

# Research Statement

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## Reaction-Diffusion Equations

**Background.** My recent research has focused on reaction-diffusion and reaction-advection-diffusion equations, as well as on related linear parabolic equations. These equations are used in modeling of reaction processes such as burning in internal combustion engines, nuclear reactions in stars, forest fires, embryogenesis, and production of ozone in the atmosphere. A simplified model for such processes, which nevertheless incorporates many important phenomena, is the PDE

$$u_t + q(x) \cdot \nabla u = \Delta u + f(x, u) \tag{1}$$

with  $u(t, x) \in [0, 1]$  the (normalized) temperature of a flammable medium or concentration of a substance. The spatial domain  $x \in D \subseteq \mathbb{R}^n$  can be bounded in some directions (e.g., an infinite cylinder) or all of  $\mathbb{R}^n$ , and one typically considers periodic or Neumann boundary conditions on  $\partial D$  (insulated boundary), although other boundary conditions can also be treated.

The temperature is subject to heat diffusion (represented by the Laplacian), advection by an incompressible flow (represented by a divergence-free vector field  $q$ ), and increase due to a reactive process. The latter is modeled by a non-linear *reaction function*  $f$ , a non-negative function with  $f(x, 0) = f(x, 1) = 0$  ( $u$  then stays between 0 and 1 if the initial datum  $u_0(x) \in [0, 1]$ ). One usually assumes  $f(x, u) = 0$  for  $u \in [0, \theta(x)]$  and  $f(x, u) > 0$  for  $u \in (\theta(x), 1)$ , with  $\theta(x) \in [0, 1]$  the location-dependent *ignition temperature*. If  $\inf_x \theta(x) > 0$ , then we call  $f$  an *ignition reaction*, otherwise  $f$  is a *monostable reaction*. A special case of the latter is a *KPP reaction* with  $0 < f(x, u) \leq \frac{\partial f}{\partial u}(x, 0)u$  for  $u \in (0, 1)$ .

There are two main directions of research involving equation (1) and its applications to natural sciences. The first is the study of *traveling fronts*, their speed of propagation and stability. Traveling fronts are special solutions modeling flame propagation in the direction of some unit vector  $e \in \mathbb{R}^n$ . They are global in time and at any fixed  $t$  converge to 0 and 1 as  $x \cdot e \rightarrow \pm\infty$ . Non-negativity of  $f$  guarantees the propagation of the reaction in the direction  $e$  at some positive speed  $c > 0$ . The simplest example, studied as early as 1937 by Kolmogorov-Petrovskii-Piskunov (KPP) and Fisher, is the 1D homogeneous case  $u_t = u_{xx} + f(u)$  and  $e = 1$ . Here traveling fronts are of the form  $u(t, x) = U(x - ct)$ , the couple  $(c, U)$  of the front speed and (constant in time) profile solving the ODE  $U'' + cU' + f(U) = 0$  with  $U(\infty) = 0$  and  $U(-\infty) = 1$ . It is not hard to show that a unique such couple  $(c^*, U_{c^*})$  exists for ignition reactions, while for monostable reactions solutions exist precisely when  $c \geq c^*$ , with  $c^* > 0$  some  $f$ -dependent *minimal front speed*.

The second area of interest is the study of the Cauchy (initial value) problem with compactly supported or spatially decaying initial data. These represent evolution of initial hot spots (e.g., of a localized explosion), and the interest is in the phenomena of *quenching* and *spreading* of reaction. Quenching (extinction) happens when  $u(t, x) \rightarrow 0$  uniformly in  $D$  as  $t \rightarrow \infty$ , having been studied particularly for ignition reactions (it can also happen for some monostable reactions [2] but never for KPP reactions). Spreading occurs when  $u(t, x) \rightarrow 1$  uniformly on compacts and typically results in a formation of fronts moving in different directions. Their

study goes back to Kanel’s 1964 work on the 1D homogeneous case which showed that for ignition reactions, solutions with large initial data spread whereas those with small data are quenched.

**Past Research.** The above questions for homogeneous media in general domains have been studied extensively in the past and (1) in this setting is by now fairly well understood. My contribution to this effort was [1] where a sharp answer has been provided to a natural question left open by Kanel’ in the 1D homogeneous case: what is the behavior of initial data with intermediate sizes? I proved that when the initial datum is the characteristic function of the interval  $[-L, L]$ , then there is  $L_0$  such that the solution  $T$  is quenched for  $L < L_0$ , spreads for  $L > L_0$ , and converges to the ignition temperature uniformly on compacts for  $L = L_0$ . This result thus provides a phase portrait for the PDE with respect to a 1-parameter family of initial data.

