Witten’s Conjecture through
Cut-and-Join Equation

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In this talk, I will present a new proof of Witten’s conjecture using localization method. Precisely, I will show that the *Cut-and-Join equation* implies ELSV-formula, the KdV-hierarchy, and the Virasoro constraint.

- Witten’s original conjecture.

- Equivalent formulations.

- Previous approaches.

- Localization method.

- Cut-and-Join equation.

- Proof of Witten’s conjecture.
Preliminaries

Let $\overline{M}_{g,n}$ be the moduli space of stable curves, i.e. it consists of Riemann surfaces of genus $g$ and $n$-marked points with certain stability condition. We can consider two natural cohomology classes on it as follows:

- **ψ-classes** : For each marked point $x_i$ where $1 \leq i \leq n$, consider the line bundle $\mathbb{L}_i$ over $\overline{M}_{g,n}$ whose fiber over $[C, x_1, \ldots, x_n] \in \overline{M}_{g,n}$ is the cotangent line $T^*_i C$ at the $i$-th marked point $x_i$. Then define $\psi_i = c_1(\mathbb{L}_i)$.

- **λ-classes** : Let $\pi : \overline{M}_{g,n+1} \rightarrow \overline{M}_{g,n}$ be the universal curve, and let $\omega_\pi$ be the relative dualizing sheaf. The Hodge bundle $\mathcal{E} = \pi_* \omega_\pi$ is a rank $g$ vector bundle over $\overline{M}_{g,n}$. Denote the Chern polynomial by

$$\Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g \lambda_g$$

and its $j$-th Chern class by $\lambda_j = c_j(\mathcal{E})$. 
Hodge integral

The *Hodge integral* is defined as the integration of these classes over the moduli space $\overline{M}_{g,n}$, i.e.

$$\int_{\overline{M}_{g,n}} \prod \psi_i^{k_i} \prod \lambda_j^{l_j}$$

where $\sum k_i + \sum j \cdot l_j = 3g - 3 + n = \dim \overline{M}_{g,n}$.

The Witten’s conjecture, in mathematical formulation, is about the governing relation of Hodge integrals which involve $\psi$-classes only: Given a quantum field theory Lagrangian $L_0$ and operators $\tau_i$, its intersection numbers $\langle \tau_0^{n_0} \tau_1^{n_1} \cdots \rangle$ amounts to the Hodge integral as

$$\langle \tau_0^{n_0} \tau_1^{n_1} \cdots \rangle = \int_{\overline{M}_{g,n}} \prod \psi_i^{k_i}$$

where $g$, $n$, and $k_i$ are determined by conditions

$$n = \sum n_p$$

$$k_i = \# \{ a \mid n_p = i \}$$

$$\sum k_i = 3g - 3 + n$$
Witten’s Original Conjecture

Motivated by the Feynman integral and its “total free energy”, E.Witten considered the generating function of the stable intersection theory on moduli space:

\[
F(t_0, t_1, \cdots) = \sum_{\{n_i\}} \prod_{i=0}^{\infty} \frac{t_{n_i}^{n_i}}{n_i!} \langle \tau_0^{n_0_0} \tau_1^{n_1_1} \tau_2^{n_2_2} \cdots \rangle.
\]

and formulated the Witten’s Conjecture as follows:

2a. The conjecture. Our basic conjecture is that \( F(t_0, t_1, \cdots) \) is determined by the following two conditions:

(1) The object \( U = \partial^2 F / \partial t_0^2 \) obeys the KdV equations.

\[
\frac{\partial U}{\partial t_n} = \frac{\partial}{\partial t_0} R_{n+1}(U, \dot{U}, \ddot{U}, \cdots),
\]

where \( \dot{U} = \partial U / \partial t_0 \), \( \ddot{U} = \partial^2 U / \partial t_0^2 \), etc., are the derivatives of \( U \) w.r.t \( t_0 \), and \( R_{n+1}(U, \dot{U}, \ddot{U}, \cdots) \) are certain polynomials in \( U \) and its \( t_0 \) derivatives that are well known in the theory of the KdV equations (and can be defined by a recursion relation that is given below).

(2) In addition, \( F \) obeys the “string equation,”

\[
\frac{\partial F}{\partial t_0} = \frac{t_0^2}{2} + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial F}{\partial t_i}.
\]

The two statements can be summarized by saying that stable intersection theory on moduli space is equivalent to the “hermitian matrix model” of two-dimensional gravity. That formulation was the original context for the conjecture.

Equivalent Formulations

1] The Virasoro constraint formulation.

