LIE GROUPS AND LIE ALGEBRAS
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Abstract. The theory of Lie groups is fundamentally different from the theory of discrete groups. We'll go over some common examples and go through the details of some proofs to get the feel for the techniques used. We'll also discuss the Lie algebra associated to a Lie group and the deeply incestuous relations between the two.

1. Lie groups

Definition. *A **Lie group** is a group G that is also a smooth manifold, such that the multiplication $G \times G \to G$, $(g,h) \mapsto gh$, and the inversion $G \to G$, $g \mapsto g^{-1}$, are smooth maps.*

Some examples of Lie groups:

- $\mathbb{R}$ is a Lie group under addition. Similarly, $\mathbb{R}^n$ is a Lie group under addition.
- S^1 is a Lie group – you can think of it as the unit circle in $\mathbb{C}$ under multiplication. It is also isomorphic to the quotient group $\mathbb{R}/\mathbb{Z}$.
- The torus T^2 is a Lie group. It is isomorphic to $S^1 \times S^1$, or to $\mathbb{R}^2/\mathbb{Z}^2$.

Fact. *If G is a Lie group, then the tangent bundle TG is always trivializable.*

Proof. In fact, any vector anywhere on G determines a unique "left-invariant vector field", as follows. Let v be a tangent vector at the identity e, so v is an element of the tangent space T_eG. Left multiplication by any element g gives a diffeomorphism L_g from G to itself; since $L_g(e) = ge = g$, L_g induces a map on the tangent space $dL_g \colon T_eG \to T_gG$. Define a vector field V as follows: at the point g of the manifold G, $V(g)$ is the tangent vector $dL_g(v) \in T_gG$. The smoothness of the group operations implies that V is a smooth vector field (with some work).

Now choose a basis $v_1, \ldots, v_n$ for the tangent space T_eG. Define vector fields $V_1, \ldots, V_n$ as above. To see that $V_1, \ldots, V_n$ give a trivialization of TG, we need to check that at each point g, the vectors $V_1(g), \ldots, V_n(g)$ are a basis for T_gG. Since L_g is a diffeomorphism, dL_g is an isomorphism. Thus since $v_1, \ldots, v_n$ form a basis of T_eG, their images $dL_g(v_1), \ldots, dL_g(v_n)$ for a basis of T_gG. But this is exactly $V_1(g), \ldots, V_n(g)$. □

Building new Lie groups. If G is a Lie group and H is a **closed** subgroup of G, then H is a Lie group. If H is a **closed** normal subgroup of G, then G/H is a Lie group. If G and H are Lie groups, then so is $G \times H$. But the image of a Lie group under a homomorphism is not always a Lie group — consider the map $\mathbb{R}/\mathbb{Z} \to (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ given by $t \mapsto (t, \sqrt{2}t)$. (This is an immersion $S^1 \to T^2$ with irrational slope; the image is dense in T^2.)

Algebraic groups. Many of the most important Lie groups arise as the group of transformations preserving some algebraic structure on a vector space.

- $\mathrm{GL}_n \mathbb{R}$, the group of $n \times n$ invertible matrices over $\mathbb{R}$, is a Lie group. (The entries of AB and A^{-1} are polynomials in the entries of A and of B, and polynomials are smooth. This actually proves that any subgroup of $\mathrm{GL}_n \mathbb{R}$ which is a smooth manifold is a Lie group — we don't need to check smoothness of the operations again.)
- $\mathrm{GL}_n^+ \mathbb{R}$, the group of matrices with positive determinant. This is the connected component of the identity in $\mathrm{GL}_n \mathbb{R}$. (Preserves the orientation of $\mathbb{R}^n$.)
- $\mathrm{SL}_n \mathbb{R}$, the group of matrices with determinant 1. (Preserves the volume element on $\mathbb{R}^n$ and the orientation.)
- $\mathrm{O}_n \mathbb{R}$, the group of matrices satisfying $AA^\top = I$. (Preserves the standard inner product on $\mathbb{R}^n$.)

There's no reason to only talk about the "standard" algebraic objects; for example, if B is *any* inner product on $\mathbb{R}^n$, then $\mathrm{O}(B)$ — the subgroup of $\mathrm{GL}_n \mathbb{R}$ preserving the inner product B — is a Lie group. All these matrix groups can be viewed as subspaces of $\mathrm{Mat}_n \mathbb{R}$, which can be identified with $\mathbb{R}^{n^2}$, and they inherit the subspace topology from $\mathbb{R}^{n^2}$.

