A VERBOSE INTRODUCTION TO MULTILINEAR ALGEBRA

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1. Résumé of Tensor Products

By **ring** we always mean a commutative ring with unity. Let R be a ring, RMod the category of modules over the ring R. If the reader is uncomfortable with the arid generality of arbitrary rings, he/she can assume that R is a field of characteristic 0 (unless explicitly stated otherwise), as this is the case for, say, the tangent space of a manifold. If the reader wants even greater arid generality, he/she can assume that we are working in the category of quasicoherent sheaves over a scheme.

Definition 1.0.1. Let M and N be R-modules. Then $M \otimes_R N$ (or just $M \otimes N$ when the ground ring R is understood) is the unique object in RMod that satisfies the following universal mapping property: There is a bilinear map $\tau : M \times N \to M \otimes N$ such that given any R-module S and any bilinear map $f : M \times N \to S$ there exists a unique R-module homomorphism $g : M \otimes N \to S$ such that $f = \tau \circ g$.

Remark 1.0.2. Bilinear maps are ubiquitous in mathematics. For just a couple of examples we have the following:

(1) Let V be a vector space over a field k. Then, *by definition*, a **bilinear form** on V is a k-bilinear map $V \times V \to k$ (for example, the standard dot product on k^n is a bilinear form). In particular, such a form is equivalent to giving a *linear* map $V \otimes V \to k$. I suppose you can think of tensor products as a way to linearize a not-quite-linear map.
(2) Let M be an R-module. By definition, this means that there is a bilinear map $R \times M \to M$ (such that $(rs)m = r(sm)$).

1.0.1. As is the case in any category, whenever an object is defined by a universal mapping property, it is unique up to unique isomorphism (this is a consequence of Yoneda's Lemma, which will be recalled in the sequel). Existence is proved as follows: we begin by taking the free module, F, on all symbols $m \otimes n$ for $m \in M$ and $n \in N$. We then quotient F by the submodule generated by elements of the form $((m_1+m_2)\otimes n - m_1 \otimes n - m_2 \otimes n)$, $(m \otimes (n_1+n_2) - m \otimes n_1 - m \otimes n_2)$, $(r(m \otimes n) - rm \otimes n)$ and $(rm \otimes n - m \otimes rn)$, for all $m, m_1, m_2 \in M, n, n_1, n_2 \in N, r \in R$.

Exercise 1.0.3. Check that the module constructed above satisfies the universal property stated in the definition.

By the construction, we see that $M \otimes N$ is generated by elements of the form $m \otimes n$. That is, every element of $M \otimes N$ is of the form

$$\Sigma_{i=1}^k m_i \otimes n_i,$$

with $m_i \in M$ and $n_i \in N$ for $1 \leq i \leq k$.

Remark 1.0.4. Let M and N be R-modules. It is tempting, but *not* necessarily true that if $m \otimes n = 0$ then either m or n is 0. For example, let $R = \mathbb{Z}$, $M = \mathbb{Q}$, and $N = \mathbb{Z}/p\mathbb{Z}$. Then, for example $1 \otimes 1 = 0$ since

$$\begin{aligned} 1 \otimes 1 &= \left(\frac{1}{p}\right) p \otimes 1 \\ &= \frac{1}{p} \otimes p \\ &= 0. \end{aligned}$$

In fact, by identical reasoning, $M \otimes N = 0$. More generally, if A is an abelian group, and $T \subset A$ is the torsion subgroup, then $\mathbb{Q} \otimes A \xrightarrow{\sim} \mathbb{Q} \otimes A/T$. We say that $\mathbb{Q}$ "kills torsion."

Remark 1.0.5. Most statements about tensor products of vector spaces carry through to tensor product of free modules (or even more generally, projective modules), but not modules in general, as we will see in the sequel.

1.1. **Basic properties of tensor products.**

Proposition 1.1.1. *Let M and N be free R-modules of rank m and n, respectively. Then, $M \otimes N$ is a free module of rank mn.*

Proof. Choose bases $\{e_i\}$ and $\{f_j\}$ for M and N. Then the elements $\{e_i \otimes f_j\}$ form a basis for $M \otimes N$. □

Proposition 1.1.2. *Tensor product is an associative, commutative, and unital operation. More precisely,*

(1) If M, N, and P are R-modules, then $(M \otimes N) \otimes P = M \otimes (N \otimes P)$.
(2) If M and N are R-modules, then $M \otimes N = N \otimes M$.
(3) The ring R considered as an R-module is a unit for $\otimes$, that is for any R-module M, $R \otimes M = M \otimes R = M$.

