suffices to choose $h$ in the Weyl chamber that expands both $W^i, W^j$ and sufficiently close to the Weyl chamber wall of $W^j$. With this choice of $h$, we have $0 < \lambda^j(A_h) < \lambda^i(A_h)$, where $\lambda^j(A_h), \lambda^i(A_h)$ are the corresponding Lyapunov exponents. Then by [18], $Dh$ expands $E^i_f$ uniformly faster than $E^j_f$ is expanded.

In summary, each $E^j_f$ is uniformly $C^1$ along $W^c_f$ and every $W^i_f$. Journé’s Lemma [41] then implies that each $E^i_f$ is a $C^1$ distribution.

By Lemma 52, it follows that $E^H = \oplus E^j_f$ is $C^1$ as well. We will prove Proposition 51 using Frobenius’ theorem. Consider two $C^1$ vector fields $X, Y$ taking values in $E^H$, and let

$$X = \sum_i X_i, Y = \sum_i Y_i$$
be their decomposition into component vector fields with respect to the $C^1$ splitting $E^H = \oplus E^j_f$. Then

\[ [X, Y] = \sum_i [X_i, Y_i] + \sum_{j \neq k} [X_j, Y_k]. \]

Since $W^i_f$ is integrable, $[X_i, Y_i]$ takes values in $E^i_f$. Similarly, since each $W^{j,k}_f$ is integrable, $[X_j, Y_k]$ takes values in $E^j_f \oplus E^k_f$. Therefore $[X, Y]$ also takes values in $E^H$. Since $E^H$ is involutive, Frobenius' theorem implies that $E^H$ is integrable.

Since $f$ is leaf conjugate to $f_0$, $f$ is an $AB$-system, as defined in Section 3.4.8. Using Theorem 13, we obtain the following corollary of Proposition 51.

**Corollary 53.** If $G$ induces a maximal Cartan Anosov action on $\mathbb{T}^{d-1}$, then

1. there exists a $G_0$-invariant continuous metric on $E^c_f$; and
2. $f$ is Hölder conjugate to $T_{A_f} \times R_\theta : \mathbb{T}^{d-1} \times \mathbb{T} \to \mathbb{T}^{d-1} \times \mathbb{T}$, for some $\theta \notin \mathbb{Q}$.

Similarly, any $h \in G_0$ is Hölder conjugate (by the same Hölder conjugacy) to a product of $T_{Ah}$ with a circle rotation.

**Proof.** Denote by $W^c_f(x_0)$ a $G_0$-fixed center leaf. Since $E^H$ is integrable, there is no open accessibility class for $f$. Then by Theorem 13, $\mathbb{T}^d$ has a product structure, i.e. $\mathbb{T}^d$ is topologically the product of $W^c_f(x_0)$ and $\mathbb{T}^{d-1}/W^c$. By Hölder continuity of $\pi$ and $W^c_f$, this product structure is Hölder continuous as well.

We consider the projection $\text{Pr}^c$ from $\mathbb{T}^d$ to $W^c_f(x_0)$ along $W^H$. Since $W^H$ is a $C^1$ foliation, $\text{Pr}^c$ is $C^1$ as well. Therefore $\text{Pr}^c_*(\text{vol}_{\mathbb{T}^d})$ is an $f$-invariant volume on $W^c_f(x_0)$ with continuous density function, and $f|_{W^c_f(x_0)}$ is $C^1$ conjugate to a circle rotation $R_\theta$. By ergodicity of $f$, the rotation number $\theta$ must be irrational.

It is easy to see that the continuous density function mentioned above gives an $f$-invariant continuous metric on $T W^c_f(x_0)$, and this pulls back via $D \text{Pr}^c|_{E^c}$ to an $f$-invariant metric on $E^c$. Since the construction of this $f$-invariant continuous metric on $E^c$ only depends on the product structure and the volume form on $\mathbb{T}^d$, it must be $G_0$-invariant. This proves (1).

For (2), we know that the action induced by $f$

- on $\mathbb{T}^d/W^c$ is Hölder conjugate to $T_{A_f}$ on $\mathbb{T}^{d-1}$; and
- on $\mathbb{T}^d/W^H$ is $C^1$-conjugate to $R_\theta$.

Using the product structure of $f$, we obtain that $f$ is Hölder conjugate to the product of $T_{A_f}$ on $\mathbb{T}^{d-1}$ with an irrational rotation $R_\theta$. The same proof also works for any $h \in G_0$ (although if $h$ is not ergodic, the rotation number might not be irrational). Therefore by the same conjugacy, $h$ is Hölder conjugate to the product of $T_{Ah}$ with a circle rotation.
6.8. Absolutely continuity of $\mathcal{W}_f^c$: volume and equilibrium states. Recall that in Section 3.7 we defined equilibrium states for a given potential. The following Proposition is a partially hyperbolic version of Theorem 20.4.1. in [48].

**Proposition 54.** Let $f: M \to M$ be a $\mathcal{C}^{1+}$, volume preserving partially hyperbolic diffeomorphism. Suppose that for any $f$–invariant ergodic measure $\nu$, the central Lyapunov exponents of $f$ with respect to $\nu$ are all zero. Then the volume $\text{vol}_M$ is an equilibrium state of the potential $\varphi := -\log J^u(f) := -\log |\det Df|_{E^u}|$.

**Proof.** The Pesin entropy formula [63] states that

$$h_{\text{vol}}(f) = \int_M \sum_i k_i \cdot \lambda_i^+(x) d\text{vol}(x),$$

where $\lambda_i$ is the $i$–th (distinct) Lyapunov exponent, $\lambda_i^+ := \max(\lambda_i, 0)$ and $k_i$ is the dimension of the Oseledec subspace corresponding to $\lambda_i$. Since $Df$ has vanished Lyapunov exponents on $E^c$. Let $\nu$ be any $f$–invariant probability measure. Since we assume that the exponents of $f$ are zero with respect to $\nu$, the unstable distribution of $f$ is the sum of the expanded Oseledec subspaces for $\nu$ (i.e. the Oseledec spaces with positive Lyapunov exponent). Setting $\nu = \text{vol}$, we obtain that the quantity

$$\sum_i k_i \cdot \lambda_i^+(x)$$

is exactly the Lyapunov exponent of the one dimensional cocycle $\log J^u(f)$ over $f$ with respect to $\text{vol}$. In particular,

$$h_{\text{vol}}(f) = \int_M \log J^u(f)(x) d\text{vol}(x).$$

Therefore $P_{\text{vol}}(\varphi) = 0$.

To complete the proof of Proposition 54, we only need to prove $P(\varphi) = 0$. In [40], the authors introduce the concept of unstable pressure $P^u(f, \psi) = P^u(\psi)$ for any continuous $\psi$ and $\mathcal{C}^1$ partially hyperbolic diffeomorphism $f$ (for definition and more details cf. [40]). The following lemma lists some useful properties of unstable pressure we need.

**Lemma 55.** (1) $P^u(\psi) \leq P(\psi)$ for any continuous $\psi$. Moreover if $f$ is $\mathcal{C}^{1+}$ and there is no positive Lyapunov exponent in the center direction with respect to any $f$–invariant ergodic measure $\nu$, then equality holds.

(2) For the potential $\varphi = -\log J^u(f)$, we have $P^u(\varphi) = 0$.

**Proof.** For (1), cf. Corollary A.2 and the paragraph right after the statement of Corollary A.2 in [40]. (2) follows from Corollary C.1 in [40].

