WOMP 2006: HOMOLOGICAL ALGEBRA
PART 2

KATE PONTO

This is quick introduction to the derived functors Tor and Ext. I will mostly discuss the case for abelian groups mentioning changes necessary for modules over commutative rings other than $\mathbb{Z}$.

1. Ext

If A and B are abelian groups, $Hom(A, B)$, the set of homomorphisms from A to B, is also an abelian group. We define $\phi + \psi$ by defining it elementwise, $(\phi + \psi)(a) = \phi(a) + \psi(a)$. The additive identity is the map that takes all elements of A to the additive identity of B.

Let B be an abelian group and $f : A \to A'$ a homomorphism of abelian groups. Then there is a homomorphism

$$f^* : Hom(A', B) \to Hom(A, B)$$

defined by $f^*(\phi) = \phi \circ f$. This makes $Hom(-, B)$ a contravariant functor. If $g : B \to B'$ is a homomorphism

$$g_* : Hom(A, B) \to Hom(A, B')$$

defined by $g_*(\psi) = g \circ \psi$. So $Hom(A, -)$ a covariant functor.

Lemma 1. *If* $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is exact then*

$$0 \to Hom(C, \mathbb{Z}) \xrightarrow{g^*} Hom(B, \mathbb{Z}) \xrightarrow{f^*} Hom(A, \mathbb{Z})$$

is also exact.

Proof. We must show that $Hom(C, \mathbb{Z}) \to Hom(B, \mathbb{Z})$ is injective and that the image of $Hom(C, \mathbb{Z}) \to Hom(B, \mathbb{Z})$ is the kernel of the map $Hom(B, \mathbb{Z}) \to Hom(A, \mathbb{Z})$.

Let $\phi \in Hom(C, \mathbb{Z})$ such that $g^*(\phi) = 0$. Then $\phi \circ g(b) = 0$ for all $b \in B$. However, g is surjective so $\phi(c) = 0$ for all $c \in C$, and $\phi = 0$.

Since $g \circ f$ is zero $f^* \circ g^*$ is also zero, and $Im(g^*) \subset Ker(f^*)$. To show the other inclusion, let $\psi \in Hom(B, \mathbb{Z})$ such that $f^*\psi = 0$. Then $\psi(f(a)) = 0$ for all $a \in A$, and we can define a homomorphism $\bar{\psi} : C \to \mathbb{Z}$ by $\bar{\psi}(c) = \psi(b)$ for some $b \in B$ with $g(b) = c$. This is well defined since for two elements b and b' such that $g(b) = g(b')$ there is an $a \in A$ with $b = b' + a$. Then $\psi(b) = \psi(b' + f(a)) = \psi(b')$. $\square$

This means that the functor $Hom(-, \mathbb{Z})$ is *left exact.* Functors that preserve exact sequences are called *exact functors.*

Consider the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n \to 0.$$

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The functor $Hom(-,\mathbb{Z})$ applied to this short exact sequence is

$$0 \leftarrow \mathbb{Z} \overset{n}{\leftarrow} \mathbb{Z} \leftarrow 0 \leftarrow 0$$

which is not exact. The functor $Hom(-,\mathbb{Z})$ is not exact.

Definition 2. A *free resolution* of an abelian group B is a chain complex

$$\ldots \to F_2 \to F_1 \to F_0 \to 0$$

of free groups with a map $F_0 \to B$ such that

$$\ldots \to F_2 \to F_1 \to F_0 \to B \to 0$$

is exact.

Remark. If $0 \to F' \to F \to F'' \to 0$ is a short exact sequence of free abelian groups then

$$0 \to Hom(F'', A) \to Hom(F, A) \to Hom(F', A) \to 0$$

is also a short exact sequence for any abelian group A. So on free abelian groups $Hom(-, A)$ is an exact functor.

A free resolution is a way of replacing a (possibly) very complicated abelian group B with much simpler groups where $Hom(-, A)$ is an exact functor.

For a chain complex C, let $H^n(C; A)$ be the (co)homology of the chain complex

$$\ldots \to Hom(C_{i-1}, A) \to Hom(C_i; A) \to Hom(C_{i+1}; A) \to \ldots$$

Lemma 3. *Given free resolutions F of B and F' of B' a homomorphism $\alpha : B \to B'$ can be extended to a chain map from F to F'. This chain map is unique up to chain homotopy.*

For any two free resolutions F and F' of B there are canonical isomorphisms

$$H^n(F; A) \cong H^n(F'; A)$$

for all n.