Often the effects of fluid motion cannot be neglected in the modeling of combustion and other chemical processes. The presence of a flow can have two diametrically opposite effects on reaction. On one hand, it can enhance front propagation or reaction spreading as wind that precipitates a fire. On the other hand, it can facilitate quenching if the flow strength is large compared to the size of the support of the initial data (the “try to light a fire in the wind” effect). Spatial dependence of the reaction function is also important in combustion models as well as in applications to biology and ecology. This is why considerable effort has recently been focused on the study of the full equation (1). In the last 15 years, significant progress has been made for periodic media ( $q$  and  $f$  spatially periodic), including questions of existence and stability of fronts, as well as of front speed-up by flows and flow-induced quenching. I will discuss below my contributions to this effort as well as some recent progress for general disordered (non-periodic) media.

The papers [2–6] studied the effects of strong periodic flows on quenching, concentrating on properties such as geometric structure of the flow and mixing properties of the dynamical system generated by it. They considered the equation

$$u_t + Aq(x) \cdot \nabla u = \Delta u + f(u) \tag{2}$$

with  $q$  a periodic flow profile and  $A \gg 1$  its amplitude. Here  $f$  is independent of  $x$  but the case of  $f$  periodic in  $x$  can be treated similarly. The incompressible flow  $q$  is called *quenching* for the reaction  $f$  if for any compactly supported initial datum  $u_0$  and all large enough  $A$ , the solution of (2) satisfies  $\|u(t, \cdot)\|_\infty \rightarrow 0$  as  $t \rightarrow \infty$ . It is *strongly (weakly) quenching* if it is quenching for any (sufficiently small) ignition reaction.

In [2, 3] quenching of ignition reactions in 2D strips  $\mathbb{R} \times [0, l]$  by shear (unidirectional) flows has been studied. The main result is a sharp characterization of the quenching shear flows.

**Theorem 1** ([3]). *The shear flow  $q(x_1, x_2) = (q_1(x_2), 0)$  on  $\mathbb{R} \times [0, l]$  is quenching for an ignition reaction  $f$  if and only if it has no plateaus (subintervals of  $[0, l]$  on which  $q_1$  is constant) longer than some critical length  $\ell(f) > 0$ . It is strongly quenching if and only if it has no plateaus.*

The second claim can be heuristically explained by noticing that for  $q$  without plateaus, large initial hot spots are quickly (before reaction can act) stretched thin by the strong flow

and then cooled below the ignition temperature by diffusion (provided  $A$  is also sufficiently large). This leads naturally to questions about *short time dynamics* (as opposed to *long time dynamics*, studied via homogenization techniques) of the solutions of the closely related linear equation

$$\phi_t + Aq \cdot \nabla \phi = \Delta \phi. \quad (3)$$

These questions have been addressed in [4–6] where sharp results on the dynamics of the solutions of (3) for large  $A$  have been obtained, along with applications to (2). In [4] we considered (3) on compact manifolds and bounded domains with Neumann boundary conditions. Here every solution  $\phi$  tends to its spatial average  $\bar{\phi}$  as  $t \rightarrow \infty$  but the question is which flows can achieve this relaxation to average on arbitrarily short time scales provided  $A$  is large. We therefore say that an incompressible flow  $q$  on  $D$  is *relaxation-enhancing* if for each initial datum  $\phi_0 \in L^1(D)$  and each  $\tau > 0$  we have  $\|\phi(\tau, \cdot) - \bar{\phi}\|_\infty \rightarrow 0$  as  $A \rightarrow \infty$ .

This is a measure of the mixing properties of the flow when coupled with the effects of diffusion. The main result of [4] is the following sharp characterization of relaxation-enhancing flows on compact manifolds and bounded domains  $D$ .

**Theorem 2** ([4]). *The incompressible flow  $q$  on  $D$  as above is relaxation-enhancing if and only if the operator  $q \cdot \nabla$  has no eigenfunctions lying in  $H^1(D)$  other than the constant functions.*

The theory of dynamical systems defines *weakly mixing* flows to be those for which the operator  $q \cdot \nabla$  has purely continuous spectrum. Thus, weakly mixing flows are relaxation enhancing but the latter class is larger due to the added effects of diffusion on mixing. Moreover, it has been shown in [4] that the relaxation-enhancing class stays unchanged when 1 and  $\infty$  in its definition are replaced by any  $p, r \in [1, \infty]$ .

