Fourier Analysis

M. Behrens

November 8, 2002

This document was prepared for electrical engineers who wanted a simple, short, and sweet introduction to fourier analysis. I have reproduced it for womp for a quick and dirty reference. Some caveats - at some points I am "gentle" to the engineers mathematics-wise, so don't interpret this language as being patronizing - I'm just too lazy to re-tex the document with mathematician audiences in mind. Also, electrical engineers use the letter j instead of i for the square root of -1. Again, I am too lazy to alter the document, changing j's to i's.

1 Motivation: Finite dimensional vector spaces

The periodic theory of Fourier analysis may be likened to the decomposition of an ordinary vector into its components with respect to a basis. To motivate the infinite dimensional situation encountered in Fourier analysis, we'll examine what happens in a very familiar setting: $\mathbb{C}^3$, three dimensional complex space. We have an inner product (like a dot product) defined by:

$$\langle (v_1, v_2, v_3), (w_1, w_2, w_3) \rangle = \sum_{k=1}^{3} v_k \overline{w}_k$$

where $v_i, w_i \in \mathbb{C}$ and the bar denotes complex conjugation:

$$\overline{x + jy} = x - jy$$

Let e_1, e_2, e_3 be the standard basis of $\mathbb{C}^3$:

$$\begin{aligned} e_1 &= (1, 0, 0) \\ e_2 &= (0, 1, 0) \\ e_3 &= (0, 0, 1) \end{aligned}$$

Then it is readily seen that the collection $\{e_k\}$ actually forms an *orthonormal* basis. That is:

$$\langle e_k, e_l \rangle = \begin{cases} 1 & \text{if } k = l, \\ 0 & \text{if } k \neq l \end{cases}$$

Then:

$$\langle (z_1, z_2, z_3), e_k \rangle = z_k$$

so we can recover the components of our vector by taking inner products with the elements of out orthonormal basis. Then we can express our vector as a sum of these components times the elements of the orthonormal basis:

$$(z_1, z_2, z_3) = \sum_{k=1}^{3} \langle (z_1, z_2, z_3), e_k \rangle e_k = \sum_k z_k e_k$$

Furthermore, given the coefficients, we can determine the length of the vector:

$$\|(z_1, z_2, z_3)\|^2 = \sum_k |z_k|^2$$

where for a complex number $z = x + jy$:

$$|z|^2 = x^2 + y^2 = z\overline{z}$$

While this process seems trivial in the finite dimensional case, it is exciting in the infinite dimensional situation.

2 Periodic Theory

Suppose that f is a square integrable periodic function defined on the real line and taking values in the complex numbers. We shall assume that f has period 2π. The square integrability condition translates to saying that:

$$\int_0^{2\pi} |f(t)|^2 dt < \infty$$

Certainly any smooth (infinitely differentiable) function satisfies this requirement, since in this case f would have to take on a maximum by continuity. We shall call the space of all smooth periodic functions which are square integrable:

$$L^2(\mathbb{T})$$

($\mathbb{T}$ stands for the 1-torus, or simply the circle, since any function which is periodic may be thought of as being defined on the circle.)

Now, we would like an orthonormal basis for $L^2(\mathbb{T})$. For starters, we need an inner product to make sense of what it means to be 'orthonormal'.

Definition. Define an inner product on $L^2(\mathbb{T})$:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)} dt$$

Our candidate for an orthonormal basis is the collection:

$$\{e^{jnt} \; : \text{ where } n \text{ is any integer (positive or negative)}\}$$

Proposition. $\{e^{jnt}\}$ forms an orthonormal basis of $L^2(\mathbb{T})$

Proof. Since we are working in an infinite dimensional vector space, being a basis means that every periodic function can be written as an *infinite* sum of the functions $\{e^{jnt}\}$. This fact is highly non-trivial, and the proof is omitted.

The orthonormality relations are much easier to see:

$$\begin{aligned}
\langle e^{jnt}, e^{jmt}\rangle &= \frac{1}{2\pi}\int_0^{2\pi} e^{jnt}\overline{e^{jmt}}dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{jnt}e^{-jmt}dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{j(n-m)t}dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \cos((n-m)t)) + j\sin((n-m)t)dt \\
&= \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m \end{cases}
\end{aligned}$$

□

The Fourier coefficients of a periodic function f are defined to be the components of f with respect to our orthonormal basis.

Definition. Let $f \in L^2(\mathbb{T})$. The *Fourier coefficients* of f are defined as:

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{jnt}dt$$

Note that in our fancy notation, we have:

$$\hat{f}(n) = \langle f, e^{jnt}\rangle$$

Also take note that we may regard the collection of Fourier coefficients as a discrete time function, that is, a complex valued function on the integers.

The key to Fourier analysis is that we can go backwards. That is, given the Fourier coefficients, we can recover the function itself.