Definition 4. For abelian groups A and C

$$Ext(C, A) := H^1(F; A)$$

where F is any free resolution of C.

By Lemma 3 $Ext(A, C)$ is well defined and $Ext(-, A)$ a contravariant functor from the category of abelian groups to the category of abelian groups. Similarly, $Ext(C, -)$ is a covariant functor from abelian groups to abelian groups.

Theorem 5. *If $0 \to A \to B \to C \to 0$ is a short exact sequence of abelian groups then*

$$\begin{aligned} 0 &\longrightarrow Hom(C, D) \longrightarrow Hom(B, D) \longrightarrow Hom(A, D) \\ &\longrightarrow Ext(C, D) \longrightarrow Ext(B, D) \longrightarrow Ext(A, D) \longrightarrow 0 \end{aligned}$$

and

$$\begin{aligned} 0 &\longrightarrow Hom(D, A) \longrightarrow Hom(D, B) \longrightarrow Hom(D, C) \\ &\longrightarrow Ext(D, A) \longrightarrow Ext(D, B) \longrightarrow Ext(D, C) \longrightarrow 0 \end{aligned}$$

are exact sequences of abelian groups.

Proof. Any abelian group B has a free resolution with only two terms. The group F_0 has a generator for each generator of B and F_1 has a generator for each relation of B.

The short exact sequence

$$0 \to A \to B \to C \to 0$$

can be extended to a short exact sequence of chain complexes

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1^A & \longrightarrow & F_1^B & \longrightarrow & F_1^C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_0^A & \longrightarrow & F_0^B & \longrightarrow & F_0^C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Applying $Hom(-, D)$ to the F_i we get a short exact sequence of chain complexes, and this short exact sequence gives a long exact sequence in homology

$$\begin{array}{l}
0 \longrightarrow H^0(F^C, D) \longrightarrow H^0(F^B, D) \longrightarrow H^0(F^A, D) \\
\longrightarrow H^1(F^C, D) \longrightarrow H^1(F^B, D) \longrightarrow H^1(F^A, D) \longrightarrow 0
\end{array}$$

The group $H^0(F^C, D)$ is isomorphic to $Hom(C, D)$ and similarly for A and B. (Try it!)

The other exact sequence is similar. □

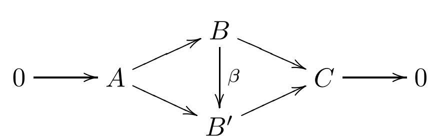
Lemma 6 (Rules for computing Ext for finitely generated abelian groups)**.** *Let B be an abelian group and A be a* finitely generated *abelian group. Then*

(1) $Ext(A \oplus A', B) \cong Ext(A, B) \oplus Ext(A', B)$
(2) $Ext(A, B) = 0$ *if A is free.*
(3) $Ext(\mathbb{Z}/n\mathbb{Z}; B) \cong B/nB$

Definition 7. Let A and C be abelian groups. An *extension* of A by C is a short exact sequence of abelian groups

$$E : 0 \to A \to B \to C \to 0.$$

We say that two extensions E and E' are equivalent if there is an isomorphism $\beta : B \to B'$ which makes the diagram



commute. We can define an addition on the set of equivalence classes of extensions and this makes this set an abelian group.[3]

Theorem 8. [3] *The abelian group of equivalence classes of extensions of A by C is isomorphic to $Ext(C, A)$.*

Remark. Let R be a commutative ring. For two R-modules M and N, let $Hom_R(M,N)$ be the R-module homomorphisms from M to N. Then $Hom_R(M,-)$ and $Hom_R(-,N)$ are both left exact functors. A short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of R-modules gives two long exact sequences

$$0 \longrightarrow Hom_R(M'',N) \longrightarrow Hom_R(M,N) \longrightarrow Hom_R(M',N)$$
$$\longrightarrow Ext^1_R(M'',N) \longrightarrow Ext^1_R(M,N) \longrightarrow Ext^1_R(M',N) \longrightarrow \cdots$$

and

$$0 \longrightarrow Hom_R(N,M') \longrightarrow Hom_R(N,M) \longrightarrow Hom_R(N,M'')$$
$$\longrightarrow Ext^1_R(N,M') \longrightarrow Ext^1_R(N,M) \longrightarrow Ext^1_R(N,M'') \longrightarrow \cdots$$

Where Ext^1_R is the first in a sequence of functors defined by

$$Ext^n_R(M'',N) := H^n(F;N)$$

with F a *projective* resolution of M''.