Fact. *If G is a Lie group, then G_0, the connected component of the identity, is a closed normal subgroup.*

Proof. A connected component is by definition a subset that is both open and closed; we show that G_0 is normal. Let g be in G_0, and let h be an arbitrary element of G. We want to show that hgh^{-1} is in G_0. Since $g \in G_0$, there exists a path p from the identity e to g. Because the group multiplication is continuous, hph^{-1} is still a path. One end is at hgh^{-1}, while the other is at $heh^{-1} = e$. Thus hph^{-1} is a path from the identity to hgh^{-1}, and thus $hgh^{-1} \in G_0$. □

Isometry groups. If M is a Riemannian manifold, then $\mathrm{Isom}(M)$, the group of isometries of M, is a Lie group. (This is a nontrivial theorem.)

Definition (aside). A Riemannian manifold M is *homogeneous* if $\mathrm{Isom}(M)$ acts transitively on M; that is, if p and q are two arbitrary points in M, there exists an isometry of M taking p onto q.

Fact. *Any homogeneous Riemannian manifold M is the quotient of a Lie group. Specifically, if $x \in M$, let $\mathrm{Stab}(x)$ be the subgroup of $\mathrm{Isom}(M)$ that fixes the point x. Then M is diffeomorphic to $\mathrm{Isom}(M)/\mathrm{Stab}(x)$ for any $x \in M$.*

Proof. This is actually elementary group theory — if G acts transitively on a set S, then S can be identified with $G/\mathrm{Stab}(s)$ for any $s \in S$. You can show that in our case, the identification $\mathrm{Isom}(M)/\mathrm{Stab}(x) \to M$ is actually a diffeomorphism. □

Fact. *If G is a Lie group with a left-invariant Riemannian metric, then G can be viewed as a subgroup of $\mathrm{Isom}(G)$.*

Proof. This is almost tautological; by definition, a left-invariant metric is a metric such that left-multiplication by g (we called it L_g) is an isometry. Since $L_g \circ L_h = L_{gh}$, the map $g \mapsto L_g$ is an isomorphism from G to a subgroup of $\mathrm{Isom}(G)$. □

2. Lie algebras

Definition. (ignore details if you feel like it) A **Lie algebra** L is a vector space (for us, always over $\mathbb{R}$) with a bilinear product $L \times L \mapsto L$, written $(x, y) \mapsto [x, y]$, satisfying certain axioms. We refer to this operation as the "bracket" on L. This operation is neither commutative nor even associative (see below). The bracket in a Lie algebra must satisfy the following properties.

- **Skew-commutativity**: for all $x, y \in L$, $[x, y] = -[y, x]$.
- **Jacobi identity**: Let D_X be the linear map defined by $D_x y = [x, y]$. Then for all $x, y, z \in L$, the Jacobi identity must hold:

  $$D_x[y, z] = [D_x y, z] + [y, D_x z]$$

  That is:

  $$[x, [y, z]] = [[x, y], z] + [y, [x, z]]$$

A homomorphism of Lie algebras $f\colon L \to L'$ is a linear map that preserves the bracket: $f([x, y]) = [f(x), f(y)]$.

Example. If A is an associative algebra (for example, you can think of $A = \mathrm{Mat}_n(R)$, the algebra of $n \times n$ matrices over R), then we can get a Lie algebra structure on A by defining the bracket by $[X, Y] = XY - YX$. This explains why we use the "commutator" notation, but it is important to remember that we cannot multiply two elements of a Lie algebra — we can only take their bracket. It is clear that this bracket is bilinear and skew-commutative, so we just need to check the Jacobi identity.

$$\begin{aligned}
[X,[Y,Z]] &\overset{?}{=} [[X,Y],Z] && + [Y,[X,Z]] \\
[X, YZ - ZY] &\overset{?}{=} [XY - YX, Z] && + [Y, XZ - ZX] \\
XYZ - XZY &\overset{\checkmark}{=} XYZ - YXZ && + YXZ - YZX \\
-YZX + ZYX &\quad\; - ZXY + ZYX && - XZY + ZXY
\end{aligned}$$

Example. Let $\mathcal{X}$ be the space of all vector fields on a given smooth manifold M. There is a natural bracket operator on $\mathcal{X}$ which I will not define; if V and W are vector fields, you can think of the vector field $[V, W]$ as somehow measuring the "failure of V and W to commute". One property of this bracket that will be useful soon: if V and W are left-invariant vector fields on a Lie group, then $[V, W]$ is left-invariant too.

The Lie algebra of a Lie group. Given a Lie group G, let $\mathfrak{g}$ be the space of all left-invariant vector fields. (We saw way back at the beginning that $\mathfrak{g}$ is isomorphic to T_eG.) I noted above that $V, W \in \mathfrak{g}$ implies $[V, W] \in \mathfrak{g}$ for the bracket on vector fields. It follows that $\mathfrak{g}$ is a sub-Lie algebra of $\mathcal{X}$; we call $\mathfrak{g}$ the Lie algebra associated to the Lie group G. Note that $\dim \mathfrak{g} = \dim T_eG = \dim G$. This definition is justified by the following two awesome facts:

Awesome Fact. If G and H are Lie groups, then any homomorphism $f\colon G \to H$ induces a linear map $df\colon T_eG \to T_eH$. If we identify T_eG with $\mathfrak{g}$ and T_eH with $\mathfrak{h}$, then $df\colon \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras. Moreover, if G is connected, then f is determined *uniquely* by its differential df.

