Fourier Series and the Fourier Transform

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1 Fourier series

Consider periodic functions, functions defined on some finite interval, or functions defined on the unit circle. All are equivalent; I prefer to think of functions defined on $[0, 1]$.

We observe:

$$\int_0^1 e^{2\pi inx}\overline{e^{2\pi imx}}dx = 1, \text{ if } n = m; \quad 0, \text{ if } n \neq m.$$

So $\{e^{2\pi inx}\}$ is an orthonormal set in the Hilbert space $L^2([0, 1])$. Let $U_n(x) = e^{2\pi inx}$.

Question. Is $\{U_n\}$ an orthonormal basis for $L^2([0, 1])$?

Answer. Yes!

Sketch of proof It's a basis if for any $f \in L^2$, there exists a $\hat{f} : \mathbf{Z} \mapsto \mathbf{C}$ such that in L^2 norm, f is the limit of the following sequence of trigonometric polynomials:

$$\left\{ \sum_{|n|\leq N} \hat{f}(n)U_n \right\}_{N=1}^{\infty}$$

I will only show that every L^2 function is the limit of *some* sequence of trigonometric polynomials.

Useful fact. If $f \in L^2(\mathbf{R})$ or if $f \in L^2([0, 1])$, then for every $\epsilon > 0$, there exists an $h \in L^2$ such that h is uniformly continuous and $||f - h||_{L^2} < \epsilon$. (You will prove this fact in first-quarter analysis.)

If f is continuous, let

$$F_k(x) = \int_0^1 f(y)Q_k(x - y)dy$$

where $Q_k(x) = c_k[1 + \cos(2\pi x)]^k = c_k[1 + \frac{1}{2}U_1(x) + \frac{1}{2}U_{-1}(x)]^k$, c_k chosen so $\int_0^1 Q_k = 1$.

Then we can write

$$Q_k(x-y) = \sum_{|n|\le k, |m|\le k} C_{k,n,m} U_n(x) U_m(y)$$

for some constants $C_{k,n,m}$. So $F_N(x)$ is a trigonometric polynomial.

But as $k \to \infty$, Q_k becomes very small away from the integers, and so $F_N(x) \to f$ pointwise if f is continuous.

So the $\{U_n\}$ are a basis; if $f \in L^2$, then $f = \sum_n \hat{f}(n) U_n$, for some $\hat{f}(n) \in \mathbf{R}$.

Note that convergence is in L^2, and in L^2 only. Fourier series in general do not converge pointwise. (They do converge pointwise if f is, for example, differentiable.)

We can write down a formula for the $\hat{f}(n)$:

$$\hat{f}(n) = \langle f, U_n \rangle = \int_0^1 e^{-2\pi i n x} f(x) dx.$$

Parseval's Inequality: Since $\{U_n\}$ is a basis,

$$\sum_n |\hat{f}(n)|^2 = \left\langle \sum_n \hat{f}(n) U_n, \sum_m \hat{f}(m) U_m \right\rangle = \langle f, f \rangle = ||f||_{L^2}^2.$$

In particular, $\hat{f}(n) \to 0$ as $n \to \infty$ for any $f \in L^2$.

Note that we may define $\hat{f}(n)$ for $f \in L^1([0,1])$, via the above integral. In this case, we still have that $\hat{f}(n) \to 0$ as $n \to \infty$. (This is the Riemann-Lebesgue Lemma.)

2 Fourier Transform

We now move on to functions defined on all of $\mathbf{R}$, rather than just $[0,1]$. If $f \in L^1(\mathbf{R})$, we define the Fourier transform $\hat{f}$ by

$$\hat{f}(\xi) = \int e^{-2\pi i x \xi} f(x)\, dx.$$

(Unless otherwise indicated, all integrals in this section are over the real number line $\mathbf{R}$.)

The Fourier transform has many nice properties. Assume that $f, g \in L^1$. Then:

- If $h(x) = f \star g(x)$, then

$$\begin{aligned} \hat{h}(\xi) &= \int e^{-2\pi i x \xi} \int f(y) g(x-y)\, dy\, dx \\ &= \left(\int e^{-2\pi i y \xi} f(y)\, dy \right) \left(\int e^{-2\pi i z \xi} g(z)\, dz \right) \\ &= \hat{f}(\xi) \hat{g}(\xi). \end{aligned}$$

  (Here $\star$ denotes convolution, that is, $f \star g(x) = \int f(y) g(x-y)\, dy$.)

- If $h(x) = f'(x)$, then $\hat{h}(\xi) = 2\pi i \xi \hat{f}(\xi)$.
- $\widehat{rf}(\xi) = r\hat{f}(\xi)$, if $r \in \mathbf{R}$; $\widehat{f+g}(\xi) = \hat{f}(\xi) + \hat{g}(\xi)$, and so the Fourier transform is a linear operator.
- If $h(x) = f(x-\alpha)$, then $\hat{h}(\xi) = \hat{f}(\xi)e^{2\pi i \alpha \xi}$.
- If $h(x) = \frac{1}{r} f\left(\frac{x}{r}\right)$, $r > 0$, then $\hat{h}(\xi) = \hat{f}(rx)$.
- If $f \in L^1$, $\hat{f}$ is continuous.[1]
- If $\phi(x) = e^{-\pi x^2}$, then $\hat{\phi} = \phi$.

