WOMP: CATEGORY THEORY

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1. Introduction

Category theory is the language of modern mathematics. It is the right level of abstraction to describe connections between varying fields of mathematics. Similarities noticed in different areas, such as Cartesian products in the categories of groups, sets and spaces, can usually be explained categorically. By proving results in a more general context we can get results that apply to widely varying fields.

2. Basics

2.1. Definitions and examples.

Definition 2.1. A category $\mathcal{C}$ is a collection, $ob(\mathcal{C})$ of objects, such that for every pair $X,Y \in ob(\mathcal{C})$ there is a set, $\mathcal{C}(X,Y)$, whose elements are called the morphisms (or maps or arrows) from X to Y. Such that the following hold

(1) There is a distinguished element $1_X \in \mathcal{C}(X,X)$ for every $X \in ob(\mathcal{C})$ called the identity map on X.
(2) For every $X,Y,Z \in ob(\mathcal{C})$ we have a map of sets $\mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) \to \mathcal{C}(X,Z)$ called the composition law of $\mathcal{C}$. This map is written $(f,g) \mapsto f \circ g = fg$.
(3) If $f \in \mathcal{C}(X,Y)$ and $g \in \mathcal{C}(Z,X)$ then $f1_X = f$ and $1_X g = g$.
(4) The composition law is associative, i.e. $(fg)h = f(gh)$ where f,g,h are composable arrows in the obvious sense.

You already know many categories. The following are all concrete categories (the objects have underlying sets and the morphisms are maps of the underlying sets) that you are probably familiar with.

Example 2.2. *Ens* : the category of sets (*Ens* is short for ensemble the French word for set) with maps set maps.

Example 2.3. *Gps* : The category of groups with homomorphisms.

Example 2.4. *AbGps* : The category of abelian groups with homomorphisms.

Example 2.5. $Vect_k$: The category of k-vector spaces with k-linear maps.

Example 2.6. Top : The category of topological spaces with continuous maps.

Example 2.7. $hTop$: The category of topological spaces with *homotopy* classes of continuous maps.

Example 2.8. *Rings* : The category of rings with unit and ring homomorphisms preserving units.

Example 2.9. *Man* : The category of topological manifolds with continuous maps.

Example 2.10. $C^\infty - Man$: The category of smooth manifolds with smooth maps.

2.2. **It's all about the maps.** Part of the philosophy of category theory is that the maps are more important than the objects. When possible a category theorist will try to describe new structures using commutative diagrams as opposed to descriptions involving elements and explicit maps of those elements. For example

Definition 2.11. A group G is an object in *Ens* with maps satisfying the following commutative diagrams:

FIGURE 2.1. Monoid Properties

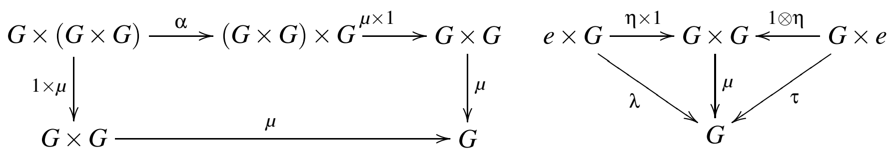
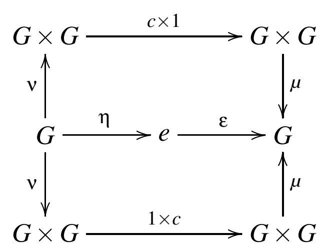


FIGURE 2.2. Invertibility



Where e is the set with one element (the terminal set categorically). The isomorphisms λ and τ are canonical (there is one natural choice). This looks a bit confusing at first but it is a good example of some useful generality. We can replace the category *Ens* in the above description with any other category such as $Top, C^{\infty} - Man$ or *Gps* and we've defined the notion of a group object in these categories.

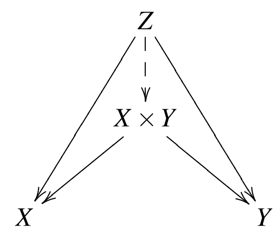
Example 2.12. A group object in *Top* is a topological group. Note that the multiplication and inversion maps have to be continuous now in order for the diagrams to be in *Top*.

Example 2.13. A group object in $C^{\infty} - Man$ is a Lie group.

Example 2.14. It turns out that a group object in *Gps* is an abelian group. This is an easy exercise in group theory.

We can also easily create a dual notion of a group from our new definition. That is a cogroup is an object G in *Ens* satisfying the diagrams above *but with all of the arrows reversed*. I know a cogroup sounds stupid but the notion does come up (e.g a cogroup object in *Rings* is a Hopf algebra which occur frequently in algebraic topology, algebraic geometry and representation theory).