Under the assumptions of Proposition 54, item (1) of Lemma 55 implies that for any $\psi \in C(M, \mathbb{R})$,

$$P^u(\psi) = P(\psi).$$

Therefore by (23) and item (2) of Lemma 55 we obtain that $P(\varphi) = 0$. This completes the proof of Proposition 54.
6.9. Absolute continuity of $W^c_\gamma$: cocycle rigidity of higher rank partially hyperbolic actions. In this section we consider a cocycle rigidity result over higher rank, partially hyperbolic abelian actions (see Section 3.8). Recall that we assume that $G$ defines a maximal, linear Anosov $\mathbb{Z}^{d-2}$-action $\alpha$ on $\mathbb{T}^{d-1}$:

$$\bar{\alpha} : \mathbb{Z}^{d-2} \to \text{SL}(d-1, \mathbb{Z}) \hookrightarrow \text{Diff}^\infty(\mathbb{T}^{d-1}).$$

We say that a $\mathbb{Z}^{d-2}$-action $\alpha$ on $\mathbb{T}^d$ is an irrational rotation extension over $\bar{\alpha}$ if for any $a \in \mathbb{Z}^{d-2}$,

$$\alpha(a) = \bar{\alpha}(a) \times R_{\theta(a)}, \quad \alpha(a)(x, y) = (\bar{\alpha}(a) \cdot x, y + \theta(a))$$

with at least one of $\theta(a)$ irrational.

Corollary 53 implies that the action of $G_0$ on $\mathbb{T}^d$ is Hölder conjugate to an irrational rotation extension $\alpha$ over $\bar{\alpha}$. From now on we fix such an $\alpha$.

**Definition 13.** A continuous function $\beta : \mathbb{Z}^{d-2} \times \mathbb{T}^d \to \mathbb{R}$ is called an additive cocycle over $\alpha$ if it satisfies

$$\beta(a + b, x) = \beta(a, \alpha(b) \cdot x) + \beta(b, x),$$

for all $a, b \in \mathbb{Z}^{d-2}$ and $x \in \mathbb{Z}^d$. A cocycle $\beta_1$ is cohomologous to another cocycle $\beta_2$ if there exists a continuous function $\Psi : \mathbb{T}^d \to \mathbb{R}$ such that

$$\beta_1(a, x) = \beta_2(a, x) + \Psi(\alpha(a) \cdot x) - \Psi(x)$$

For a maximal $\mathbb{Z}^{d-2}$—Anosov action $\bar{\alpha}$ on $\mathbb{T}^{d-1}$, it is known that any Hölder continuous cocycle over $\bar{\alpha}$ is cohomologous to a constant cocycle (cf. [53]). We generalize this result to the partially hyperbolic case. A cocycle $\beta$ on $\mathbb{T}^{d-1} \times \mathbb{T}$ is constant on $\mathbb{T}^{d-1}$ if $\beta(a, x) = \beta(a, y)$ whenever $x, y$ have the same $\mathbb{T}$-component, i.e. they lie on the same leaf of the horizontal $\mathbb{T}^{d-1}$-foliation $\{T^{d-1} \times \{t\} : t \in \mathbb{T}\}$.

**Proposition 56.** Let $\alpha$ be an irrational rotation extension over a maximal, linear Anosov $\mathbb{Z}^{d-2}$-action $\bar{\alpha}$ on $\mathbb{T}^{d-1}$. Then any Hölder continuous cocycle over $\alpha$ is cohomologous to a cocycle that is constant on $\mathbb{T}^{d-1}$.

**Proof.** The proof of Proposition 56 is an application of the periodic cycle functionals argument for higher rank actions developed in [20, 50] (cf. [87] for the rank-1 case). It is easy to see that there are $\alpha$—invariant distributions $E^i$ such that

1. $\dim E^i = 1$.
2. Each $E^i$ is smooth and tangent to a smooth foliation $\mathcal{F}^i$.
3. For any pair $i \neq j$, there exists $a \in \mathbb{Z}^{d-2}$ such that $E^i \oplus E^j \subset E^a := E^a_{\alpha(a)}$.
4. $\oplus E^i = T^{d-1}$ (i.e., the $E^i$ span the tangent bundle to the horizontal leaves in $\mathbb{T}^{d-1} \times \mathbb{T}$).

A $\mathcal{F}_1^{\cdots, d-1}$—path is an ordered set of points in $\mathbb{T}^d$ such that every two consecutive points lie in a single leaf of one of the foliations $\mathcal{F}^i$. It is easy to see that any two sufficiently close points $x, y$ on the same leaf of the horizontal $\mathbb{T}^{d-1}$ foliation can be connected by a $\mathcal{F}_1^{\cdots, d-1}$—path consisting of no more than $d - 1$ pieces of length bounded by $O(d_{\mathbb{T}^{d-1}}(x, y))$. 
An ordered set of points \( x_1, \ldots, x_N; x_{N+1} = x_1 \in M \) is called an \( \mathcal{F}^{1,\ldots,d-1} \)-cycle of length \( N \) if for every \( i = 1, \ldots, N \), there exists \( j(i) \in \{1, \ldots, d-1\} \) such that \( x_{i+1} \in \mathcal{F}^{j(i)}(x_i) \).

Now consider an arbitrary Hölder cocycle \( \beta \) over \( \alpha \). To simplify notations we denote the action \( \alpha(a) \cdot x \) by \( ax \) and \( \alpha(ka) \cdot x \) by \( kax \) for \( k \in \mathbb{N} \); for \( a \in \mathbb{Z}^{d-2} \), we denote the function \( \beta(a, \cdot): M \to \mathbb{R} \) by \( \beta_a \).

**Definition 14.** Suppose that for some \( j \in \{1, \ldots, d-1\} \) and \( a \in \mathbb{Z}^{d-2} \), \( \alpha(a) \) either contracts or expands \( \mathcal{F}^j \) (i.e. \( a \) is not in the Weyl chamber wall corresponding to \( E^j \)). The \( \beta \)-potential \( P_j^\beta(x, y)(\beta) \) of \( y \) with respect to \( x \) is defined by

\[
P_j^\beta(x, y)(\beta) := \begin{cases} 
\sum_{k=0}^{\infty} \beta_a(kax) - \beta_a(kay), & \text{if } \alpha(a) \text{ contracts } E^j \\
\sum_{k=-\infty}^{1} \beta_a(kax) - \beta_a(kay), & \text{if } \alpha(a) \text{ expands } E^j.
\end{cases}
\]

This can be written in the more compact form as follows:

\[
P_j^\beta(x, y)(\beta) = * \sum (\beta_a(kax) - \beta_a(kay)),
\]

where \( ax := \alpha(a) \cdot x \),

\[
*: = *_{(j, a)} := \begin{cases} 
-, & \text{if } \alpha(a) \text{ expands } E^j \\
+, & \text{if } \alpha(a) \text{ contracts } E^j,
\end{cases} \in \{+, -\}
\]

\[
\sum_+ := \sum_{k=0}^{\infty}, \text{ and } \sum_- := \sum_{k=-\infty}^{1}.\]

The following summarizes some important properties of the \( \beta \)-potential.

**Lemma 57.** [Proposition 2 of [20]]

(1) If \( \beta \) is Hölder continuous (smooth) then \( P_j^\beta(x, y)(\beta) \) is uniformly continuous in \((x, y)\) and uniformly Hölder continuous (smooth) along the leaves of \( \mathcal{F}^j \).

(2) \( P_j^\beta(x, y)(\beta) = P_i^\beta(x, y)(\beta) \) if \( a, b \) are not in the Weyl chamber wall corresponding to \( E^j \).

**Definition 15.** Given \( a \in \mathbb{Z}^{d-2} \) not in any Weyl chamber wall and an \( \mathcal{F}^{1,\ldots,d-1} \)-cycle \( C \) of length \( N \), we define the periodic cycle functional on the space of Hölder cocycles over \( \alpha \) by

\[
F_a(C)(\beta) = \sum_{i=1}^{N} P_j^{(i)}(x_i, x_{i+1})(\beta)
\]

where \( j(i) \in \{1, \ldots, d-1\} \) and \( x_{i+1} \in \mathcal{F}_j^{(i)}(x_i) \).

**Remark 58.** Item (2) of Lemma 57 implies that for any \( a, b \in \mathbb{Z}^{d-2} \) that don’t lie in a Weyl chamber wall, we have

\[
F_a(C)(\beta) = F_b(C)(\beta).
\]

Write \( F(C)(\beta) \) to denote the common value of \( F_a(C)(\beta) \), for \( a \) not in a chamber wall.

The following lemma is the heart of the proof of Proposition 56.
Lemma 59. Suppose $\beta$ is a Hölder continuous cocycle over $\alpha$ such that $F(C)(\beta) = 0$ for all $F^{1,\ldots,d-1}$ cycles $C$. Then $\beta$ is cohomologous to a cocycle that is constant on $\mathbb{T}^{d-1}$.

