WOMP CATEGORIES PART II, 2007

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Abstract. These notes outline the second part of the WOMP 2007 talk on category theory. We'll explore the themes of naturality and universality. In particular, universal properties provide important examples of adjoint pairs of functors. An overwhelmingly useful and widely pervasive example of universality gives us the notion of products and coproducts, and their generalization, limits and colimits.

1. Naturality

When doing mathematics, we often speak of some construction being "natural". This notion was actually used for quite a while before it was formalized, and the desire to do so was one of the leading motivations for the creation of the language of category theory.

One very concrete example is the determinant. Taking the determinant of a matrix doesn't depend on the particular choice of ring that your matrix entries lie in—the construction is "natural". More formally, consider the functors $\mathrm{GL}(n,-)$ and $(-)^\times\colon \mathit{CommRng} \to \mathit{Grp}$ from commutative rings to groups. The first takes a ring R to the matrix group $\mathrm{GL}(n,R)$ and the second takes a ring R to its group of multiplicative units $R^\times$. The naturality of the determinant then states that for a ring homomorphism $\varphi\colon R \to S$, the following diagram commutes, i.e. that the determinant is defined the same way regardless of the base ring:

$$\begin{tikzcd} \mathrm{GL}(n,R) \arrow[r, "\det"] \arrow[d] & R^\times \arrow[d] \\ \mathrm{GL}(n,S) \arrow[r, "\det"'] & S^\times \end{tikzcd}$$

Naturality is made precise in category theory with the notion of natural transformation:

Definition 1.1. Let $F, G\colon \mathscr{C} \to \mathscr{D}$ be two functors from the category $\mathscr{C}$ to the category $\mathscr{D}$. A *natural transformation* $\tau\colon F \to G$ consists of a collection of morphisms (of $\mathscr{D}$) $\tau_c\colon F(c) \to G(c)$, one for each object c of $\mathscr{C}$, such that for every morphism $f\colon c \to d$ in $\mathscr{C}$, the following diagram commutes:

$$\begin{tikzcd} F(c) \arrow[r, "\tau_c"] \arrow[d, "F(f)"'] & G(c) \arrow[d, "G(f)"] \\ F(d) \arrow[r, "\tau_d"'] & G(d) \end{tikzcd}$$

The morphisms τ_c are called the *components* of the natural transformation τ.

Another aspect of natural transformations is that they are "morphisms of functors", since they provide a way of functorially comparing functors. The existence of identity natural transformations and composition of natural transformations should be fairly clear, so we can consider the category whose *objects* are functors $F\colon \mathscr{C} \to \mathscr{D}$ (for fixed categories $\mathscr{C}$ and $\mathscr{D}$) and whose morphisms are natural transformations between functors. This category is often denoted $\mathscr{D}^{\mathscr{C}}$.

2. Universality

We already saw one example of universality in the first talk: the abelianization $G/[G,G]$ of a group G is universal among maps from G to abelian groups. Another example is given by quotient groups. Suppose that H is a normal subgroup of a group G. The quotient group G/H is described by the universal property that for any group X and group homomorphism $\varphi\colon G \to X$ such that H is sent to the identity in X under φ, the map φ factors uniquely through the quotient map $\pi\colon G \to G/H$, as described by the following commutative diagram:

$$\begin{tikzcd}
G \arrow[rr, "\varphi"] \arrow[dr, "\pi"] & & X \\
& G/H \arrow[ur, dashed] & \\
H \arrow[uu, "\triangledown"] \arrow[rr] & & \{e\} \arrow[uu] \arrow[ul]
\end{tikzcd}$$

As with any object defined by a universal property, one must construct the quotient group explicitly, say via cosets and such, in order to show that it exists. Also, once an object defined by a universal property is shown to exist, it is automatically unique up to isomorphism, via standard diagram-theoretic arguments.

One nice thing about this understanding of quotient groups is that it naturally leads us to define quotient objects in other categories, even ones where we might not of thought to define quotient objects originally. Given a closed subspace A of a topological space X, the quotient space X/A is defined by the universal property described by the following diagram:

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr, "\pi"] & & Z \\
& X/A \arrow[ur, dashed] & \\
A \arrow[uu] \arrow[rr] & & * \arrow[uu] \arrow[ul]
\end{tikzcd}$$

Although quotients of groups and quotients of spaces are "the same" categorically, their descriptions within their respective categories are quite different. This is a common theme in category theory.

Universality can be encoded into adjoint functors. Let $SubGrp$ be the category whose objects are pairs (G,H), where H is a normal subgroup of the group G, and whose morphisms $(G,H) \to (G',H')$ are group homomorphisms $\varphi\colon G \to G'$ such that $\varphi(H) \subset H'$. Taking quotient groups is then a functor $SubGrp \to Grp$. There

is a forgetful functor $Grp \to SubGrp$ which sends a group X to the pair $(X, \{e\})$, where $\{e\}$ is the trivial subgroup. The universal property for quotient groups is exactly the statement that these functors are adjoint to each other:

$$Grp(G/H, X) \cong SubGrp((G, H), (X, \{e\})).$$

For completeness, I feel compelled to include the formal definition of adjoint functors. Hopefully, this slightly terse and perhaps foreboding definition will be clear after having almost always already seen the concept illustrated thus far.