Awesome Fact. If G and H are Lie groups, and G is both connected and simply connected, then *every* homomorphism $\mathfrak{g} \to \mathfrak{h}$ is the differential of some group homomorphism $G \to H$.

It is important to note that while a Lie group has a unique Lie algebra associated to it, there may be many different Lie groups all having isomorphic Lie algebras.

Examples of Lie algebras of certain Lie groups. The Lie algebra of the Lie group $\mathbb{R}^n$ is just the vector space $\mathbb{R}^n$ with trivial bracket: $[x, y] = 0$ for all x, y. Such a Lie algebra, where the bracket is always trivial, is called *abelian*, for obvious reasons.

Fact. *If G is connected, then G is an abelian Lie group if and only if $\mathfrak{g}$ is an abelian Lie algebra.*

For $G = \mathrm{GL}_n\,\mathbb{R}$ we have $\mathfrak{g} = \mathfrak{gl}_n\mathbb{R} \approx \mathrm{Mat}_n\,\mathbb{R}$, the Lie algebra of $n \times n$ matrices with bracket given by $[A, B] = AB - BA$. Every matrix group will be of this form: for example, $\mathfrak{sl}_n\mathbb{R}$ is the Lie algebra of $n \times n$ matrices with trace 0, with bracket again given by $[A, B] = AB - BA$. For completeness, $\mathfrak{so}_n\mathbb{R}$ is the Lie algebra of $n \times n$ matrices satisfying $A + A^\top = 0$, with bracket as above. (Fun fact: $\mathfrak{so}_3\mathbb{R}$ is isomorphic to the Lie algebra $\mathbb{R}^3$, where the bracket of v and w is given by the cross product: $[v, w] = v \times w$.) A sub-Lie algebra of $\mathrm{Mat}_n\,\mathbb{R}$ is called a matrix Lie algebra.

Note on simply connected Lie groups. You might wonder, in the second awesome fact above, whether it's a big deal that G must be simply connected. It's not so bad, and here's one reason why. If G is a Lie group with Lie algebra $\mathfrak{g}$, then its universal cover $\widetilde{G}$ is also a Lie group, and it turns out $\widetilde{G}$ has Lie algebra $\mathfrak{g}$ too. $\widetilde{G}$ is simply connected by definition, so our awesome fact applies to $\widetilde{G}$ and $\mathfrak{g}$. In fact, if G' is any other Lie group with Lie algebra $\mathfrak{g}$, then its universal cover is isomorphic to $\widetilde{G}$.

The matrix exponential. I defined Lie algebras abstractly, but I've only talked about Lie algebras that arise from Lie groups. You might wonder if I'm neglecting a huge class of Lie algebras; fortunately it turns out that every finite-dimensional Lie algebra is the Lie algebra of some Lie group. The matrix exponential map is one way to see this.

We define a map $e^X\colon \mathrm{Mat}_n\,\mathbb{R} \to \mathrm{GL}_n\,\mathbb{R}$ by

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

with convergence in the vector space $\mathrm{Mat}_n\,\mathbb{R}$. This series always converges, and in fact it converges to an invertible matrix. If A and B commute as matrices, then $e^{A+B} = e^A e^B$ (exercise!); thus the inverse of e^X is e^{-X}.

Say that L is a sub-Lie algebra of $\mathrm{Mat}_n\,\mathbb{R}$. Then the image of L under the exponential will be a subgroup G_L of $\mathrm{GL}_n\,\mathbb{R}$—and the Lie algebra of G_L is isomorphic to L. So every matrix Lie algebra comes from some matrix Lie group. Now we quote Ado's theorem, which says that every abstract Lie algebra is isomorphic to a matrix Lie algebra, and we're done.

Example. Consider the matrix Lie algebra L consisting of matrices of the form $\begin{bmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}$.

We want to find a Lie group whose Lie algebra is L, so we use the matrix exponential. But this is really easy, since the infinite series above actually has only finitely many terms (L consists of *nilpotent* matrices). If $A \in L$ is $\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}$, then $A^2 = \begin{bmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $A^3 = 0$. Thus

$$e^A = I + A + \frac{A^2}{2} = \begin{bmatrix} 1 & a & b + (ac/2) \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

It is clear that such matrices make up the entire group of upper-triangular matrices with 1's on the diagonal; this is called the Heisenberg group, and its Lie algebra is indeed L.