Proof. The proof of Lemma 59 is similar to that of [20] Proposition 4, or cf. [50]. For completeness we give a proof here.

The vertical circle $S := \{0\} \times \mathbb{T} \subset \mathbb{T}^{d-1} \times \mathbb{T}$ is fixed by $\alpha$. For any $x \in S$ and any $y$ in the same $\mathbb{T}^{d-1}$-horizontal leaf as $x$, and for any $a \in \mathbb{Z}^{d-2}$ that is not in any Weyl chamber wall, we define

$$F_a(S(x, y))(\beta) := \sum_{i=1}^N P_a^j(i)(x_i; x_{i+1})(\beta),$$

where $S(x, y)$ is some $F^{1,\ldots,d-1}$-path connecting $x$ and $y$. It is easy to see that such a path $S(x, y)$ can be chosen so that

$$N \leq d - 1, \quad \text{and} \quad d_{F_a}(x_i, x_{i+1}) \leq O(d_{\mathbb{T}^n}(x, y)).$$

Then from the assumption that $F_a(C)(\beta) = 0$ for any $F^{1,\ldots,d-1}$ cycle $C$, we obtain that $F_a(S(x, y))(\beta)$ only depends on the points $x$ and $y$ and not on the choice of the path $S(x, y)$, so we may write $F_a(x, y)$ instead. This gives a well-defined function $\Psi_x : \mathbb{T}^{d-1} \times \{x\} \to \mathbb{R}$:

$$(25) \quad \Psi_x(y) = F_a(x, y).$$

Lemmas 57 and 24 imply that $\Psi_x(y)$ is continuous in $(x, y)$. For any $x \in S$ and $y \in \mathbb{T}^{d-1} \times \{x\}$, we choose a $F^{1,\ldots,d-1}$-path $S(x, y) = (x = x_1, \ldots, y = x_N)$, $N \leq d - 1$ from $x$ to $y$ and note that, because $\alpha$ preserves the horizontal $\mathbb{T}^{d-1}$ foliation, the path

$$aS(x; y) = (ax = ax_1, \ldots, ay = ax_N)$$

is a $F^{1,\ldots,d-1}$-path connecting $ax$ and $ay$. Then

$$\Psi_{ax}(ay) = F_a(ax, ay) = \sum_{i=1}^{N-1} P_a^j(i)(ax_i, ax_{i+1}) = \sum_{i=1}^{N-1} (\sum_{k}^* \beta_a((k + 1)ax_i) - \beta_a((k + 1)ax_{i+1})) = F_a(x, y) - \sum_{i=1}^{N-1} (\beta_a(x_i) - \beta_a(x_{i+1})) = \Psi_x(y) - \beta_a(x) + \beta_a(y).$$

Therefore we have

$$\beta_a(y) = \beta_a(x) + \Psi_{ax}(ay) - \Psi_x(y),$$

which means $\beta_a(\cdot)$ is cohomologous to a function that is constant on $\mathbb{T}^{d-1}$ via the continuous transfer function $\Psi_x(\cdot)$.

By item (2) of Lemma 57 and (25) we know that for any $b$ not in a Weyl chamber wall, we have

$$\Psi_x(y) = F_a(x, y) = F_b(x, y).$$
Therefore by the argument above we obtain that
\[ \beta_b(y) = \beta_b(x) + \Psi_{bx}(by) - h_x(y) \]
holds.

Now we consider an arbitrary \( c \in \mathbb{Z}^{d-2} \). It is not hard to prove that \( c \) can be written as a sum of elements \( a_i, i = 1, 2 \) that do not lie in a Weyl chamber wall. Therefore
\[
\beta(c, y) = \beta(a_1 + a_2, y) = \beta(a_1, a_2) + \beta(a_2, y) \\
= \beta(a_1, a_2x) + \Psi_{(a_1+a_2)x}((a_1 + a_2)y) - \Psi_{a_2x}(a_2y) + \beta(a_2, x) + \Psi_{a_2x}(a_2y) - \Psi_x(y) \\
= \beta(a_1 + a_2, x) + \Psi_{(a_1+a_2)x}((a_1 + a_2)y) - \Psi_x(y) \\
= \beta(c, x) + \Psi_{cx}(cy) - \Psi_x(y).
\]

Therefore \( \beta \) is cohomologous to a cocycle that is constant on \( \mathbb{T}^{d-1} \). \( \square \)

To complete the proof of Proposition 56 by Lemma 59 we need only prove:

**Lemma 60.** Suppose \( \beta \) is a Hölder continuous cocycle over \( \alpha \), then \( F(\mathcal{C})(\beta) = 0 \), for every \( \mathcal{F}^{1,\ldots,d-1} \)-cycle \( \mathcal{C} \).

**Proof.** The proof is basically the same as that in [20]. Here we only give an outline. First it is easy to see (cf. Proposition 5 in [20]) that if a \( \mathcal{F}^{1,\ldots,d-1} \)-cycle \( \mathcal{C} \) is contained in a stable leaf for some element of the action then \( F(\mathcal{C})(\beta) = 0 \).

Now consider an arbitrary \( \mathcal{F}^{1,\ldots,d-1} \)-path
\[ \mathcal{P} : x = x_1, x_2, \ldots, x_{N-1}, x_N = y. \]
Suppose there exist \( i, j, 1 \leq i < j \leq N \) such that all points \( x_k, k = i, i+1, \ldots, j \) lie in the same stable leaf for some element of the action. Let \( x'_i = x_i, \ldots, x'_s = x_j \) be a \( \mathcal{F}^{1,\ldots,d-1} \)-path that lies in the same stable manifold as \( x_i, \ldots, x_j \). Define the path \( \mathcal{P}' \) by
\[ \mathcal{P}' : x, x_i, x'_i, \ldots, x'_s, x_j, \ldots, y. \]
Substituting a \( \mathcal{F}^{1,\ldots,d-1} \)-path \( \mathcal{P} \) from \( x \) to \( y \) by the path \( \mathcal{P}' \) is called an allowed substitution. A sequence of \( \mathcal{F}^{1,\ldots,d-1} \)-cycles \( \mathcal{C} = \mathcal{C}_1, \ldots, \mathcal{C}_m \) constitutes a reduction via allowed substitutions of \( \mathcal{C} \) if the substitution of \( \mathcal{C}_i \) by \( \mathcal{C}_{i+1} \) is an allowed substitution for \( i = 1, \ldots, m-1 \). In particular, if \( \mathcal{C}_m \) is a trivial one-point cycle the reduction is called a trivialization via allowed substitutions. Clearly \( F(\mathcal{C})(\beta) = 0 \) for any Hölder cocycle \( \beta \), if \( \mathcal{C} \) can be trivialized via allowed substitution.

We claim that any \( \mathcal{F}^{1,\ldots,d-1} \)-cycle \( \mathcal{P} : x = x_1, x_2, \ldots, x_{N-1}, x_N = x \) can be trivialized via allowed substitution. In fact it is known for totally non-symplectic (TNS) Anosov actions on tori or nilmanifolds (cf. [20] and the references therein). In our case since each \( \mathcal{F}^{1,\ldots,d-1} \)-cycle is contained in a horizontal \( \mathbb{T}^{d-1} \)-leaf and the action \( \bar{\alpha} \) on \( \mathbb{T}^{d-1} \) is maximal (and hence TNS), the the proof is essentially the same. Thus \( F(\mathcal{C})(\beta) = 0 \) for any \( \mathcal{F}^{1,\ldots,d-1} \)-cycle \( \mathcal{C} \), completing the proof of Lemma 60. \( \square \)

Together, Lemmas 60 and 59 complete the proof of of Proposition 56. \( \square \)
6.10. **Absolute continuity of** $\mathcal{W}_f^T$: **uniqueness of the measure of maximal entropy.** Consider the diffeomorphism $T_A \times R_\theta : \mathbb{T}^{d-1} \times \mathbb{T} \to \mathbb{T}^{d-1} \times \mathbb{T}$ where $R_\theta$ is an irrational rotation on circle and $A \in \text{SL}(d-1, \mathbb{Z})$ is hyperbolic. The following lemma is known; we give a proof for completeness.