The Witten conjecture states that the intersection theory on the moduli space of stable curves is governed by the KdV-hierarchy. It is also shown to be equivalent to the following Virasoro constraint:

\[ L_n \cdot F = 0, \quad n \geq -1 \]

where \( L_n \) denote the Virasoro operators.

\[
L_{-1} = -\frac{1}{2} \frac{\partial}{\partial \tilde{t}_0} + \sum_{k=1}^{\infty} \left( k + \frac{1}{2} \right) \tilde{t}_k \frac{\partial}{\partial \tilde{t}_{k-1}} + \frac{1}{4} \tilde{t}_0^2
\]

\[
L_0 = -\frac{1}{2} \frac{\partial}{\partial \tilde{t}_1} + \sum_{k=0}^{\infty} \left( k + \frac{1}{2} \right) \tilde{t}_k \frac{\partial}{\partial \tilde{t}_k} + \frac{1}{16}
\]

\[
L_n = -\frac{1}{2} \frac{\partial}{\partial \tilde{t}_{n-1}} + \sum_{k=0}^{\infty} \left( k + \frac{1}{2} \right) \tilde{t}_k \frac{\partial}{\partial \tilde{t}_{k+n}} + \frac{1}{4} \sum_{i=1}^{n} \frac{\partial^2}{\partial \tilde{t}_{i-1} \partial \tilde{t}_{n-i}}, \quad \tilde{t}_n = t_{2n+1}
\]

[E. Witten, *On the Kontsevich model and other models of two-dimensional gravity*, Proceeding of the XXth International Conference on Differential Geometric Methods in Theoretical Physics]

E.Verlinde, H.Verlinde, and R.Dijkgraaf derived the following recursion relation for the correlation functions of the topological gravity

\[
\langle \tilde{\sigma}_n \prod_{k \in S} \tilde{\sigma}_k \rangle_g = \sum_{k \in S} (2k + 1) \langle \tilde{\sigma}_{n+k-1} \prod_{l \neq k} \tilde{\sigma}_l \rangle_g
\]

\[
+ \frac{1}{2} \sum_{a,b} \left[ \langle \tilde{\sigma}_a \tilde{\sigma}_b \prod_{l \in S} \tilde{\sigma}_l \rangle_{g-1} \right.
\]

\[
+ \sum_{\tilde{g},X} \langle \tilde{\sigma}_a \prod_{k \in X} \tilde{\sigma}_k \rangle_{\tilde{g}} \langle \tilde{\sigma}_b \prod_{l \in X^c} \tilde{\sigma}_l \rangle_{g-\tilde{g}} \right]
\]

where \( \tilde{\sigma}_n = (2n + 1)!!\psi^n \) and

\[
\langle \prod_{k \in S} \tilde{\sigma}_k \rangle_g = \left[ \prod_{k \in S} (2k + 1)!! \right] \int_{\mathcal{M}_{g,|S|}} \prod_{k \in S} \psi^k
\]

Physically, they considered the contact terms which occur when two operators collide in the form

\[
O_i(z) \cdot O_j(0) \sim O_k(0) \delta^2(z)
\]

and showed that the above recursion relation is equivalent to the Virasoro constraint.
Previous Approaches

1] Matrix model and Kontsevich’s proof

Witten conjecture was first proved by M. Kontsevich using combinatorial model of the moduli space and matrix model.

The string partition function $\tau(t)$:

$$\tau(t) = \exp \sum_{g=0}^{\infty} \langle \exp \sum_n t_n O_n \rangle_g$$

admits an integral representation which involves the following integral over a $N \times N$ Hermitian matrix $Y$ of the form

$$\tau(Z) = \rho(Z)^{-1} \int dY \cdot \exp \text{Tr} \left[ -\frac{1}{2} ZY^2 + \frac{i}{6} Y^3 \right]$$

where $Z$ is a second $N \times N$ Hermitian matrix, and $\rho(Z)$ is the one-loop integral

$$\rho(Z) = \int dY \cdot \exp \left[ -\frac{1}{2} \text{Tr} ZY^2 \right]$$

This requires a map from the matrix $Z$ to the coupling constants $t_n$, which is given by

$$t_n = -\frac{1}{n} \text{Tr} \ Z^{-n}$$
2] Approaches using ELSV formula

- A. Okounkov and R. Pandharipande gave a different approach using the enumeration of branched covering of \( \mathbb{P}^1 \) through the ELSV-formula:

\[
\frac{H_{g,\mu} \cdot |\text{Aut } \mu|}{(2g - 2 + |\mu| + l(\mu))!} = \prod_{i=1}^{l(\mu)} \frac{\mu_i^{\mu_i} \mu_i!}{\mu_i!} \int_{\mathcal{M}_{g,l(\mu)}} \Lambda_g^V(1) \prod (1 - \mu_i \psi_i)
\]

and using Kontsevich’s matrix model. Here \( H_{g,\mu} \) is the single Hurwitz number with ramification type at \( \infty \) given by the partition \( \mu \).