The paper [5] extends this result to time-periodic flows  $q(t + \alpha, x) = q(t, x)$ , with the role of  $q \cdot \nabla$  played by the time- $\alpha$  map generated by the flow, obtained by solving the ODE  $\dot{X} = q(t, X)$ . It also provides a construction of a time-periodic flow on the torus which is relaxation-enhancing while being Hamiltonian (and thus not relaxation-enhancing) at any *fixed time*  $t$ . This demonstrates the informal principle that time-dependence of flows typically results in increased mixing.

Applications to quenching require similar results in the whole space and these have been pursued in [6]. We again let  $\phi_0 \in L^1(\mathbb{R}^n)$ , meaning that now  $\bar{\phi} = 0$  and thus we ask when  $\|\phi(\tau, \cdot)\|_\infty \rightarrow 0$  for any  $\tau > 0$  as  $A \rightarrow \infty$ . It is proved in [6] that periodic flows on  $\mathbb{R}^n$  which are relaxation-enhancing on their cells of periodicity, are also relaxation-enhancing and strongly quenching on  $\mathbb{R}^n$ . This is, however, not a necessary condition. In fact, the main result of [6] is the following sharp characterization of relaxation-enhancing periodic flows on  $\mathbb{R}^2$ .

**Theorem 3** ([6]). *The incompressible spatially periodic flow  $q$  on  $\mathbb{R}^2$  with a cell of periodicity  $Q$  is relaxation-enhancing if and only if it leaves no bounded open subset of  $\mathbb{R}^2$  invariant and the operator  $q \cdot \nabla$  on  $Q$  has no eigenfunctions in  $H^1(Q)$  except possibly those with eigenvalue 0.*

Here  $Q$  is a torus (a rectangle with opposite sides identified) and thus a compact manifold. It is also proved in [6] that periodic relaxation-enhancing flows on  $\mathbb{R}^2$  are strongly quenching, and that those which do leave a bounded open set invariant as well as those which give rise to a  $C^2(Q)$ -eigenfunction with a non-zero eigenvalue are not.

Finally, the opposite question of “slowdown of diffusions” by stirring has been studied in [7]. Somewhat surprisingly, it has been showed that the expected exit time of a diffusing particle from a bounded simply connected domain  $D \subseteq \mathbb{R}^2$  may be *increased* by the introduction of an incompressible flow if (and only if)  $D$  is not a disc.

The papers [8–12] have addressed the second area of interest, properties of traveling fronts for (2) on  $\mathbb{R}^n$  with spatially periodic flows. As we have already mentioned, the presence of a flow in (2) typically enhances the speed of front propagation. The main focus was on obtaining estimates on this speed, both for fixed amplitudes  $A$  and in the limit of strong flows  $A \rightarrow \infty$ . In all of these works the customary zero mean condition  $\int_Q q(x)dx = 0$  has been assumed — in other cases front speeds grow proportionally to  $A$ . It has been proved by Xin and Berestycki-Hamel that if such  $q$  is periodic, then for each direction  $e \in \mathbb{R}^n$  fronts moving in that direction exist. The results mirror those in the 1D homogeneous case described earlier. For ignition reactions, the front speed  $c_e^*(Aq, f)$  is unique while for monostable reactions fronts with any speed in an infinite interval  $[c_e^*(Aq, f), \infty)$  exist. In the later case the minimal speed  $c_e^*(Aq, f)$  is the most important one (and most physical) as it is the speed of spreading of solutions with compactly supported initial data.

The paper [8] has treated reactions of KPP type and studied asymptotics of  $c_e^*(Aq, f)$  as  $A \rightarrow \infty$ . It is well known that the front speed can grow at most linearly with  $A$  and a characterization of periodic flow profiles which achieve this rate (and thus are most effective at speeding up fronts) for KPP reactions has been provided by Berestycki-Hamel-Nadirashvili. In [8] it was proved that, in fact, exact asymptotic  $\lim_{A \rightarrow \infty} c_e^*(Aq, f)/A$  always exists (this is also proved for ignition reactions when  $q$  is a shear flow [9]) and the limit has been evaluated. These results are the only ones so far where exact asymptotics of  $c_e^*(Aq, f)$  have been found.

