This paper studies discrete half-line Schrödinger operators with sparse potentials, in particular the relationship between the spacing of barriers, their height, and spectral type of the operator. Constructed are potentials with exponentially spaced identical barriers such that in some interval of energies the spectral measure is purely singular continuous and has fractional Hausdorff dimension (strictly between 0 and 1). Employed are results of Jitomirskaya-Last on the relation between the dimension of the spectral measure and the growth of eigenfunctions. By using large deviations methods this growth is estimated (with a small error) for all energies outside of a set of dimension α < 1. As a consequence, the results hold for all boundary conditions. In the case of a corresponding random model (with randomness in the position of the barriers) these techniques are employed to prove and compute an exact Hausdorff dimension of the spectral measure where it is singular continuous — in the center of [−2, 2] — and show dense pure point spectrum in the rest of the interval.


We study the Case sum rules for Jacobi matrices — self-adjoint tri-diagonal semi-infinite matrices with real \( \{b_n\}_{n=1}^{\infty} \) on the diagonal and positive \( \{a_n\}_{n=1}^{\infty} \) on the first sub- and super-diagonal. We consider mainly the case \( a_n - 1, b_n \to 0 \) so that the essential spectrum of the matrix is [−2, 2] and we let \( d\mu(x) = w(x)dx + d\mu_{\text{sing}}(x) \) be its spectral measure. The sum rules are formulae that relate sums in terms of the coefficients \( a_n, b_n \) and spectral quantities — Szegő-type integrals (e.g., the Szegő integral \( \int_{-2}^{2} \sqrt{E_j - \frac{1}{2}} \frac{\log(w(x))}{4 - x^2} dx \)) and Lieb-Thirring-type sums involving eigenvalues \( E_j \) of the matrix outside of [−2, 2] (e.g., \( \sum_j \sqrt{|E_j| - \frac{1}{2}} \)). We establish situations where the sum rules are valid, extending results of Killip-Simon. Applications include an extension of Shohat’s characterization of matrices obeying the Szegő condition (summability of the Szegő integral above) to cases with possible infinite point spectrum, under the a priori assumption \( \sum_j \sqrt{|E_j| - \frac{1}{2}} < \infty \), and various general necessary conditions for the validity of the Szegő condition. We also show that if \( \lim n(a_n - 1) = \alpha \) and \( \lim nb_n = \beta \) exist and \( 2\alpha < |\beta| \), then the Szegő condition fails and streamline a rather lengthy proof of the Killip-Simon characterization of spectral measures of Hilbert-Schmidt perturbations of the “free” matrix, which says that \( a_n - 1, b_n \in L^2 \) if and only if \( \mu \) has only eigenvalues outside of [−2, 2], \( \int_{-2}^{2} \frac{\log(w(x))}{4 - x^2} dx \) is summable, and \( \sum_j |E_j| - 2)^{3/2} < \infty \).


We prove a conjecture of Askey from 1979 on the Szegő condition (see 2. above) for Coulomb perturbations of the free Jacobi matrix \( (a_n = 1 + \alpha/n + O(n^{-1-\varepsilon}) \) and \( b_n = \beta/n + O(n^{-1-\varepsilon}) \)) and some related results. First various general sufficient conditions for the validity of the Szegő condition are obtained. These apply, in particular, to matrices with \( |a_n - 1|, |b_n| \) “regularly” decaying to 0. Then simultaneous control of the movement of an infinite number of eigenvalues under certain perturbations and results from 2. are employed to treat \( O(n^{-1-\varepsilon}) \) perturbations. It is shown that the Szegő condition holds if and only if \( 2\alpha \geq |\beta| \) in the case above (the “only if” part is from 2.). In addition, similar results on one-sided Szegő conditions are provided (summability of the Szegő integral at −2 or 2 only).

With the notation from 2., assume that \( a_n - 1 \in \ell^2 \) and \( b_n \to 0 \) for a Jacobi matrix. We show that if \( b_n \notin \ell^4 \), \( b_{n+1} - b_n \in \ell^2 \), then \( \sum_j |E_j| - 2)^{5/2} = \infty \), and if \( b_n \in \ell^4 \), \( b_{n+1} - b_n \notin \ell^2 \), then \( \int_{-2}^2 \ln(w(x))(4 - x^2)^{3/2} \, dx = -\infty \). We also show that if \( a_n - 1, b_n \in \ell^3 \), then the above integral is summable if and only if \( a_{n+1} - a_n, b_{n+1} - b_n \in \ell^2 \). We prove these and other results by deriving general sum rules in which the Szegő-type integrals and the eigenvalues appear on opposite sides of the equation.


We consider a probability measure \( d\mu(\theta) = w(\theta) \frac{d\theta}{2\pi} + d\mu_{\text{sing}}(\theta) \) on the unit circle \( \partial \mathbb{D} \), naturally identified with \([0, 2\pi)\), and let \( \{\alpha_n\}_{n=0}^\infty \) be its Verblunsky coefficients (with \( \alpha_n \in \mathbb{D} \)). That is, if \( \{\Phi_n(z)\}_{n\geq 0} \) are the monic orthogonal polynomials for \( \mu \), then \( \Phi_{n+1}(z) = z\Phi_n(z) - \alpha_n\Phi_n^*(z) \) with \( \Phi_n^*(z) = z^n\Phi_n^*(1/\overline{z}) \). Our interest is in a conjecture of Simon which says that if \( Q(z) \equiv \sum_{m=0}^N q_m z^m \) is a polynomial, then with \( \bar{Q}(z) \equiv \sum_{m=0}^N \bar{q}_m z^m \) and \( \delta \) the left shift operator \( (\delta \alpha)_n = \alpha_{n+1} \) one has

\[
|Q(e^{i\theta})|^2 \log w(\theta) \in L^1(d\theta) \iff (\bar{Q}(\delta)\alpha)_n \in \ell^2 \text{ and } \alpha_n \in \ell^{2k+2},
\]

where \( k \) is the largest of the multiplicities of those roots of \( Q \) that lie on the unit circle. We apply the techniques from 2., 3., 4. above to this case and prove that the conjecture holds for \( N = 2 \).