**Lemma 61.** The volume $\text{vol}_{\mathbb{T}^d}$ on $\mathbb{T}^{d-1} \times \mathbb{T}$ is the unique measure of maximal entropy of $T_A \times R_\theta$.

**Proof.** Suppose $\mu$ is a measure of maximal entropy for $T_A \times R_\theta$. We determine the projections of $\mu$ to $\mathbb{T}^{d-1}$ and $T$, $\text{Pr}^T_{\mathbb{T}^{d-1}}(\nu)$ and $\text{Pr}^T_T(\mu)$ respectively. Clearly $\text{Pr}^T_T(\mu)$ is Lebesgue measure on $T$ since it is invariant under the irrational rotation $R_\theta$.

Next consider $\text{Pr}^T_{\mathbb{T}^{d-1}}(\nu)$. The Lyapunov exponents $\{\lambda(i) \mid i = 1, 2, \ldots\}$ of any $T_A \times R_\theta$-invariant measure $\nu$ are independent of the choice of $\nu$. Therefore by the Pesin entropy formula and Ledrappier-Young’s result [56, Theorem A], we obtain that $\mu$ must be both a Gibbs-u and Gibbs-s state, i.e. $\mu$ has Lebesgue disintegrations along both $\mathcal{W}^s$ and $\mathcal{W}^u$. Moreover, by the formula of the density functions of Gibbs states along (un)stable leaves (cf. [64]) we have that the conditional density of $\mu$ along stable and unstable leaves are constant functions (by linearity of $A$).

Consequently, the projection $\text{Pr}^T_{\mathbb{T}^{d-1}}(\mu)$ also has constant density functions along $\mathcal{W}^s_A$, $\mathcal{W}^u_A$. Moreover it is $A$-invariant; therefore $\text{Pr}^T_{\mathbb{T}^{d-1}}(\mu) = \text{vol}_{\mathbb{T}^{d-1}}$, by the uniqueness of SRB measures for transitive Anosov diffeomorphisms.

In summary, $\text{Pr}^T_{\mathbb{T}^{d-1}}(\mu) = \text{vol}_{\mathbb{T}^{d-1}}$ and $\text{Pr}^T_T(\mu) = \text{vol}_T$. Therefore $\mu$ must be $\text{vol}_{\mathbb{T}^d}$, since any zero entropy system is disjoint from a Bernoulli dynamical system (cf. [33]).

The following proposition is a corollary of Proposition [56]

**Proposition 62.** Suppose that $f$ satisfies the hypotheses of Theorem 6, and let $G_0, G \subset \text{Diff}(\mathbb{T}^d)$ be the finitely generated abelian groups defined in Section 6.7. If $G$ defines a maximal linear Anosov action, then the volume $\text{vol}_{\mathbb{T}^d}$ is the unique measure of maximal entropy of $f$.

**Proof.** By the discussion in Section 6.8 and Corollary 53 we know $\text{vol}_{\mathbb{T}^d}$ is an equilibrium state of the potential $\varphi := -\log J^u(f)$ for $f$. We define the cocycle $\beta := -\log J^u$ over the action of $G_0$ as follows. For $f_1 \in G_0, x \in \mathbb{T}^d$, we set

$$\beta(f_1, x) := -\log |\det Df_1|_{E^u(f)}(x)|.$$  

Clearly $\beta$ is a cocycle over the action of $G_0$, and $\beta(f, x) = \varphi(x)$, for all $x$.

The action of $G_0$ is Hölder conjugate to the algebraic action $\alpha$ defined in Section 6.9. By Proposition 56 we know any Hölder continuous cocycle over $\alpha$ is cohomologous to a cocycle that is constant on $\mathbb{T}^{d-1}$. Therefore $\beta$ must be cohomologous to a cocycle that is constant on each horizontal $\mathcal{W}^H$-leaf. In particular, there exist continuous functions $\psi, \Psi : \mathbb{T}^d \to \mathbb{R}$, such that

$$\varphi = \psi + \Psi \circ f - \Psi,$$

(26)
and \( \psi(x) = \psi(y) \) whenever \( x, y \) lie in the same \( \mathcal{W}^H \)-leaf.

As in the proof of Corollary 53, we denote by \( \mathcal{W}_f^c(x_0) \) a \( G_0 \)-fixed center leaf, and let \( \Pr^c : \mathbb{T}^d \rightarrow \mathcal{W}_f^c(x_0) \) be the projection along the horizontal foliation \( \mathcal{W}^H \). Then \( \psi \) defined in (26) induces a well-defined continuous function \( \psi^c \) on \( \mathcal{W}_f^c(x_0) \) such that

\[
(27) \quad \psi = \psi^c \circ \Pr^c.
\]

Now we claim that for any \( f \)-invariant measure \( \mu \), \( \int_{\mathbb{T}^d} \varphi \, d\mu \) is independent of \( \mu \). Indeed

\[
\int_{\mathbb{T}^d} \varphi \, d\mu = \int_{\mathbb{T}^d} (\psi \circ \Psi \circ f - \Psi) \, d\mu \quad \text{(by (26))}
\]

\[
= \int_{\mathbb{T}^d} \psi \, d\mu \quad \text{(since \( \mu \) is \( f \)-invariant)}
\]

\[
= \int_{\mathcal{W}_f^c(x_0)} \psi^c \, d\Pr^c_\mu(\mu) \quad \text{(since \( \psi \) is constant along each horizontal leaf).}
\]

But \( f|_{\mathcal{W}_f^c(x_0)} \) is conjugate to an irrational rotation, and hence is uniquely ergodic, and \( \Pr^c_\mu(\mu) \) is \( f \)-invariant on \( \mathcal{W}_f^c(x_0) \). Therefore \( \int_{\mathcal{W}_f^c(x_0)} \psi^c \, d\Pr^c_\mu(\mu) \) (and hence \( \int_{\mathbb{T}^d} \varphi \, d\mu \)) is independent of the choice of \( \mu \). Write \( s(\varphi) \) for the value \( \int_{\mathbb{T}^d} \varphi \, d\mu \) of this integral.

Since \( \vol_{\mathbb{T}^d} \) is an equilibrium state of the potential \( \varphi \), we have that

\[
P_{\vol}(\varphi) = \sup_{\mu \text{ is } f \text{-inv}} P_\mu(\varphi)
\]

\[
= \sup_{\mu \text{ is } f \text{-inv}} h_\mu(f) + \int_\mu \varphi
\]

\[
= \sup_{\mu \text{ is } f \text{-inv}} h_\mu(f) + s(\varphi) \quad \text{(since } \int_\mu \varphi = s(\varphi), \text{ which is independent of } \mu).}
\]

But

\[
P_{\vol}(\varphi) = h_{\vol}(f) + \int_{\vol} \varphi = h_{\vol}(f) + s(\varphi).
\]

Therefore \( h_{\vol}(f) = \sup_{\mu \text{ is } f \text{-inv}} h_\mu(f) \), which implies \( \vol_{\mathbb{T}^d} \) is a measure of maximal entropy of \( f \). But by Corollary 53 we know \( f \) is conjugate to \( T_A \times R_\theta \), for some \( \theta \notin \mathbb{Q} \), therefore by Lemma 61 \( \vol_{\mathbb{T}^d} \) is the unique measure of maximal entropy of \( f \). \( \square \)

As a corollary, the conjugacy between \( f \) and \( T_A \times R_\theta \) identifies the measure of maximal entropy \( \vol_{\mathbb{T}^d} \) of \( T_A \times R_\theta \) with the measure of maximal entropy \( \vol_{\mathbb{T}^d} \) of \( f \). Recall that \( \vol_{\mathbb{T}^d} \), the measure of maximal entropy of \( T_A \times R_\theta \) is the product of \( \Pr^T_\mu(\vol_{\mathbb{T}^d}) \) and \( \Pr^{d-1}_\mu(\vol_{\mathbb{T}^d}) \). Therefore \( \vol_{\mathbb{T}^d} \), the measure of maximal entropy of \( f \), is the product of \( \Pr^c_\mu(\vol_{\mathbb{T}^d}) \) and \( \Pr^H(\vol_{\mathbb{T}^d}) \), where \( \Pr^H \) is the projection from \( \mathbb{T}^d \) to \( \mathbb{T}^d/\mathcal{W}_f^c \) along \( \mathcal{W}_f^c \).