**Theorem 4** ([8]). *Let  $q$  be a mean-zero incompressible periodic flow on  $\mathbb{R}^n$  with a cell of periodicity  $Q$ , let  $\mathcal{I}$  be the set of all  $w \in H^1(Q)$  satisfying  $q \cdot \nabla w = 0$ , and let  $f$  be a KPP reaction. Then*

$$\lim_{A \rightarrow \infty} \frac{c_e^*(Aq, f)}{A} = \sup_{\substack{w \in \mathcal{I} \\ \|\nabla w\|_2^2 \leq f'(0)\|w\|_2^2}} \frac{\int_Q (q \cdot e)w^2 dx}{\|w\|_2^2}. \quad (4)$$

In particular, linear speed-up is equivalent to the existence of  $w \in \mathcal{I}$  for which the integral in (4) is positive. Combining this with Theorem 2 shows that flows which are relaxation-enhancing on their cell of periodicity (and thus very efficient at mixing) are not the most efficient at speeding up fronts.

The question of finding estimates on the front speeds for arbitrary  $A$  has been addressed for KPP reactions in [10] and for general reactions in [12]. For periodic flows in two dimensions,  $c_e^*(Aq, f)$  has been showed to be comparable to the square root of the *effective diffusivity* of the flow. The latter describes the *long term* diffusive rate of the solutions of (3) in the direction  $e$  and is given by  $D_e(Aq) = 1 + \int_Q |\nabla w_e|^2 dx$  where  $w_e$  solves  $-\Delta w_e + Aq \cdot \nabla w_e = Aq \cdot e$  on  $Q$ . We then have the following result.

**Theorem 5** ([10, 12]). *For all  $A, q, f, e$  with  $q$  a mean-zero incompressible periodic flow on  $\mathbb{R}^2$  and  $f$  any reaction, there exist  $C_1(f), C_2(f) > 0$  such that*

$$C_1(f)\sqrt{D_e(Aq)} \leq c_e^*(Aq, f) \leq C_2(f)\sqrt{D_e(Aq)}. \quad (5)$$

We note that the effective diffusivity has been well-studied in the homogenization theory and, unlike the front speed, it is relatively easily computable for a given flow using the above formula. In particular, its strong flow asymptotics as  $A \rightarrow \infty$  are available (up to bounded factors) for different classes of periodic flows in two dimensions such as shear, cellular, cat-eye, and others. This leads to corresponding estimates on  $c_e^*(Aq, f)$  via Theorem 5.

In addition, (5) can be used to provide a sharp characterization of those periodic flow profiles in two dimensions which are able to arbitrarily speed up fronts as their amplitude grows.

**Theorem 6** ([10,12]). *Let  $q$  be a mean-zero incompressible periodic flow on  $\mathbb{R}^2$  with a cell of periodicity  $Q$  and let  $f$  be any reaction. Then*

$$\lim_{A \rightarrow \infty} c_e^*(Aq, f) = \infty \tag{6}$$

*if and only if there is no  $w \in H^1(Q)$  satisfying  $q \cdot \nabla w = q \cdot e$ .*

In particular, this property is inherent to the flow and independent of the reaction or its type. Moreover, [12] also shows that (6) is equivalent to  $q$  being weakly quenching.

Although significant progress has been achieved in the study of periodic media, until recently very little has been known about traveling fronts in general disordered media. The reason is that in the former case, the fronts have a profile that is time-periodic in a frame moving at some front speed  $c > 0$ , which provides an ansatz that transforms the parabolic PDE into a more tractable degenerate elliptic PDE. The first results for disordered media were proofs of existence and uniqueness of fronts for the simple 1D inhomogeneous model

$$u_t = u_{xx} + g(x)f_0(u), \tag{7}$$

with  $f_0$  an ignition reaction and  $g$  bounded away from 0 and  $\infty$ . Mellet-Nolen-Roquejoffre-Ryzhik-Sire proved that up to a translation in time, there is a unique traveling front moving to the right (and another to the left) and that it is globally stable. That is, *front-like* initial data (converging to 1 as  $x \rightarrow -\infty$  and exponentially quickly to 0 as  $x \rightarrow \infty$ ) approach some time-translate of this front.