Exercise 1.1.3. Prove these statements. Note that item (3) is proved in the sequel (Proposition 1.1.10). [Hint: These can be proven directly, but a sexier approach is as follows: use the universal property to get maps both ways, then use uniqueness to show the compositions are the identity.]

Remark 1.1.4. Note that in the above proposition we use an equals sign instead of an isomorphism symbol. This is because the isomorphism is "natural" in the sense that it comes straight from the universal property–no choices were made. (In fancy language, there is a natural equivalence of the two corresponding functors on RMod.)

An equals sign is often used in this situation, and, at least in the opinion of the author, no problems can arise from this abuse of notation, *as long as one keeps track of the maps involved.* Here are a couple of convincing examples:

(1) If X and Y are sets, we white $X \times Y = Y \times X$, keeping in mind that by equality we mean the switching coordinates map.
(2) Over $\mathbb{Q}$, we do not hesitate to write $\frac{7}{14} = \frac{2}{4}$, keeping mind that what we mean here is that numerator and denominator can be multiplied to get equivalent expressions.

Proposition 1.1.5. *Let $\{M_i\}_{i\in I}, N$ be R-modules. Then,*

$$N \otimes (\bigoplus_{i\in I} M_i) = \bigoplus_{i\in I}(N \otimes M_i).$$

Proof. We show the isomorphism for I finite, as that is the only case we will use. We leave the general case to the reader (see, *e.g.*, [Lang]). By induction, it suffices to prove for cardinality of $I = 2$. We construct maps going both ways. We have an evident bilinear map $(M_1 \oplus M_2) \times N \to (M_1 \otimes N) \oplus (M_2 \otimes N)$. Namely, send $(m_1 + m_2, n)$ to $m_1 \otimes n + m_2 \otimes n$ (feel free to check that this is indeed bilinear). This induces a map $f : (M_1 \oplus M_2) \otimes N \to (M_1 \otimes N) \oplus (M_2 \otimes N)$.

To go the other way, recall that, in general, a map from a direct sum of modules is the same as a map from each module separately (some people might even tell you that that's the *definition* of a direct sum of modules). So, to construct a map $g : (M_1 \otimes N) \oplus (M_2 \otimes N) \to (M_1 \oplus M_2) \otimes N$ it suffices to construct a maps from $M_1 \otimes N$ and $M_2 \otimes N$. Let's do the former. There is again an evident bilinear map $M_1 \times N \to (M_1 \oplus M_2) \otimes N$, namely $(m_1, n) \mapsto (m_1 + 0) \otimes n$.

Thus, we have maps f and g going both ways. The reader can check that these are inverses of each other. □

After this proposition the notation of direct *sum* and tensor *product* is more transparent.

Lemma 1.1.6. *The tensor product is "functorial." More precisely, let M be an R-module. We define a map $M \otimes - : R\mathrm{Mod} \to R\mathrm{Mod}$ as follows. On objects $(M \otimes -)(N) = M \otimes N$, and if $f : N \to L$ is a morphism in RMod, then $(M \otimes -)(f) = 1 \otimes f : M \otimes N \to M \otimes L$ is defined on generators by $(1 \otimes f)(x \otimes l) = x \otimes f(l)$. Then $(M \otimes -)(1_N) = 1_{M\otimes N}$ and $(M \otimes -)(g \circ f) = (M \otimes -)(g) \circ (M \otimes) - (f)$. That is, $M \otimes -$ is a functor.*

Exercise 1.1.7. Prove this.

Proposition 1.1.8. *The functor*

$$M \otimes - : R\mathrm{Mod} \to R\mathrm{Mod}$$

is ***right exact****. Concretely, if*

$$0 \to N' \xrightarrow{f} N \xrightarrow{g} N'' \to 0$$

is a short exact sequence then

$$M \otimes N' \to M \otimes N \to M \otimes N'' \to 0$$

is also exact.