In particular, since \( \Pr^c_\mu(\vol_{\mathbb{T}^d}) \) is absolutely continuous with respect to the Lebesgue measure on \( \mathcal{W}_f^c(\bar{x}_0) \) (since \( \Pr^c \) is \( C^1 \)), it follows that \( \vol_{\mathbb{T}^d} \) has Lebesgue disintegration along \( \mathcal{W}_f^c \). This completes the proof of Proposition 40 which implies Proposition 37.
7. Proof of Theorem 6: non-Cartan case

7.1. Some basic properties. In this section we will complete the proof of Theorem 6, i.e. remove the extra assumption in Proposition 37 that all eigenvalues of $A_f$ are real.

For $f_0$ as in Theorem 6, we denote by
\[ \lambda^1(f_0) > \cdots > \lambda^i(f_0) > \cdots \]
the distinct Lyapunov exponents of $f_0$ and the corresponding $Df_0$-invariant Lyapunov splitting by
\[ T_{T^d} = \oplus E^i_{f_0} \oplus E^c_{f_0}. \]

Since $f$ is $C^1$-close to $f_0$, $Df$ preserves a corresponding $Df$-invariant dominated splitting
\[ T_{T^d} = \oplus E^i_f \oplus E^c_f. \]

In general $E^i = E^i_f$ in (29) might not be 1-dimensional.

As in the proof of Proposition 37, we define $G, G_0$ as in Section 6.1. Lemma 39 holds for $f$, except Case II in item (3) is replaced by:

II'. $G$ induces a maximal Anosov (but not necessarily Cartan) action on $T^{d-1}$.

As in Section 6 since $G$ is a finitely generated abelian group, without loss of generality we may assume that $G, G_0$ are torsion free, and
\[ h \mapsto A_h, h \in G_0 \]
is a group isomorphism. Define $\pi: T^d \to T^{d-1}$ as in Lemma 25, and note that Lemma 41 holds for $\pi$. Moreover, since the proofs of Lemmas 43 and 44 did not use the condition that all the eigenvalues of $f_0$ are real, the conclusions of Lemmas 43 and 44 hold for $f$ as well.

Our goal is to prove Proposition 42 in Section 6 in this setting, i.e., to show that under the assumption that $G$ is a finite index subgroup of $Z_{SL(d-1,\mathbb{Z})}(A_f)$ (which implies that $G$ induces a maximal Anosov action on $T^{d-1}$), the volume $\text{vol}_{T^d}$ has Lebesgue disintegration along $W^c_f$.

It is easy to see that if we prove Proposition 42 and Lemma 50 under the assumptions on $f$ of Theorem 6, then the rest of the proof is similar to that of Proposition 37 since the discussions in Sections 6.7, 6.8, 6.9 also work for general $f_0$ in Theorem 6.

7.2. Proof of Proposition 42 and Lemma 50. In the rest of the section we prove Proposition 42 and Lemma 50 under the assumption on $f$ in Theorem 6. Most of the ideas here are the same as in Section 6.6.

Recall that in Section 7.1 we order $E^i_f$ and $E^i_{f_0}$ in $i$ by increasing size of Lyapunov exponents. We consider the distributions $E^i_f$ and $E^i_{f_0}$.

Since $G$ induces a maximal Anosov action on $T^{d-1}$, there exists a Weyl chamber adjacent to that of $A_f$ such that for any element $h \in G_0$ with $A_h$ in this chamber, the signs of all the exponents of $A_f$ and $A_h$ are the same except one exponent corresponding to $E^1_{f_0}$. We
denote by $\mathcal{W}^s_{T_{A_f}}$ the $T_{A_f}$-invariant foliation of $T^{d-1}$ that is tangent to $E^s_{T_{A_f}}$. Here $E^s_{T_{A_f}}$ is the $T_{A_f}$-invariant Lyapunov distribution corresponding to $E^s_{f_0}$ in a canonical way.

Then, as in Step 1 of Section 6.6 we define a topological $G_0$-invariant foliation $\mathcal{W}^#$ that is the lift of $\mathcal{W}^s_{T_{A_f}}$ by $\pi^{-1}$ on $\mathcal{W}^s_f$-leaves. The key step in this section is the following proposition:

**Proposition 63.** $\mathcal{W}^#$ is a foliation with $C^1$ leaves, and $T\mathcal{W}^# = E^1_f$.

**Proof.** First, (30) holds for $\mathcal{W}^#$ as in Section 6.6. By Lemma 30, for any $G_0$-invariant ergodic measure $\nu$, $\mathcal{W}^#(x)$ coincides with the (global) Pesin stable manifold $W_{h|_{\mathcal{W}^#}}(x, gl)$ of $x$ (for the dynamics $h|_{\mathcal{W}^#}$) for $\nu$-almost every $x$. Therefore $\mathcal{W}^#(x, loc)$ is tangent to $E^s_h \cap E^u_f$ at $x$ for $\nu$-almost every $x$, where $E^s_h$ is the Oseledec stable space of $(Dh, \nu)$ and $E^u_f$ is the unstable distribution of $f$. This defines a $G_0$-invariant measurable distribution $(E^s_h \cap E^u_f, \nu)$, for any $G_0$-invariant measure $\nu$. For the case $\nu = \text{vol}(= \text{vol}_{T^d})$, we have

**Lemma 64.** The measurable distribution $E^s_h \cap E^u_f$ coincides with $E^1_f$, $\text{vol}$-almost everywhere.

**Proof.** 

**Case 1:** $\dim E^s_h \cap E^u_f (= \dim E^1_f) = 1$. Then as in the proof of Lemma 45 it is easy to show that there exists $E^1_f$ such that $E^s_h \cap E^u_f \subset E^1_f$, $\text{vol}$-almost everywhere. As in Step 2 of Section 6.6, using the absolute continuity of $\mathcal{W}^#$ and recurrence under iteration by $f$, we can find a full volume set $K \subset T^d$ such that for any $x \in K$, $\mathcal{W}^#(x)$ is a $C^1$ submanifold tangent to $E^1_f$ everywhere.

Now we claim that $i = 1$. The proof is similar to that of Lemma 47. Suppose $i \neq 1$. By H"{o}lder continuity of $\pi$, there exist positive constants $\epsilon_1, C_1$ such that for any $x$ and any $y \in \mathcal{W}^#(x)$ with $d_{\mathcal{W}^#}(x, y) \leq \epsilon_1$,

$$d_{T^d}(\pi(x), \pi(y)) \leq C_1 \epsilon_1. \tag{30}$$

Now we pick an arbitrary $x \in K$ and consider the $C^1$ submanifold $\mathcal{W}^#(x)$. Since $\mathcal{W}^#(x)$ is everywhere tangent to $E^1_f$, $f$ expands $\mathcal{W}^#$ with rate slower than $e^{\lambda_1(f_0) + \eta}$ for some $\eta \ll \lambda_1(f_0) - \lambda_1(f_0)$ (by smallness of $d_{C^1}(f, f_0)$). We choose $y \in \mathcal{W}^#(x, loc)$ such that $d_{\mathcal{W}^#}(x, y) \leq \epsilon_1$. As in the proof of Lemma 47, for $n$ large, $\pi(f^n(x)), \pi(f^n(y))$ can be connected by a $W^1_{T_{A_f}}$ path with length less than $O(C_1 \epsilon_1 e^{n(\lambda_1 + \eta)})$. But as in the proof of Lemma 47, since $T_{A_f}$ expands $W^1_{T_{A_f}}$ leaves at a constant rate $e^{\lambda_1(f_0)}$, $\pi(f^n(x)), \pi(f^n(y))$ cannot be linked by a $W^1_{T_{A_f}}$ path with length $o(e^{n\lambda_1(f_0)})$, giving a contradiction. Therefore $i$ must be 1.