The methods of this work are, however, specific to (7). They only apply in one dimension, with no flow present, and critically depend on the ignition temperature being independent of  $x$  (whence the choice  $f(x, u) = g(x)f_0(u)$ ). In [13] I have developed a new method that applies to (1) on general cylindrical domains in  $\mathbb{R}^n$  and with general ignition reactions  $f$ .

**Theorem 7** ([13]). *Let  $q$  be a mean-zero incompressible periodic flow on  $D = \mathbb{R} \times \Omega \subset \mathbb{R}^n$  with  $\Omega$  bounded and let  $f$  be a (non-periodic) reaction satisfying  $f_0(u) \leq f(x, u) \leq f_1(u)$  for some ignition reactions  $f_0 \leq f_1$  with possibly different ignition temperatures. Then there exists a unique (up to a translation in time) traveling front for (1) moving to the right (and another moving to the left). Moreover, the front is increasing in time, globally stable, and solutions with compactly supported initial data which spread form two wave-fronts moving right and left, each approaching some time translate of the corresponding right- and left-moving front.*

Thus behavior of very general solutions of the PDE is described in this result. In addition, the existence part also applies to some monostable reactions  $f$  (with  $f_1$  monostable rather than

ignition), provided  $f_1'(0)$  is not too large relative to  $f_0$  (these fronts are, however, not unique even in homogeneous media). The bound on  $f_1'(0)$  is, in fact, essential — Roquejoffre and I have constructed examples of media with  $\frac{\partial f}{\partial u}(x, 0)$  allowed to be large for some  $x$ , for which no fronts exist [14].

Finally, note that in general there is no well-defined front speed in disordered media and the position of the front can move with a (time-dependent) rate  $c(t) \in [c_0, c_1]$ , where  $c_0, c_1$  are the speeds of the (unique right-moving) fronts for  $f_0, f_1$ . Nevertheless, it is proved in [13] that if  $f$  is a stationary ergodic reaction, then fronts have almost surely a deterministic speed  $c^* \in [c_0, c_1]$ .

**Future Directions.** I will now briefly describe some of the problems I am currently working on or plan to address in the near future. A natural question is a generalization of Theorem 7 to domains unbounded in more than one direction (such as the whole space). Fronts are not unique in this setting, even for homogeneous ignition reactions, but one might still hope for existence of fronts and their stability with respect to small perturbations. Also of interest are estimates on the deviation of the front profile from a hyperplane normal to the direction of its propagation, and convergence of general solutions to fronts. Further questions in this vein, with a close relation to homogenization of random reaction-diffusion equations, include existence of a deterministic front speed for random reactions and its dependence on the direction of the propagation of the front.

Another very interesting and important case is that of *active combustion*, with a feedback of reaction on the advecting flow  $q$ . This can be modeled by the Navier-Stokes-Boussinesq system, where  $q$  is not prescribed but rather a solution of the equations of fluid dynamics, driven by the buoyancy force caused by the gravity-induced rise of light hot matter and sinking of dense cold one. The resulting motion can have dramatic effects on the propagation of combustion such as speed-up of fronts and enhanced quenching due to increased mixing. Many natural and deep questions arise and some should be accessible by current techniques, at least in two dimensions.

Finally, traveling fronts in combustion models are a particular instance of an invasion of one equilibrium of an evolution (e.g.,  $u \equiv 0$ ) by another ( $u \equiv 1$ ). Such phenomena occur in other models which can be approached via techniques developed to study reaction-diffusion fronts. One such example is the spatial Smoluchowski equation which arises in the description of nematic liquid crystalline polymers. My recent work with Constantin [15] focused on identification of equilibrium solutions of this equation. The study of transitions between them will be a natural next step.

## Spectral Theory and Orthogonal Polynomials

**Background.** My research in spectral theory has focused on the direct and inverse spectral problems for Schrödinger operators and orthogonal polynomials. The free discrete Schrödinger operator  $J_0$  on the halfline is the semi-infinite matrix with 1 on off-diagonals  $(n-1, n)$ ,  $(n, n-1)$  and 0 elsewhere (the discrete Laplacian plus twice identity), acting on the Hilbert space  $\ell^2(0, 1, \dots)$ . If the ones are replaced by a sequence  $a_n > 0$  and the diagonal zeros by  $b_n \in \mathbb{R}$ , then one obtains a (tri-diagonal symmetric) Jacobi matrix  $J$ . Let  $\mu$  be the spectral measure of the vector  $\delta_0 = (1, 0, 0, \dots)$  for  $J$  and let  $P_n(x)$  be the orthogonal polynomials (on the real line –