- Recently, M. E. Kazarian and S. E. Lando obtained an algebro-geometric proof by using the ELSV-formula and the PDEs which govern the generating series of Hurwitz numbers to derive the KdV-equation.

3] Approach using Weil-Petersen volumes

M. Mirzakhani derived the Virasoro constraint using Weil-Petersen volumes on moduli spaces of bordered Riemann surfaces.
Localization method

Let $\overline{M}_g(\mu, \mathbb{P}^1)$ be the moduli space of relative stable morphisms, i.e. it consists of mapping

$$f : [C; x_1, \ldots, x_{l(\mu)}] \rightarrow (\mathbb{P}^1, \infty)$$

with prescribed ramification $\mu$ at the divisor $\infty$

$$f^{-1}(\infty) = \sum \mu_i x_i$$

We can induce $S^1$-action on this space from the natural $S^1$-action on $\mathbb{P}^1$ given by $t \cdot [u, v] = [tu, v]$. Then localization method can be applied and we can express an integration of any natural class $\omega$ as the summation over fixed locus of $S^1$-action:

$$\int_{\overline{M}_g(\mu, \mathbb{P}^1)} \omega = \sum_{F : \text{fixed locus}} \int_F i^*_F(\omega) e(N_F)$$

Also the moduli space of relative stable morphisms admits the branching morphism:

$$\text{Br} : \overline{M}_g(\mu, \mathbb{P}^1) \rightarrow \text{Sym}^r \mathbb{P}^1 \cong \mathbb{P}^r$$

where $r = 2g - 2 + |\mu| + l(\mu) = v.\text{dim} \overline{M}_g(\mu, \mathbb{P}^1)$. 
**Cut-and-Join equation**

Consider the hyperplane class \( H \in H^*(\mathbb{P}^r) \) with prescribed values \( H(p_k) = k \) for \( k = 0, \ldots, r \) and \( p_k \) is the \( k \)-th fixed point of \( \mathbb{P}^r \). Let \( \omega = Br^* \prod_{k=0}^{r-2} (H - k) \). The localization method applied to \( \omega \) yields:

\[
r \Gamma_r = \Gamma_{r-1}
\]

where \( \Gamma_r \) and \( \Gamma_{r-1} \) denote the contributions from the fixed locus that are mapped under the branching morphism to \( p_r \) and \( p_{r-1} \), respectively.

This is the Cut-and-Join equation, and is of same type as the single Hurwitz number formula:

\[
H_{g,\mu} = \sum_{\nu \in J(\mu)} I_1(\nu) H_{g,\nu} + \sum_{\nu \in C(\mu)} I_2(\nu) H_{g-1,\nu}
\]

\[
+ \sum \left( 2g_1 - 2 + |\nu^1| + l(\nu^1) \right) I_3(\nu^1, \nu^2) H_{g_1,\nu^1} H_{g_2,\nu^2}.
\]

except that the single Hurwitz numbers are replaced by the Hodge integrals

\[
H_{g,\mu} = \frac{r!}{|\text{Aut } \mu|} \frac{l(\mu)}{\prod_{i=1}^{l(\mu)} \mu_i^{\mu_i}} \int_{\mathcal{M}_{g,l(\mu)}} \frac{\Lambda^\vee_g(1)}{\prod(1 - \mu_i \psi_i)}
\]

Hence the Cut-and-Join equation recovers the ELSV-formula since both sides satisfy identical recursion relation with same initial values.
Proof of Witten’s conjecture

Application of degree analysis to the combinatorial data (i.e. Cut-and-Join equation):

- **Combinatorics:** Cut and join of cycles:

  - **Cut:** a \((i + j)\)-cycle is cut into an \(i\)-cycle and a \(j\)-cycle, denote the set by \(C(\mu)\):