Definition 2.1. Let $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{C}$ be functors (note that they go in opposite directions). We say that F and G form a pair of *adjoint functors*, or more specifically that G is the *right adjoint* of F and that F is the *left adjoint* of G, if for each pair of objects c of $\mathscr{C}$ and d of $\mathscr{D}$ we have the following isomorphism of sets of morphisms:

$$\mathscr{D}(F(c), d) \cong \mathscr{C}(c, G(d)).$$

Furthermore, we insist that these bijective correspondences be natural[1] in the entries c and d.

3. Products and Coproducts

Consider the cartesian product of sets. Also consider the cartesian product of groups, vector spaces, modules, and topological spaces. In fact, there is a cartesian product of categories: given categories $\mathscr{C}$ and $\mathscr{D}$, their product $\mathscr{C} \times \mathscr{D}$ is the category with objects pairs (c, d) of one object from $\mathscr{C}$, one from $\mathscr{D}$, and morphisms $(c, d) \to (c', d')$ pairs of morphisms $c \to c'$ in $\mathscr{C}$ and $d \to d'$ in $\mathscr{D}$. These examples are all defined in the same way, in some sense, and they are all examples of the (categorical) product.

Definition 3.1. Let a and b be objects of a category $\mathscr{C}$. Their *product* is the object $a \prod b$ (or $a \times b$) of $\mathscr{C}$, which comes with morphisms $p\colon a \prod b \to a$ and $q\colon a \prod b \to b$, defined by the following universal property: given any object c of $\mathscr{C}$ and morphisms $\alpha\colon c \to a$ and $\beta\colon c \to b$, there exists a unique morphism $\alpha \prod \beta\colon c \to a \prod b$ making the following diagram commute:

$$\begin{tikzcd} a & a \prod b \arrow[l, "p"'] \arrow[r, "q"] & b \\ & c \arrow[ul, "\alpha"] \arrow[u, dashed] \arrow[ur, "\beta"'] & \end{tikzcd}$$

Whenever we make a definition in category theory, there is always a dual definition (or codefinition) where we turn all the arrows around. Hence the coproduct:

Definition 3.2. Let a and b be objects of a category $\mathscr{C}$. Their *coproduct* is the object $a \coprod b$ of $\mathscr{C}$, which comes with morphisms $i\colon a \to a \coprod b$ and $j\colon b \to a \coprod b$, defined by the following universal property: given any object c of $\mathscr{C}$ and morphisms

[1] To be precise, this means that for a fixed object c of $\mathscr{C}$, there is a natural transformation between the represented functors $\mathscr{D}(F(c), -)$ and $\mathscr{C}(c, G(-))$, and similarly for functors $\mathscr{D}(F(-), d)$ and $\mathscr{C}(-, G(d))$.

$\alpha\colon a \to c$ and $\beta\colon b \to c$, there exists a unique morphism $\alpha \coprod \beta\colon a \coprod b \to c$ making the following diagram commute:

$$\begin{tikzcd} a \arrow[r, "i"] \arrow[dr, "\alpha"'] & a \coprod b \arrow[d, dashed] & b \arrow[l, "j"'] \arrow[dl, "\beta"] \\ & c & \end{tikzcd}$$

Examples of coproducts include:

- In *Sets*: the disjoint union $X \coprod Y$,
- In *AbGrp*: the direct sum $A \oplus B$,
- In *Grp*: the free product $G * H$,
- In *Top*: the disjoint union $X \coprod Y$,
- In *CommRng*: the tensor product $R \otimes_{\mathbb{Z}} S$,
- In Top_* (based topological spaces): the wedge sum $X \vee Y$ (the join of two spaces by identifying their basepoints).

As an instantiation of the mantra "adjoints arise everywhere", let's see how the universal property for (co)products can be encoded into an adjunction. Recall the product category $\mathscr{C} \times \mathscr{C}$. Taking the product of a pair of objects in $\mathscr{C}$ defines a functor $\mathscr{C} \times \mathscr{C} \to \mathscr{C}$. In the other direction, the *diagonal functor* $\Delta\colon \mathscr{C} \to \mathscr{C} \times \mathscr{C}$ takes an object c to the pair (c, c). The universal property for the product $a \times b$ of two objects can then be written as an adjunction between these two functors:

$$\mathscr{C} \times \mathscr{C}(\Delta c, (a, b)) \cong \mathscr{C}(c, a \times b).$$

The coproduct gives an adjunction between the same functors, but with the left and right sides of the pair switched:

$$\mathscr{C}(a \amalg b, c) \cong \mathscr{C} \times \mathscr{C}((a, b), \Delta c).$$