**Case 2:** $\dim E^s_h \cap E^u_f (= \dim E^1_f) > 1$. Suppose that $E^s_h \cap E^u_f$ does not coincide $\text{vol}$-a.e. with $E^1_f$. Then we have

**Lemma 65.** The measurable distribution $E^s_h \cap E^u_f$ has non-trivial intersection (over a positive volume set) with $\bigoplus_{i>1} E^i_f$. 
Proof. Suppose that $E_h^s \cap E_f^u$ has trivial intersection with $\oplus_{i>1} E_f^i$, vol-almost everywhere. By Lusin’s theorem there is a compact set $K_2$ with positive volume and a positive constant $\delta_2$ such that for any $x \in K_2$, 

$$\angle(E_h^s \cap E_f^u(x), E_f^1(x)) > \delta_2. \tag{31}$$

If $x$ and if $f^{n_k}(x) \in K_2$ for some $n \geq 1$, then

$$\angle(E_h^s \cap E_f^u(f^n(x)), E_f^1(f^n(x))) > \delta_2. \tag{32}$$

On the other hand since $\oplus E_f^i$ is a dominated splitting, $E_h^s \cap E_f^u$ is $Df$–invariant and has trivial intersection with $\oplus_{i>1} E_f^i$, we have that

$$\lim_{n \to \infty} \angle(E_h^s \cap E_f^u(f^n(x)), E_f^1(f^n(x))) = \angle(Df^n(E_h^s \cap E_f^u(x)), Df^n(E_f^1(x))) = 0.$$

If $x \in K_2$ is recurrent, then this contradicts (32). Since almost every $x \in K_2$ is recurrent, this gives a contradiction. \qed

As a corollary, as in Step 2 of Section 6.6, using the absolute continuity of $W^\# \cap I \to W^\#(x)$ such that for any $x \in K$, $W^\#(x)$ is a $C^1$ submanifold and $TW^\#(x)$ has non-trivial intersection with $\oplus_{i>1} E_f^i$ everywhere. Here we use the continuity of $TW^\# \cap \oplus_{i>1} E_f^i$.

Moreover, by the Cauchy-Peano existence theorem, for $x \in K$, there exists a $C^1$ path $\gamma : I \to W^\#(x)$ such that for any $t \in I$, $\gamma(t) \in \oplus_{i>1} E_f^i \cap TW^\#$. Denote $\gamma(0), \gamma(1)$ by $z_0, z_1$, respectively. As in Case 1, $f^n(z_0)$ and $f^n(z_1)$ can be linked by a $C^1$ path $f^n(\gamma)$ in $W^\#(x)$ of length $O(e^{n(l^2(f_0) + \eta)})$ for some $\eta_1 \ll \lambda^1(f_0) - \lambda^2(f_0)$; this implies that $\pi(f^n(z_0))$ and $\pi(f^n(z_1))$ can be linked by a $W^1_{T \gamma_f}$–path of length $O(e^{n(l^2(f_0) + \eta)})$ (since (30) holds here). On the other hand, since $\pi(f^n(z_1)) = T^n_{\gamma_f}(\pi(z_1)), i = 0, 1$, it follows that $\pi(f^n(z_0))$ and $\pi(f^n(z_1))$ cannot be connected by a $W^1_{T \gamma_f}$–path of length $o(e^{n\lambda'(f_0)})$, which is a contradiction. \qed

The rest of the proof of Proposition 63 is basically the same as that in the corresponding parts in Section 6.6, especially Steps 3 and 4. By the same proof of Lemma 65 we know that for any $G_0$–invariant ergodic measure $\nu$, if $(E_h^s \cap E_f^u, \nu)$ does not coincide with $E_f^1$, then it must have non-trivial intersection with $\oplus_{i>1} E_f^i$, and so the local Pesin manifold of $\nu$ cannot lie in the cone field around $E_f^1$ (since a distribution with non-trivial intersection with $\oplus_{i>1} E_f^i$ is bounded away from $E_f^1$). Then as in Step 3 of Section 6.6, by using an approximation argument we can show that the local Pesin manifold of $\nu$ cannot be tangent to a subspace with a non-trivial intersection with $\oplus_{i>1} E_f^i$, which contradicts our assumption.

In summary, for any $G_0$–invariant ergodic measure $\nu$, $(E_h^s \cap E_f^u, \nu) = E_f^1$. Mimicking the proof of Step 4 in Section 6.6 one can prove that $Dh$ uniformly contracts $E_f^1$ and $W^\#$ is tangent to $E_f^1$ everywhere. In addition, by the same proof of Step 5 in Section 6.6 we
obtain that \( h \) uniformly expands \( \oplus_{E_j \subset E_j^s, j \neq 1} E_j^i \) and uniformly contracts \( E_j^s \), and \( W^\# \) is the stable manifold within \( W^\#_j \) for \( h \).

As in the proof of Proposition 63 by induction one can show that the lift of \( W^{\#}_{TA_f} \) by \( \pi^{-1} \) on \( W^\#_j \) is a foliation with \( C^1 \) leaves and tangent to \( E^s_j \) everywhere; we denote this foliation by \( W^\#_j \). From the proof of Proposition 63 we get that \( E^s_j \oplus E^c_i \) is integrable, since center holonomy preserves the \( W^\#_j \) foliation. Therefore the leaf conjugacy \( h^c \) maps each \( W^c_{f_0} \) to \( W^c_{f_j} \) respectively. Again by Lemma 26 the H"older exponent for \( h^c \) and \( (h^c)^{-1} \) are close to 1. The rest of the proofs of Proposition 42 and Lemma 50 are the same as in Section 6.6.

8. Proof of Theorem 4

Proof of Theorem 4. Recall that under the assumptions of Theorem 4, \( f \in \text{Diff}_v^\infty(T^d) \) is a \( C^1 \)–small ergodic perturbation of \( f_0 \), where \( f_0 \) is an isometric extension over a hyperbolic automorphism of the torus defined in Theorem 4.

As in Section 7.1, we denote by \( \lambda^i(f_0) \) the distinct Lyapunov exponents of \( f_0 \) (ordered in \( i \) by increasing size) and \( T \mathbb{T}^d = \oplus E^s_j \oplus E^c_i \) the corresponding \( f_0 \)-invariant Lyapunov splitting. Let \( T \mathbb{T}^d = \oplus E^s_j \oplus E^c_i \) be the corresponding \( f \)-invariant dominated splitting. It is easy to see that up to a coordinate change \( Df_0|_{E^s_{f_0}} \) is conformal for each \( i \), and hence the cocycles \( Df_0^{-1}|_{E^s_{f_0}}, Df_0|_{E^c_i} \) have point (Mather) spectrums. As a consequence,

Lemma 66. If \( d_{C^1}(f,f_0) \) is sufficiently small then the cocycles \( Df^{-1}|_{E^s_j}, Df|_{E^c_i} \) satisfy the narrow band condition defined in Section 3.6.

Proof. If \( d_{C^1}(f,f_0) \) is small then \( E^s_j \) is close to \( E^s_{f_0} \). Since \( Df_0|_{E^s_{f_0}} \) has point Mather spectrum, the Mather spectrum of \( Df|_{E^s_j} \) for each \( i \) is contained in an arbitrarily narrow band, which implies Lemma 66.

Since \( f \) is leaf conjugate to \( f_0 \), there is an \( f \)-fixed center leaf \( W^c_j(x_0) \). As in the proof of Lemma 39, \( Z_{\text{Diff}^r(T^d)}(f) \) is virtually \( Z_0 \), where

\[ Z_0 := \{ h \in Z_{\text{Diff}^r(T^d)}(f), h \text{ preserves the orientation of } W^c_j, \text{ and } h(W^c_j(x_0)) = W^c_j(x_0) \}. \]

By Lemma 25 there is a H"older continuous fiber bundle \( \pi : T^d \to T^{d-1} \) such that \( \pi \circ f = T_{A_f} \circ \pi \) and the fibers of \( \pi \) are the center leaves of \( f \). For any \( h \in Z_0 \), \( h \) preserves this fiber bundle structure and there is an automorphism \( T_{A_h} : T^{d-1} \to T^{d-1} \) such that \( \pi \circ h = T_{A_h} \circ \pi \).