Proof. That $1 \otimes g$ is surjective and $(1 \otimes g) \circ (1 \otimes f)$ is 0 are straight forward and left as exercises. It remains to show that $\ker(1 \otimes g) \subseteq \mathrm{Im}\,(1 \otimes f)$, or equivalently, that $M \otimes N/(\mathrm{Im}\,(1 \otimes f)) \to M \otimes N''$ is injective, and hence an isomorphism. We define a map $M \otimes N'' \to M \otimes N/(\mathrm{Im}\,(1 \otimes f))$ as follows. For a generator $x \otimes y \in M \otimes N''$, choose $\tilde{y} \in N$ such that $\tilde{y} = y$. Then, we map $x \otimes y \mapsto x \otimes \tilde{y}$. The reader can check that this is indeed well-defined, and it is clearly an inverse, whence the result. □

Remark 1.1.9. Tensor product is *not* in general exact. That is, if $N' \to N$ is an injection of R-modules, $N' \otimes M \to N \otimes M$ need not be injective. For example, $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$ is injective, but $\mathbb{Z} \otimes \mathbb{Z}/p\mathbb{Z} \xrightarrow{p\otimes 1} \mathbb{Z} \otimes \mathbb{Z}/p\mathbb{Z}$ is the zero map.

The failure of tensor to be exact is measured by the Tor sequence, which will be covered in the talk on homological algebra. When $M \otimes -$ is an exact functor, we call M **flat**.

Proposition 1.1.10. *Let M be an R-module. Then $M \otimes R = M$*

Proof. We define a map $M \otimes R \to M$ by $m \otimes r \mapsto rm$. This has an inverse given by $M \to M \otimes R$; $m \mapsto m \otimes 1$. □

We have the following generalization of Proposition 1.1.1.

Corollary 1.1.11. *If M^n denotes the n-fold direct sum of M, then $M \otimes R^n = M^n$.*

Proof. This follows immediately from Propositions 1.1.5 and 1.1.10. □

1.2. **Calculation of tensors of abelian groups.** As an application of the propositions so far, we calculate the the tensor product of two finitely generated $\mathbb{Z}$-modules (*i.e.*, abelian groups). We recall the Fundamental Theorem of Finitely Generated Abelian Groups:

Theorem 1.2.1. *Let M be a finitely generated $\mathbb{Z}$-module. Then*

$$M \cong \mathbb{Z}^r \oplus \mathbb{Z}/p_1^{s_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_k^{s_k}\mathbb{Z}$$

for some integers $r, s_1, \ldots, s_k$ and distinct prime numbers $p_1, \ldots, p_k$.

Given the Fundamental Theorem, by Propositions 1.1.11 and 1.1.5, to calculate the tensor product of two finitely generated $\mathbb{Z}$-modules, we are reduced to the following lemma.

Lemma 1.2.2. *Let p and q be primes, $k \leq l$ nonnegative integers. Then*

$$\mathbb{Z}/p^k\mathbb{Z} \otimes \mathbb{Z}/q^l\mathbb{Z} \cong \begin{cases} 0 & \text{if } p \neq q \\ \mathbb{Z}/p^k\mathbb{Z} & \text{if } p = q. \end{cases}$$

Proof. We consider the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{p^k} \mathbb{Z} \to \mathbb{Z}/p^k\mathbb{Z},$$

and apply the functor $\mathbb{Z}/q^l\mathbb{Z} \otimes -$. By the right exactness of tensor, we get an exact sequence

$$\mathbb{Z}/q^l\mathbb{Z} \xrightarrow{p^k} \mathbb{Z}/q^l\mathbb{Z} \to \mathbb{Z}/p^k\mathbb{Z} \otimes \mathbb{Z}/q^l\mathbb{Z} \to 0$$

(here we have identified $\mathbb{Z} \otimes \mathbb{Z}/q^l\mathbb{Z} = \mathbb{Z}/q^l\mathbb{Z}$ via the maps provided in the proof of Proposition 1.1.10.

Now if $p \neq q$, $\mathbb{Z}/q^l\mathbb{Z} \xrightarrow{p^k} \mathbb{Z}/q^l\mathbb{Z}$ is an isomorphism, so $\mathbb{Z}/p^k\mathbb{Z} \otimes \mathbb{Z}/q^l\mathbb{Z} = 0$. If $q = p$, then it is clear that the cokernel of $\mathbb{Z}/p^l\mathbb{Z} \xrightarrow{p^k} \mathbb{Z}/p^l\mathbb{Z}$ is $\mathbb{Z}/p^k\mathbb{Z}$. □

Exercise 1.2.3. Using the properties shown for tensors, show that this corollary does suffice to describe tensor products of finitely generated abelian groups.