  - **Join:** an \(i\)-cycle and a \(j\)-cycle are joined to an \((i + j)\)-cycle, denote the set by \(J(\mu)\):
• **Degree Analysis:** Scale the partition $\mu$

$$\mu_i = N \cdot x_i, \quad \text{for some } x_i \in \mathbb{R} \text{ and } N \in \mathbb{N}$$

and let $N$ tend to $\infty$. The asymptotic behaviour of Cut-and-Join equation can be obtained by using the asymptotic formula:

$$e^{-n} \sum_{p+q=n} \frac{pp+k+1 \cdot qq+l+1}{p!q!} \rightarrow \frac{1}{2} \left[ \frac{(2k + 1)!!(2l + 1)!!}{2^{k+l+2}(k + l + 2)!} \right] n^{k+l+2} + o(n^{k+l+2})$$

$$e^{-n} \sum_{p+q=n} \frac{pp+k+1 \cdot qq-1}{p!q!} \rightarrow n^{k+\frac{1}{2}} - \left[ \frac{(2k + 1)!!}{2^{k+1}k!} \right] n^k + o(n^k)$$

[Proof is an application of integration by parts and Stirling’s formula $n! \sim \sqrt{2\pi}e^{-n}n^{n+1/2}(1 + 1/12n + \cdots)$]

and the stratification given by

$$\int_{\mathcal{M}_{g,n}} \frac{\Lambda^\vee_g(1)}{\prod (1 - \mu_i \psi_i)} = \left[ \sum_k \prod x_i^{k_i} \int_{\mathcal{M}_{g,n}} \prod \psi_i^{k_i} \right] N^{3g-3+n} + \text{lower } N\text{-degree terms}$$
The resulting relation is a system of vanishing equations between linear Hodge integrals. Its first non-trivial equation is the recursion relation of topological gravity obtained by E. Verlinde, H. Verlinde, and R. Dijkgraaf.

- The highest $N$-degree gives a trivial identity:

$$0 = (x_1 + \cdots + x_n) \prod x_i^{k_i-1/2} \int_{\mathcal{M}_{g,n}} \prod \psi_i^{k_i} - (x_1 + \cdots + x_n) \prod x_i^{k_i-1/2} \int_{\mathcal{M}_{g,n}} \prod \psi_i^{k_i}$$

- The second highest $N$-degree gives a recursion relation between Cut-and-Join graphs:

$$0 = \sum_{i=1}^{n} \left[ (2k_i + 1)!! \frac{x_i^{k_i}}{2^{k_i+1}k_i!} \prod_{j \neq i} x_j^{k_j-1/2} \int_{\mathcal{M}_{g,n}} \prod \psi_j^{k_j} - \sum_{j \neq i} (x_i + x_j)^{k_i+k_j-1/2} \prod_{l \neq i,j} x_l^{k_l-1/2} \int_{\mathcal{M}_{g,n-1}} \psi_i^{k_i+k_j-1} \prod \psi_l^{k_i} - \frac{1}{2} \sum_{k+l=k_i-2} (2k + 1)!!(2l + 1)!! \frac{x_i^{k_i}}{2^{k+l+2} (k + l + 2)!} \prod_{j \neq i} x_j^{k_j-1/2} \int_{\mathcal{M}_{g,n+1}} \psi_1^{k_1} \prod \psi_j^{k_j} \int_{\mathcal{M}_{g_1,n_1}} \psi_1^{k_1} \prod \psi_j^{k_j} \int_{\mathcal{M}_{g_2,n_2}} \psi_1^{k_1} \prod \psi_j^{k_j} \right]$$

- Lower $N$-degree strata will give relations for Hodge integrals involving non-trivial $\lambda$-class in terms of lower-dimensional ones.
After taking the Laplace transformations,

\[
\int_0^\infty x^{k-1/2} e^{-x/2s} \, dx = (2k - 1)!! \, s^{k+1/2}
\]

\[
\int_0^\infty x^k e^{-x/2s} \, dx = k! \, (2s)^{k+1}
\]

we recover the recursion relation for the correlation functions of the topological gravity.

\[
\langle \tilde{\sigma}_n \prod_{k \in S} \tilde{\sigma}_k \rangle_g = \sum_{k \in S} (2k + 1) \langle \tilde{\sigma}_{n+k-1} \prod_{l \neq k} \tilde{\sigma}_l \rangle_g \\
+ \frac{1}{2} \sum_{a,b} \left[ \langle \tilde{\sigma}_a \tilde{\sigma}_b \prod_{l \in S} \tilde{\sigma}_l \rangle_{g-1} \\
+ \sum_{\tilde{g},X} \langle \tilde{\sigma}_a \prod_{k \in X} \tilde{\sigma}_k \rangle_{\tilde{g}} \langle \tilde{\sigma}_b \prod_{l \in X^c} \tilde{\sigma}_l \rangle_{g-\tilde{g}} \right]
\]