Consider the group \( Z^c \) generated by the center-fixing elements in \( Z_0 \) and let \( H = \{ A_h : h \in Z_0 \} \). Then \( Z_0 \) is a group extension of \( H \) by \( Z^c \).
If the disintegration of volume along $W^c_f$ leaves is not Lebesgue, then Theorem 4 is a corollary of Theorem 6. Assume that $\text{vol}_{T^d}$ has Lebesgue disintegration along $W^c_f$. By Theorem 13, one of the following cases holds:

**Case 1:** $f$ is accessible. Then by [3] the disintegration of $\text{vol}_{T^d}$ has a continuous density function on the leaves of $W^c_f$. Therefore $f$ is topologically conjugate to a rotation extension over $A_f$. More precisely, there is a continuous function $\psi$ on $T^{d-1}$ and a map $\Phi : T^d \to T^d$ such that

$$\Phi^{-1} \circ f \circ \Phi(x, y) = (T_{A_f}(x), R_{\psi(x)}(y)).$$

For any $h \in Z^c$, by commutativity and transitivity of $f$, there exists $\rho = \rho_h \in T$ such that

$$\Phi^{-1} \circ h \circ \Phi(x, y) = (x, R_h(y)).$$

Moreover similar to the topological argument in the proof of Lemma 17 we have that each $W^c_f$ is $h$–invariant.

As in the proof of Theorem 3 or by [2], we can construct a continuous vector field $X$ tangent to $W^c_f$ that is induced by the continuous density function of the volume along $W^c_f$. Consider the flow $\{\varphi_t, t \in \mathbb{R}\}$ generated by $X$: then $\varphi_t$ commutes with $f$ for any $t \in \mathbb{R}$, $\varphi_1 = \text{id}$ and $h = \varphi_{\rho_h}$ for any $h \in Z^c$. Moreover, as in [2], accessibility implies that $X$ is $C^\infty$ along $W^c_f$ (see also [87], where it is proved that any holonomy-invariant section of an $r$–bunched cocycle is $C^r$).

As in the the proof of Theorem 8, we set $D := \{t \in T : \varphi_t \in Z^c\}$. There are two possibilities:

1. $D < T$ is discrete. Then $Z^c$ is finite. By Lemma 16 the group $H$ is abelian with rank $\leq \ell$, where $\ell$ is defined in Theorem 4.
   - If $H$ has rank less than $\ell$, then by finiteness of $Z^c$, there is no subgroup of $Z_0$ projecting onto $Z^c$. Since $Z_{\text{Diff}^r(\mathbb{T}^d)}(f)$ is virtually $Z_0$, it follows that no subgroup of $Z_{\text{Diff}^r(\mathbb{T}^d)}(f)$ projects onto $Z^c$ either, which implies that Theorem 4 holds in this case.
   - If $H$ has rank $\ell$, then as in the proof of Theorem 6 we can construct partially hyperbolic elements in all the Weyl chambers of the action of $Z_0$, which implies that $E^u_f \oplus E^s_f$ is jointly integrable, contradicting the accessibility of $f$.

2. $D < T$ is dense. Lemma 66 implies that the triple $(f, \varphi_t, X)$ satisfies the hypotheses of Proposition 31, applying this result, we obtain that $D = \mathbb{R}$, $X$ is a $C^\infty$ vector field and so $\varphi_t$ is a $C^\infty$ flow. Therefore $W^c_f$ is a smooth foliation, and $f$ is smoothly conjugate to a skew product $g_\rho$. By Lemmas 15 and 16 it is easy to see that $Z_{\text{Diff}^r(\mathbb{T}^d)}(f)$ is virtually $Z^c \times T$ for some $\ell \leq \ell_0(A_{f_0})$.

**Case 2:** The diffeomorphism $f$ is topologically conjugate to $T_{A_f} \times R_\theta$, i.e. there is a homeomorphism $\Phi : T^d \to T^d$ such that $\Phi^{-1} \circ f \circ \Phi = T_{A_f} \times R_\theta$, for some $\theta \notin \mathbb{Q}$, and $E^u_f \oplus E^s_f$ is jointly integrable.

Since $f$ is center bunched, as in the proof of Theorem 8 $E^u_f \oplus E^s_f$ is tangent to the $C^1$ foliation $W^H$. For any $x \in \mathbb{T}^d$, we denote by $\text{Pr}_x^c$ the projection from $\mathbb{T}^d$ to $W^c_f(x)$ along
\( \mathcal{W}^H \) and let \( \mu_x := \text{Pr}^c_x (\text{vol}_{\mathbb{T}^d}) \). Then the family \( \{ \mu_x, x \in \mathbb{T}^d \} \) is \( f \)-invariant, i.e.

\[
(33) \quad (f|_{\mathcal{W}^c})_* \mu_x = \mu_{f(x)}
\]

The \( C^1 \)-ness of \( \mathcal{W}^H \) implies that the family of measures \( \{ \mu_x, x \in \mathbb{T}^d \} \) along \( \mathcal{W}^c \)-leaves have continuous density functions. Therefore \( f \) is \( \infty \)-bunched, which implies \( \mathcal{W}^c \) has \( C^\infty \) leaves, and the stable and unstable holonomies between center leaves are uniformly smooth.

Since \( \mathcal{W}^c, \mathcal{W}^s \) have uniformly smooth leaves, Journé’s lemma implies that \( \mathcal{W}^H \) has uniformly smooth leaves as well. In summary, \( \mathcal{W}^H \) is a smooth foliation. Then by construction of \( \{ \mu_x \} \) we know that the measures \( \{ \mu_x, x \in \mathbb{T}^d \} \) have \( C^\infty \) densities along \( \mathcal{W}^c \).

Therefore we can construct a continuous vector field \( X \), tangent to \( \mathcal{W}^c \) and \( C^\infty \) along \( \mathcal{W}^c \) leaves, which is induced by the density functions of \( \{ \mu_x, x \in \mathbb{T}^d \} \). Again we denote by \( \{ \varphi_t, t \in \mathbb{R} \} \) the flow generated by \( X \). Then \( \varphi_1 = \text{id} \).

As in Case 1, for any \( h \in \mathbb{Z}^c \), by commutativity and transitivity of \( f \) there exists \( \rho = \rho_h \in \mathbb{T} \) such that \( \Phi^{-1} \circ h \circ \Phi(x, y) = (x, H_\rho(y)) \), where \( \Phi \) is the conjugacy defined in the beginning of Case 2. Moreover \( h = \varphi_\rho \) for any \( h \in \mathbb{Z}^c \).

The rest of the proof for Case 2 is similar to that of Case 1. Again we take the set \( D := \{ t \in \mathbb{T}, \varphi_t \in \mathbb{Z}^c \} \), and consider the following cases.

1. \( D < \mathbb{T} \) is discrete. Then \( \mathbb{Z}^c \) is finite. We consider the group \( H \) as in Case 1.
   
   a. If \( H \) has rank less than \( \ell \), then by exactly the same proof as in Case 1 there is no subgroup of \( \mathbb{Z}_{\text{Diff}^{\ell}(\mathbb{T}^d)}(f) \) projecting onto \( \mathbb{Z}^c \).
   
   b. If \( H \) has rank \( \ell \), first we claim that the action of \( \mathbb{Z}_{\text{Diff}^{\ell}(\mathbb{T}^d)}(f) \) on \( \mathbb{T}^d \) is \( C^\infty \) (a priori it is only \( C^r \)). The reason is that for any \( g \in \mathbb{Z}_{\text{Diff}^{\ell}(\mathbb{T}^d)}(f) \), \( g \) preserves the smooth density on \( \mathcal{W}^c \) (induced by \( \{ \mu_x, x \in \mathbb{T}^d \} \)) and the smooth normal form on \( \mathcal{W}^s_f \) and \( \mathcal{W}^u_f \). Therefore by Journé’s lemma, \( g \) is uniformly smooth.
   