6. L. Golinskii and A. Zlatoš, *Coefficients of orthogonal polynomials on the unit circle and higher order Szegő theorems*, submitted.

We provide, for the first time, an expression of the coefficients of the orthogonal polynomials \( \Phi_n \) in terms of the Verblunsky coefficients \( \alpha_n \) (see 5. above). If \( \log w \in L^1(d\theta) \), we do the same for the Fourier coefficients of \( \log w \). As an application we prove that if \( \alpha_n \in \ell^4 \) and \( Q(z) \) is a polynomial, then (cf. 5.)

\[
|Q(e^{i\theta})|^2 \log w(\theta) \in L^1(d\theta) \iff (\bar{Q}(\delta)\alpha)_n \in \ell^2.
\]

We also study relative ratio asymptotics \( \Phi_{n+1}^*(\mu)/\Phi_n^*(\mu) - \Phi_{n+1}^*(\nu)/\Phi_n^*(\nu) \) of the reversed polynomials \( \Phi_n^* \) for measures \( \mu, \nu \) and provide a necessary and sufficient condition in terms of the Verblunsky coefficients of \( \mu, \nu \) for this difference to converge to zero uniformly on compact subsets of \( \mathbb{D} \). We use this to obtain (simple proofs of) known and new results on the ratio asymptotics \( \Phi_{n+1}^*(\mu)/\Phi_n^*(\mu) \).

II. PDEs and Reaction-diffusion equations


We consider a model describing premixed combustion in the presence of fluid flow — the reaction-diffusion equation \( T_t + u \cdot \nabla T - \Delta T = f(T) \) for the temperature \( T \), with passive advection \( u(x) \) and ignition type reaction term \( f \) in an infinite cylindrical domain. We are interested in the question of which velocity profiles are able to quench (extinguish) any initially compactly supported flame, provided the velocity’s amplitude is adequately large. Even for shear flows, the solution turns out to be surprisingly subtle. We provide the following sharp characterization of “quenching” shear flows. The flow can quench any initial datum if and only if the velocity profile does not have a plateau (an interval where it is constant) larger than a certain critical size. The efficiency of quenching depends strongly on the geometry of the flow. We discuss the cases of slowly and quickly varying flows, proving rigorously scaling laws that have been observed earlier in numerical experiments. The results require new estimates on the behavior of the solutions to advection-enhanced diffusion equations (also known as passive scalar in physical literature), a classical model describing a wealth of phenomena in nature. The techniques include probabilistic and PDE estimates, in particular, applications of Malliavin calculus and the central limit theorem for martingales.
We consider the reaction-diffusion equation $T_t = T_{xx} + f(T)$ on $\mathbb{R}$ with $T_0(x) \equiv \chi_{[-L,L]}(x)$ and $f(0) = f(1) = 0$. In 1964 Kanel’ proved that if $f$ is an ignition non-linearity, then $T \to 0$ as $t \to \infty$ when $L < L_0$, and $T \to 1$ when $L > L_1$. We answer the open question of the relation of scales $L_0$ and $L_1$ by showing that $L_0 = L_1$. We also determine the large time limit of $T$ in the critical case $L = L_0$, thus providing the phase portrait for the above PDE with respect to a 1-parameter family of initial data. Analogous results for combustion and bistable non-linearities are proved as well.

We consider an advection-diffusion equation on a compact Riemannian manifold $M$. We study enhancement of diffusion on $M$ by a strong incompressible fluid flow. Our main result is a sharp description of the class of flows that make the deviation of the solution from its average arbitrarily small in an arbitrarily short time, provided that the flow amplitude is large enough. The condition on such flows is expressed naturally in terms of the spectral properties of the dynamical system associated with the flow. In particular, we find that weakly mixing flows always enhance dissipation in this sense. The proofs apply in a more general setting and are based on spectral techniques. In particular, they employ the RAGE theorem describing evolution of a quantum particle corresponding to the continuous spectral subspace of the hamiltonian, which is related to a classical theorem of Wiener on Fourier transforms of measures. Our results carry over to bounded domains in $\mathbb{R}^n$, and are also applied to the study of quenching in reaction-advection-diffusion equations on manifolds and bounded domains.

III. Discrete models of the Euler equation

Two discrete models for the Euler equation describing incompressible fluid dynamics are considered. These models are infinite coupled systems of ODEs for the functions $u_j$ which can be thought of as wavelet coefficients of the fluid velocity. The first model has been proposed and studied by Katz and Pavlović. The second has been recently discussed by Waleffe and goes back to Obukhov studies of the energy cascade in developed turbulence. These are the only basic models of this type satisfying some natural scaling and conservation conditions. We prove that the Katz-Pavlović model leads to finite time blowup for any initial datum, while the Obukhov model has a global solution for any sufficiently smooth initial datum.
IV. Graph Theory


These are some earlier papers — from my undergraduate research in topological graph theory. They deal with construction and characterization of certain regular maps, that is, regular embeddings of graphs in two-dimensional manifolds. In particular, bipartite graphs and graphs with multiple edges are considered. In addition, a way of constructing large graphs with controlled diameter, based on the technique of voltage assignments, is provided in 13.