Theorem. Let $f \in L^2(\mathbb{T})$. Then:

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{jnt}$$

Proof. We know that since $\{e^{jnt}\}$ forms a basis of $L^2(\mathbb{T})$, f may be written as a sum of these basis elements:

$$f(t) = \sum_n c_n e^{jnt}$$

for some complex numbers c_n. We just need to prove that $c_n = \hat{f}(n)$. But:

$$\begin{aligned}\hat{f}(n) &= \langle f(t), e^{jnt} \rangle \\ &= \langle \sum_m c_m e^{jmt}, e^{jnt} \rangle \\ &= \sum_m c_m \langle e^{jmt}, e^{jnt} \rangle \\ &= c_n\end{aligned}$$

The last equality follows from the fact that the e^{jnt}'s are orthonormal. □

We conclude the study of the periodic case with Parseval's identity, which relates the size of a function to the size of its Fourier coefficients.

Theorem. Let $f \in L^2(\mathbb{T})$. Then:

$$\frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 dt = \sum_n |\hat{f}(n)|^2$$

Proof. We just compute:

$$\begin{aligned}\frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 dt &= \langle f, f \rangle \\ &= \langle \sum_n \hat{f}(n) e^{jnt}, \sum_m \hat{f}(m) e^{jmt} \rangle \\ &= \sum_{n,m} \hat{f}(n) \overline{\hat{f}(m)} \langle e^{jnt}, e^{jmt} \rangle \\ &= \sum_n |\hat{f}(n)|^2\end{aligned}$$

where the last equality holds by orthonormality. □

3 Discrete Time Fourier Analysis

The discrete time case is really just the periodic case backwards, with the sign on j reversed. We shall consider the collection of all discrete time complex valued square integrable functions. By square integrable, we mean:

$$\sum_n |x[n]|^2 < \infty$$

We shall denote this space as:

$$L^2(\mathbb{Z})$$

Where $\mathbb{Z}$ stands for the integers.

Definition. Define, for $x \in L^2(\mathbb{Z})$, the Fourier transform X of x to be the periodic function:

$$X(\omega) = \sum_n x[n] e^{-jn\omega}$$

We very swiftly work through the results that we essentially already proved in the last section. We have a Fourier inversion result, which says that our discrete time function may be recovered from our Fourier transform:

Theorem. Let $x \in L^2(\mathbb{Z})$ and X be its Fourier transform. Then:

$$x[n] = \frac{1}{2\pi} \int_0^{2\pi} X(\omega) e^{jn\omega} d\omega$$

We get a Parseval's identity:

Theorem. Let $x \in L^2(\mathbb{Z})$ and X be its Fourier transform. Then:

$$\sum_n |x[n]|^2 = \frac{1}{2\pi} \int_0^{2\pi} |X(\omega)|^2 d\omega$$

Piece of cake, since we already know these for the periodic theory.

4 Fourier Transform: Non-discrete Case

We wish to consider the collection of all complex valued functions on the real line which are square integrable. Here we assume nothing about periodicity. By square integrable, we mean:

$$\int |f(t)|^2 dt < \infty$$

where if the limits of integration are left off we take it to be an integral from $-\infty$ to ∞. We shall denote this space:

$$L^2(\mathbb{R})$$

It is more difficult to fully understand what the meaning is of the Fourier transform on $\mathbb{R}$. In some sense, it takes a time distribution and yields a frequency distribution. Then, given the frequency distribution, you can reconstruct the time distribution.

Definition. Let $f \in L^2(\mathbb{R})$. The Fourier transform of f is denoted $\hat{f}$, and is also an element in $L^2(\mathbb{R})$, defined by:

$$\hat{f}(\omega) = \int f(t) e^{-j\omega t} dt$$

There is a Fourier inversion theorem, which says that given the Fourier transform of a function, one can recover the actual function.

Theorem. Let $f \in L^2(\mathbb{R})$. Then:

$$f(t) = \frac{1}{2\pi} \int \hat{f}(\omega) e^{j\omega t} d\omega$$

The proof of the Fourier inversion theorem, if done rigorously, is lengthy, and will not be included here. There is also a result that vaguely reminds one of the Parseval's identity, called the Plancharel theorem:

Theorem. Let $f \in L^2(\mathbb{R})$. Then:

$$\int |f(t)|^2 dt = \int |\hat{f}(\omega)|^2 d\omega$$

So the size of the Fourier transform is the same as that of the original function. Define the convolution of $f, g \in L^2(\mathbb{R})$ to be:

$$f * g(t) = \int f(s) g(t-s) ds$$

Then Fourier transform possesses the following pleasant property: it turns convolution into something much nicer.

Exercise. Let $f \in L^2(\mathbb{R})$ and define $g(t) = f(t-x)$. Express $\hat{g}(\omega)$ in terms of $\hat{f}(\omega)$.

Exercise. Express $\widehat{f * g}(\omega)$ in terms of $\hat{f}(\omega)$ and $\hat{g}(\omega)$.