2. Tor

Let A and B be abelian groups. Then $A \otimes B$, the tensor product of A and B, is the abelian group generated by $a \otimes b$ for $a \in A$ and $b \in B$ subject to the relations that

(1) $(a+a') \otimes b = a \otimes b + a' \otimes b$ and $a \otimes (b+b') = a \otimes b + a \otimes b'$
(2) $an \otimes b = a \otimes nb$ for any $n \in \mathbb{Z}$.

For abelian groups A and B, $A \otimes -$ and $- \otimes B$ are covariant functors.

Lemma 9. *For each short exact sequence of abelian groups*

$$0 \to A \to B \to C \to 0$$

there is an exact sequence

$$D \otimes A \to D \otimes B \to D \otimes C \to 0$$

This means that the functor $D \otimes -$ is *right exact*.

Lemma 10. *For any two free resolutions F and F' of an abelian group A there are canonical isomorphisms $H_n(F \otimes B) \cong H_n(F' \otimes B)$ for all $n \in \mathbb{Z}$ and all abelian groups B.*

If C_* is a chain complex of abelian groups and B is an abelian group $C_* \otimes B$ is the chain complex

$$\ldots \to C_{i+1} \otimes B \overset{\partial \otimes 1}{\to} C_i \otimes B \overset{\partial \otimes 1}{\to} C_{i-1} \otimes B \to \ldots$$

Definition 11. For abelian groups A and B, $Tor(A,B)$ is defined to be $H_1(F \otimes B)$ for some free resolution F of A.

Lemma 12. *For each short exact sequence of abelian groups*

$$0 \to A \to B \to C \to 0$$

there is an exact sequence

$$0 \to Tor(D,A) \to Tor(D,B) \to Tor(D,C) \to D \otimes A \to D \otimes B \to D \otimes C \to 0$$

Lemma 13 (Rules for computing Tor)**.** *Let A and B be abelian groups.*

(1) $Tor(A,B) = Tor(B,A)$
(2) $Tor(\oplus_i A_i, B) \cong \oplus_i Tor(A_i, B)$
(3) $Tor(A,B) = 0$ *if A or B is free, or more generally torsion free.*
(4) $Tor(A,B) \cong Tor(T(A), B)$ *where $T(A)$ is the torsion subgroup of A.*
(5) $Tor(\mathbb{Z}/n\mathbb{Z}, A) = Ker(n : A \to A)$.

Remark. Let R be a commutative ring and let M and N be R-modules. The tensor product of M and N over R, $M \otimes_R N$, is the free R-module generated by $m \otimes n$ $m \in M$, $n \in N$ subject to the relations

- $(m + m') \otimes n = m \otimes n + m' \otimes n$
- $m \otimes (n + n') = m \otimes n + m \otimes n'$
- $mr \otimes n = m \otimes rn$ for $r \in R$.

Then $- \otimes_R N$ and $M \otimes_R -$ are right exact functors.

A short exact sequence of R-modules

$$0 \to M' \to M \to M'' \to 0$$

gives a long exact sequence

$$\ldots \to Tor_1^R(N, M') \to Tor_1^R(N, M) \to Tor_1^R(N, M'') \to N \otimes M' \to N \otimes M \to N \otimes M'' \to 0$$

where Tor_1^R is the first in a sequence of functors defined by

$$Tor_n^R(M; N) := H_n(F \otimes_R N)$$

with F a projective resolution of M.

References

[1] Hatcher, Allen. Algebraic topology. Cambridge University Press, Cambridge, 2002. Really an Algebraic Topology book, but includes basic definitions as they are needed.
[2] Hu, Sze-Tsen. Introduction to Homological Algebra. Holden-Day, Inc., San Francisco, 1968. Older reference, but starts at the beginning and includes lots of details.
[3] Mac Lane, Saunders. Homology. Reprint of the 1975 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Also an older reference, less approachable than Hu, but more approachable than Weibel.
[4] Weibel, Charles A. An introduction to homological algebra. Cambridge Studies in Advanced Mathematics, 38. Cambridge University Press, Cambridge, 1994. A useful book but probably not the place to start.