Corollary 1.2.4. *Let m and n be any integers, and write $d = \gcd(m, n)$. Then,*

$$\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/d\mathbb{Z}.$$

2. Hom: The *yin* to Tensor's *yang*

Let M and N be R-modules. Denote by $\operatorname{Hom}_R(M, N)$ (or simply $\operatorname{Hom}(M, N)$) the R-module whose elements are R-module homomorphisms with source M and target N. The abelian group structure is given by $(f+g)(m) = f(m) + g(m)$ and for $r \in R$, $(rf)(m) = rf(m)$.

We use this to define a new functor on RMod. Fix an R-module M. We define a functor, denoted $\operatorname{Hom}(M, -)$, as follows. On objects N, the functor maps N to the R-module $\operatorname{Hom}(M, N)$, and if $\phi : N \to P$, then the functor sends ϕ to the map $\phi_* : \operatorname{Hom}(M, N) \to \operatorname{Hom}(M, P)$ given by $\phi_*(f) = \phi \circ f$.

Exercise 2.0.5. Check this defines a functor.

Remark 2.0.6. There is an analogous functor $\operatorname{Hom}(-, M)$ (with obvious notation) that is **contravariant**.

We will nail down a precise connection between $\otimes$ and Hom in the sequel, but for now let us say imprecisely that Hom shares much in common with $\otimes$. For example, we have the following familiar looking propositions:

Proposition 2.0.7. *Let M be an R-module. Then* $\operatorname{Hom}(R, M) = M$.

Proposition 2.0.8. *Let $\{M_i\}_I$ and N be R-modules. Then*

$$\operatorname{Hom}\Big(\bigoplus_{i\in I} M_i, N\Big) = \prod_{i \in I} \operatorname{Hom}(M_i, N).$$

Remark 2.0.9. Of course, if the index set I is finite, then the direct product on the right hand side is in fact a direct sum.

Proposition 2.0.10. *The functor* $\operatorname{Hom}(M, -)$ *is* ***left exact****. Concretely, if*

$$0 \to N' \xrightarrow{f} N \xrightarrow{g} N'' \to 0$$

is a short exact sequence then

$$0 \to \operatorname{Hom}(M, N') \to \operatorname{Hom}(M, N) \to \operatorname{Hom}(M, N'')$$

is also exact.

Remark 2.0.11. It probably goes without saying, but it is hardly ever true that $\operatorname{Hom}(M, N) = \operatorname{Hom}(N, M)$.

Remark 2.0.12. As before, the failure of Hom to be exact is measured by the Ext sequence, which will also be covered in the homological algebra talk. When $\operatorname{Hom}(M, -)$ is exact, we call M **projective** (and when $\operatorname{Hom}(-, M)$ is exact, we call M **injective**).

All the above propositions can be proven directly (and it may very well be a good exercise to do so). However, the amazing fact is that they can all be formally deduced from the corresponding facts about tensor products. To tie these two seemingly disparate functors together, we have the following fundamental fact.

Theorem 2.0.13 (Adjointness of $\otimes$ and Hom)**.** *Let M, N, and P be any R-modules. Then,*

$$\operatorname{Hom}(M \otimes N, P) = \operatorname{Hom}(M, \operatorname{Hom}(N, P)).$$

Proof. We construct maps each way. Suppose given a map $f : M \otimes N \to P$. We define a map $g : M \to \operatorname{Hom}(N, P)$ by $g(m)(n) = f(m \otimes n)$. Conversely, given g, we define f by the formula $f(m \otimes n) = g(m)(n)$. ☐

Remark 2.0.14. The term **adjointness** (which you probably came across in the category theory talk) is used to evoke the analogy with vector spaces. Recall that if $T : V \to V$ is a linear transformation of a vector space V, and V is equipped with a bilinear form $(\, , \,)$, we say that $U : V \to V$ is the **adjoint** of T if for any $v, w \in V$, $(Tv, w) = (v, Uw)$. In fact, one can think of Hom as a "bilinear form" on RMod, albeit a nonsymmetric and rather degenerate one.

For a flavor of how to use adjointness to prove facts about Hom once we know the analogous fact about $\otimes$, let us prove Proposition 2.0.7. We need to recall the (following imprecise) statement of Yoneda's Lemma.

Theorem 2.0.15 (Yoneda's Lemma)**.** *Let M and M' be R-modules, and suppose that for every R-module T, $\operatorname{Hom}(T, M) = \operatorname{Hom}(T, M')$. Then $M = M'$.*

Proof. As usual, "equality" here means natural isomorphism. One gets maps going both ways by inserting for T, M and M'. ☐

Remark 2.0.16. The motivation behind Yoneda's Lemma is again in functional analysis, where one can understand a function by integrating it against "test functions." For this reason, one often calls the T in the Yoneda's Lemma a **test object**.