   Therefore the action by \( \mathbb{Z}_0 \) is actually smooth and volume preserving on \( \mathbb{T}^d \). Moreover since \( H \) has rank \( \ell \), following the proof of Theorem 6 we can construct partially hyperbolic elements in all the Weyl chambers of the action of \( \mathbb{Z}_0 \). Then by [23], the action of \( \mathbb{Z}_0 \) is rigid, see Appendix for the precise statement of the result. In particular \( f \) is smoothly conjugate to \( T_A \times R_\theta \) for some \( \theta \notin \mathbb{Q} \).

2. \( D < \mathbb{T} \) is dense. By the same proof as in Case 1 we can prove that \( f \) is smoothly conjugate to an isometric extension \( (x, y) \mapsto (g(x), y + \rho(x)) \). The accessibility condition is not necessary in this case since we already obtained that the vector field \( X \) is uniformly smooth along \( \mathcal{W}^c \).

Case 3: There exists \( n \geq 0 \) such that the torus \( \mathbb{T}^d \) is the union of countably many open, \( f^n \)-invariant accessibility classes \( \{ U_\alpha \} \) whose boundary consists of compact, \( f^n \)-invariant horizontal leaves \( \{ \mathcal{W}^H_\beta \} \), \( \beta \in \mathbb{K} \). The action of \( f^n \) on the space of horizontal leaves given by the map \( p \) in Theorem 13 is semiconjugate to the identity on \( S^1 \). By [3], within each accessibility class \( U_\alpha \), \( \text{vol}_{\mathbb{T}^d} \) has Lebesgue disintegration with continuous density function along \( \mathcal{W}^c_f \).
As in Cases 1 and 2, we consider the groups $Z_0, Z^c$ and $H$. First we claim that $Z^c$ is a finite group. For any $h \in Z^c$, Proposition 2.3 implies that $h \mathcal{W}_f^u = \mathcal{W}_f^u, h \mathcal{W}_f^s = \mathcal{W}_f^s$, and so $h$ maps (open) accessibility classes to (open) accessibility classes. The map $h^H$ induced by $h$ and $p$ on space of horizontal leaves $\{\mathcal{W}_f^H\}_{\beta \in \mathcal{K}}$ is semiconjugate to a homeomorphism $g_h$ of $S^1$.

Let $c = \max_\alpha \text{vol}(U_\alpha)$, and let $U_1, \ldots, U_N$ be the open accessibility classes of volume $c$. By ergodicity of $f$, $h$ is volume preserving. It follows that for any $i$, the image $h(U_i)$ must be some $U_j$, and so $h^{N_1}(U_i) = U_i$. Then within each $U_i$, $h^{N_1}$ must preserve the continuous density function of $\text{vol}_{T^d}$ along $\mathcal{W}_f^c$, and so $h^{N_1}$ must be the identity on $\bigcup_{i=1}^N U_i$. By the same argument, for every open accessibility class $U_\alpha$, there exists $k_\alpha$ such that $h^{k_\alpha}|_{U_\alpha} = id|_{U_\alpha}$. This implies that the homeomorphism $g_h$ has the property that for every $x \in S^1$ there exists $k$ such that $g_h^k(x) = x$. But this implies that there exists $m$ such that $g_h^m = id$. This in turn implies that $h^m = id$.

Now consider the group of homeomorphisms \{\$g_h : h \in Z^c\$\} on $S^1$: each element has finite order. Since Homeo($S^1$) satisfies the Burnside property, this implies that the group has finite order. This implies by the preceding argument that $Z^c$ is finite as well.

If $H$ has rank less than $\ell$, then by the same proof of (1)(a) in Case 1, there is no subgroup of $\mathcal{Z}_{\text{Diff}}(f)$ that projects onto $Z_\ell$. If $H$ has rank $\ell$, then as in Cases 1 and 2, following the proof of Theorem 6 we can construct partially hyperbolic elements in all the Weyl chambers of the action of $Z_0$, which implies the joint integrability of $E_f^u \oplus E_f^s$, contradicting the assumption of Case 3. □

**Appendix A. Global rigidity of conservative partially hyperbolic abelian actions on the torus**

For completeness, we state a result in [23] used in Section 8. The basic setting of this section is the following. Suppose $\alpha : \mathbb{Z}^k \to \text{Diff}_{\text{vol}}^\infty(T^d)$ is a smooth, volume preserving ergodic abelian action. We assume that there exists at least one $a \in \mathbb{Z}^k$ such that $\alpha(a)$ is a fibered partially hyperbolic diffeomorphism and all the partially hyperbolic elements of $\alpha$ preserve a common circle center foliation $\mathcal{W}^c$.

As in the discussion in Section 3.8, the sum of the stable and unstable distributions $E^H := E^u_a \oplus E^s_a$ of a single partially hyperbolic element $\alpha(a)$ is $\alpha$–invariant. Then we can consider the Lyapunov exponents (functionals) $\chi_i$ and hyperbolic Weyl chamber pictures induced by the cocycle $D\alpha|_{E^H}$ with respect to $\text{vol}_{T^d}$.

In [32], the authors proved that any irreducible higher rank smooth Anosov action on nilmanifold is smoothly conjugate to an algebraic action under the extra assumption that in each Weyl chamber there exists an Anosov element. Similar condition for the existence of sufficiently many Anosov elements are assumed in almost all the previous rigidity results for abelian actions. In the following theorem we assume the existence of a partially hyperbolic element in each hyperbolic Weyl chamber (induced by the cocycle $D\alpha|_{E^H}$ with respect to $\text{vol}$).
Theorem 16. Suppose there is no pair of Lyapunov functionals \( \chi_i, \chi_j \) and \( c \in (-\infty, \frac{1}{2}] \cup [2, \infty) \) such that \( \chi_i = c \chi_j \). Then \( \alpha \) is smoothly conjugate an algebraic action, i.e. the product of linear Anosov action on \( T^{d-1} \) and rotation actions on \( T^1 \).

To apply Theorem 16 we need the following lemma:

**Lemma 67.** The action of \( Z_0 \) in (1)(b) of Case 2 in Section 8 satisfies all the conditions in Theorem 16.

**Proof.** Recall that in (1)(b) of Case 2 in Section 8 we obtain that the action of \( Z_0 \) on \( T^d \) is abelian, \( C^\infty \) and volume preserving. Every element \( h \) in \( Z_0 \) preserves the common center foliation \( W^c \) and there is a Hölder continuous fiber bundle \( \pi : T^d \to T^{d-1} \) such that for any \( h \in Z_0 \), there is a linear automorphism \( T_{A_h} : T^{d-1} \to T^{d-1} \) such that \( \pi \circ h = T_{A_h} \circ \pi \).

In addition, the group \( H := \{ A_h, h \in Z_0 \} \) has rank at least \( \ell \). Then by Lemma 33 \( H \) induces a maximal Anosov linear action on \( T^{n-1} \). As a consequence, the action of \( H \) is totally non-symplectic (i.e., there are no negatively proportional Lyapunov functionals) and conformal on each coarse Lyapunov foliation. By Lemma 48 and the discussion in Section 6.4, the Lyapunov functionals of the action of \( Z_0 \) must be close to that of \( H \) (or see discussions in Step 7. of Section 6.6), therefore \( Z_0 \) must be TNS and satisfy the \( \frac{1}{2} \)-pinching condition. In particular, in (2)(b) of Case 2 of Section 8 \( Z_0 \) satisfies all the conditions in Theorem 16. \( \square \)

This has the following corollary.

**Corollary 68.** The action of \( Z_0 \) is smoothly conjugate to a product of a linear Anosov action on \( T^{d-1} \) and a rotation action on \( T^1 \). In particular, in (1)(b) of Case 2 in Section 8 \( f \) is smoothly conjugate to \( A_f \times R_\theta \) for some \( \theta \in \mathbb{R} \).

**References**

(Damjanović) Department of mathematics, Kungliga Tekniska högskolan, Lindstedtsvägen 25, SE-100 44 Stockholm, Sweden.
E-mail address: ddam@kth.se

(Wilkinson) Department of mathematics, the University of Chicago, Chicago, IL, USA, 60637
E-mail address: wilkinso@math.uchicago.edu

(Xu) Department of mathematics, the University of Chicago, Chicago, IL, USA, 60637
E-mail address: dishengxu@math.uchicago.edu