That last example allows us to prove the Fourier inversion formula.

Theorem 1 *If $g \in L^1$ is continuous at $x \in \mathbf{R}$, and if either*

- *$\hat{g}$ is also in L^1, or*
- *$\hat{g} \geq 0$ everywhere and $x = 0$,*

then

$$g(x) = \int e^{2\pi i x \xi} \hat{g}(\xi)\, d\xi.$$

Proof Let $\phi(x) = e^{-\pi x^2}$, $\phi_r(x) = \frac{1}{r}\phi\left(\frac{x}{r}\right)$. Note that $\int \phi_r = 1$ for all $r > 0$.

Then if g is continuous, $g(x) = \lim_{r \to 0} g \star \phi_r(x)$.

So:

$$\begin{aligned} \int e^{2\pi i x \xi} \hat{g}(\xi)\, d\xi &= \int \lim_{r \to 0} e^{2\pi i x \xi} \hat{g}(\xi) \phi(r\xi)\, d\xi \\ &= \lim_{r \to 0} \int e^{2\pi i x \xi} \hat{g}(\xi) \phi(r\xi)\, d\xi. \end{aligned}$$

This is where we use our conditions on $\hat{g}$. Switching limits with integrals is an interesting subject you will look at in first-quarter analysis.

Now,

$$\begin{aligned} \int e^{2\pi i x \xi} \phi(r\xi) \hat{g}(\xi)\, d\xi &= \int e^{2\pi i x \xi} \phi(r\xi) \int e^{-2\pi i y \xi} g(y)\, dy\, d\xi \\ &= \int g(y) \int \phi(r\xi) e^{-2\pi i (y-x)\xi}\, d\xi\, dy \\ &= \int g(y) \phi_r(x-y)\, dy \end{aligned}$$

So

$$g(x) = \lim_{r \to 0} \int g(y) \phi_r(x-y)\, dy = \int e^{2\pi i x \xi} \hat{g}(\xi)\, d\xi.$$

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[1] We will later define $\hat{f}$ for $f \in L^2$ as well as L^1. This result will *not* hold there.

In the section on Fourier series, it was the L^2 theory that was interesting. Unfortunately, we can only define the Fourier transform for $f \in L^1$. So now we look at functions in $L^1 \cap L^2$.

We have a very useful and interesting result:

Theorem 2 *(Plancherel's Theorem) If $f \in L^1 \cap L^2$, then $\hat{f} \in L^2$ as well, with*

$$||f||_{L^2} = ||\hat{f}||_{L^2}.$$

Proof Let $\tilde{f}(x) = \overline{f(-x)}$, and $g(x) = f \star \tilde{f}(x)$.

Now,

$$g(0) = \int f(x)\overline{f(-(0-x))}\,dx = ||f||_{L^2}^2$$

and

$$\hat{g}(x) = \hat{f}(\xi)\widehat{\tilde{f}}(\xi) = |\hat{f}(\xi)|^2,$$

so if we could apply our previous theorem (with $x = 0$), we would be done.

We need only show that $g \in L^1$ and g continuous. But

$$\begin{aligned} \int |g(x)|dx &= \int \left| \int f(y)\overline{f(y-x)}\,dy \right| dx \\ &\leq \int |f(y)| \int |f(y-x)|\,dx\,dy \leq ||f||_{L^1}^2 \end{aligned}$$

and so $g \in L^1$.

Recall our useful fact: if $f \in L^2$, then for every $\epsilon > 0$, there is some $h \in L^2$ such that h is uniformly continuous and $||f - h||_{L^2} < \epsilon$. Let δ be such that $|h(x+\delta) - h(x)| < \epsilon$ for all x.

So

$$\begin{aligned} |g(x+\delta) - g(x)| &= \left| \int \overline{f(y)}[f(y-x-\delta) - f(y-x)]\,dy \right| \\ &\leq \int |f(y)||h(y-x-\delta) - h(y-x)|\,dy \\ &\quad + \int |f(y)||f(y-x-\delta) - h(y-x-\delta)|\,dy \\ &\quad + \int |f(y)||f(y-x) - h(y-x)|\,dy \\ &\leq \epsilon||f||_{L^1} + 2\epsilon||f||_{L^2} \end{aligned}$$

and so g is continuous. Thus we are done. ∎

We can use these to extend the Fourier transform (and its inverse) to all of L^2.

If $f \in L^2$, there is some $\{f_n\} \subset L^1 \cap L^2$ such that $f_n \to f$ in L^2. For example, let $f_n(x) = f(x)$ if $|x| < n$ and 0 otherwise; clearly $f_n \to f$ in L^2, and since $|f| \leq \max(1, |f|^2)$,

$$\int |f_n(x)|dx \leq \int_{-n}^{n} 1\, dx + \int |f|^2\, dx \leq 2n + ||f||_{L^2}^2$$

and so $f_n \in L^1$.

Then by Plancherel's theorem, $\{\hat{f_n}\}$ is a Cauchy sequence in L^2, and since L^2 is a complete metric space, $\lim_{n\to\infty} \hat{f_n}$ exists in L^2-norm, and so we can define $\hat{f} = \lim_{n\to\infty} \hat{f_n}$.

Note that this means the Fourier transform of an L^2 function is an L^2 function. If $f \notin L^1$, then $\hat{f}$ may not be continuous, and its value at a given point is completely arbitrary.