Exercise 2.0.17. If you know the "real" Yoneda's Lemma, make the above statement more precise, and show that it is an equivalent formulation.

Proof of Proposition 2.0.7. Let T be a test object in RMod. Then, by adjointness, we have

$$\begin{aligned} \operatorname{Hom}(T, \operatorname{Hom}(R, M)) &= \operatorname{Hom}(T \otimes R, M) \\ &= \operatorname{Hom}(T, M). \end{aligned}$$

Thus, by Yoneda's Lemma, we have that $\operatorname{Hom}(R, M) = M$. ☐

Exercise 2.0.18. Recall that the equalities above are all natural equivalences (including the use of Yoneda's Lemma). Work through these maps to recover the usual natural map $\operatorname{Hom}(R, M) \xrightarrow{\sim} M$.

Exercise 2.0.19. Prove other statements about Hom using adjointness.

3. Some Important "Formulas" That Only Work for Free Modules

Definition 3.0.20. Let V be a free module of finite rank over a ring R. We define the **dual** of V, denoted V^*, by $V^* = \operatorname{Hom}(V, R)$. If $f : V \to W$ is linear, we write $f^* : W^* \to V^*$ for the induced map.

Remark 3.0.21. Of course, we could make the above definition for any R-module M, but that would be silly if M is not free. For example, if $M = \mathbb{Z}/n\mathbb{Z}$, then $\operatorname{Hom}(M, \mathbb{Z}) = 0$. Duality also works perfectly well for projective modules, if you are into that sort of thing. In fact, probably everything we state regarding free modules is true for projective modules as well.

Remark 3.0.22. Although V and V^* have the same dimension, and hence are abstractly isomorphic, it is a mortal sin to identify one with the other. However, we do have the identification below.

Theorem 3.0.23. *Let V be a free R-module of finite rank. Then $V = (V^*)^*$.*

Proof. This is the same as for vector spaces. We have a natural map $V \to (V^*)^*$ given by $x \mapsto [\mathrm{ev}_x : \phi \mapsto \phi(x)]$. This map is clearly injective, and so is an isomorphism since $\dim V = \dim V^*$. $\square$

The following is a very useful formula.

Theorem 3.0.24. *Let V and W be a free R-modules of finite rank. Then $V \otimes W^* = \mathrm{Hom}\,(W, V)$.*

Proof. We define a map $V \otimes W^* \to \mathrm{Hom}\,(W, V)$ by the formula $v \otimes \phi \mapsto [w \mapsto \phi(w)v]$. This is injective and so an isomorphism by dimension count. $\square$

Yet another useful formula:

Corollary 3.0.25. *Let V and W be free R-modules of finite rank. Then $(V \otimes W)^* = V^* \otimes W^*$.*

Exercise 3.0.26. Prove this. [Hint: use Theorem 3.0.24 adjointness.]

Example 3.0.27. Theorem 3.0.24 allows us to define the trace of a linear transformation without having to choose a basis. Namely, let V be a free R-module of finite rank, and let $T : V \to V$ be an R-module map. Then T gives rise to an element of $V \otimes V^*$. There is a natural pairing map $V \otimes V^* \to R$ given by $v \otimes \phi \mapsto \phi(v)$. Then the trace of T is defined to be the image of T under this map.

Exercise 3.0.28. By choosing a basis of V, the reader can check that this agrees with the usual definition of trace.

Example 3.0.29. Suppose we are given a bilinear map $\omega : V \times W \to k$. Then, ω induces a linear map $\omega : V \otimes W \to k$. By adjointness, we have:

$$\begin{aligned} \mathrm{Hom}(V \otimes W, k) &= \mathrm{Hom}(V, \mathrm{Hom}(W, k)) \\ &= \mathrm{Hom}(V, W^*). \end{aligned}$$

We say ω is **perfect** if the resulting map $V \to W^*$ is an isomorphism. Thus, the map ω allows us to identify V with the dual space of V.

References

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[4] M. Atiyah and I. Macdonald, *Introduction to Commutative Algebra*, Perseus Books, Cambridge, MA, 1969.
The best book from which to learn commutative algebra, period. Grossly overpriced 128 page paperback; stick it to the Man by photocopying it. This is a book you can (and should!) read cover to cover and do every exercise.