FIGURE 2.3. Products



We can describe many constructions in a very universal way. For example the product of two objects $X, Y \in ob(\mathcal{C})$ is an object $X \times Y$ satisfying the following commutative diagram for every Z.

Where the dotted line indicates that such a map always exists and there is only one such map making the diagram commute.

This gives one definition that works in every category. We can also dualize it to define a coproduct. In all of the above examples the product is the ordinary Cartesian product, but the coproducts vary widely.

Example 2.15. In $Ens, Top, Man, C^\infty - Man$ and $hTop$ the coproduct is the disjoint union.

Example 2.16. In $AbGrps$ and $Vect_k$ it is the direct sum.

Example 2.17. In *Rings* it is the tensor product.

Example 2.18. In *Gps* it is the free product.

2.3. **Abstract Categories.** There are also more abstract categories. In these categories the morphisms may not be maps of sets which can be a bit counterintuitive at first.

Example 2.19. P is a poset we can create a corresponding category with one object for each element of P and if $a \leq b$ then we define a morphism $a \to b$. Transitivity gives us our composition law and since $a \leq a$ we have an identity for each object.

Example 2.20. Let $\mathcal{C}$ be the category with two objects a, b and one non-identity morphism $a \to b$. It's a pretty boring category mostly.

Example 2.21. Let $\mathcal{C}$ be a category with one object. The set of endomorphisms has a multiplication coming from the composition law. This multiplication is associative and has a unit by definition. This defines a monoid. Similarly, we can define a category with one object from a monoid.

Example 2.22. Let $\mathcal{C}$ be a category with one object and whose morphisms are all isomorphisms. That is for every f there exists f^{-1} such that $ff^{-1} = f^{-1}f = id$. By a similar construction we can see that such categories are in one to one correspondence with groups.

These examples show that abstract categories can be interesting by themselves. They also lend themselves more easily to generalization (e.g. a groupoid is a category, with possibly many objects, where every morphism is an isomorphism or more informally a groupoid is a "group with many objects").

3. FUNCTORS

We would like to get a bit metaphysical and talk about the category of (small) categories, *Cat*. What we are lacking is the notion of a morphism between two categories. This is where functors come in. Functors are the way in which we connect different areas of mathematics and a big part of how we use abstract categories.

Definition 3.1. A functor F, between two categories $\mathcal{C},\mathcal{D}$, assigns to each $X \in ob(\mathcal{C})$ an object $F(X) \in ob(\mathcal{D})$ and to every morphism $f \in \mathcal{C}(X,Y)$ a morphism $F(f) \in \mathcal{D}(F(X),F(Y))$ such that

(1) $F(fg) = F(f)F(g)$ for two composable morphisms f and g.
(2) $F(id_X) = id_{F(X)}$.

Example 3.2. We can define the abelianization functor $F : Gps \to AbGps$ by $F(G) = G/[G,G]$. Since maps of groups send commutators to commutators we see that a map $f : H \to G$ induces a map $F(f) : H/[H,H] \to G/[G,G]$.

Example 3.3. If we have two categories as in example 2.21 a functor between them is a morphism of monoids. Similarly, if we have two categories as in example 2.22 a functor between them defines a homomorphism of groups.

Example 3.4. Let $\mathcal{C}$ be as in 2.22 then a functor $F : \mathcal{C} \to Vect_k$ defines a representation of the corresponding group. That is $F(*)$ is a vector space and for each morphism of $\mathcal{C}$ gives an automorphism of that vector space. The definition of a functor says that this assignment respects the identity map (corresponding to the identity element) and composition (corresponding to the multiplication).

Example 3.5. The functor $\pi_1 : Top \to Gps$ which assigns each topological space its fundamental group.

Example 3.6. Homology theories are usually functors (don't worry if you don't know what they are).

Example 3.7. $\hom_k(\_\_,k) = (\_\_)^* : Vect_k \to Vect_k$ isn't really a functor. If we have a map $X \to Y$ then we get a map $Y^* \to X^*$. This is called a *contravariant functor*. Or better yet (because differential geometers have ruined a perfectly good term by using it in precisely the opposite way) we have $(\_\_)^* : Vect_k^{op} \to Vect_k$ is a functor. Where $\mathcal{C}^{op}$ is the *opposite category* of $\mathcal{C}$, defined to have the same objects but all of the arrows go in the opposite direction. Functors of the form $\hom(\_\_,T)$ or $\hom(T,\_\_)$ are called *representable* (and are represented by T).