OPRL) for  $\mu$ . One then has the recursion relation  $xP_n(x) = a_{n+1}P_{n+1}(x) + b_nP_n(x) + a_nP_{n-1}(x)$  with  $P_{-1} \equiv 0$  (i.e.,  $\{P_n(x)\}_{n=0}^\infty$  is the generalized Dirichlet eigenfunction of  $J$  at energy  $x$ ). This sets up a 1-to-1 correspondence between probability measures on  $\mathbb{R}$  (with finite moments) and Jacobi matrices. Determination of properties of one from the other (in this setting as well as in that of continuous Schrödinger operators) has been one of the main directions of research in mathematical physics.

**Past Research.** In the paper [16] I have constructed a class of bounded Jacobi matrices which have purely singular continuous spectra with fractional Hausdorff dimensions. Most of my other research has focused on Jacobi matrices which are compact perturbations of  $J_0$ . That is  $a_n \rightarrow 1$  and  $b_n \rightarrow 0$ , which also means that the essential spectrum of  $J$  is the interval  $[-2, 2]$ . At its center are the *sum rules*, also called trace formulae, relations involving the spectral measure on one side and the  $a_n, b_n$  on the other. They appeared first time in the work of Case, and saw a recent surge of interest following works of Deift-Killip and Killip-Simon. In [17] we extend these works by developing a new method which is used to prove, in full generality, the following Szegő sum rule.

**Theorem 8** ([17]). *If  $a_n \rightarrow 1$ ,  $b_n \rightarrow 0$ ,  $\mu'$  is the density of the a.c. part of  $\mu$ , and  $E_k$  are the eigenvalues of  $J$  outside  $[-2, 2]$ , then*

$$\sum_n \log(a_n) = \frac{1}{2\pi} \int_{-2}^2 \log\left(\frac{2\pi\mu'(x)}{\sqrt{4-x^2}}\right) \frac{dx}{\sqrt{4-x^2}} + \sum_k \log\left(\frac{|E_k| + \sqrt{E_k^2 - 4}}{2}\right). \quad (8)$$

The quantities in (8) may be infinite and this result implies, in particular, that if two of the quantities in the sum rule are finite, then so is the third. Since the integral in (8) is essentially the Szegő integral  $Z(J) = \int_{-2}^2 \log(\mu'(x))(4-x^2)^{-1/2} dx$  and the eigenvalue sum is equiconvergent with the Lieb-Thirring sum  $E(J) = \sum_k \sqrt{|E_k| - 2}$ , the sum rule is a means to relate these important quantities to the matrix coefficients. Its applications include a characterization of  $J$  that obey the Szegő condition  $Z(J) > -\infty$  under the a priori assumption  $E(J) < \infty$  (one always has  $Z(J) < \infty$ ), as well as various general necessary conditions for the validity of the Szegő condition.

In [18] I addressed a conjecture of Askey from 1979 on the validity of the Szegő condition for Coulomb (i.e., of order  $n^{-1}$ ) perturbations of  $J_0$ . I showed that whether the condition holds depends on the ratio of the coupling constants of the diagonal and off-diagonal perturbations, and was even able to accommodate additional  $O(n^{-1-\varepsilon})$  errors by deriving general sufficient conditions for the finiteness of the Szegő integral. This was possible by obtaining simultaneous control of the movement of an infinite number of eigenvalues under certain finite rank perturbations. The results also extend to other power-law perturbations of the free matrix. The paper [19] extends the methods from the previous works and derives an infinite family of sum rules, involving Szegő-like integrals with different weights and Lieb-Thirring sums with different powers. The sum rules yield various necessary conditions for these quantities to be finite.

The papers [20, 21] deal with similar questions about the relation of a measure and the corresponding coefficients for orthogonal polynomials on the unit circle (OPUC). In this setting  $\mu$  is a probability measure on the unit circle  $\partial\mathbb{D}$ , with orthogonal polynomials  $\Phi_n(z)$  and the

role of the  $a_n, b_n$  is played by the Verblunsky coefficients  $\alpha_n \in \mathbb{D}$ . The first OPUC sum rule goes back to the works of Szegő and Verblunsky, and it shows that  $\log \mu'(\theta) \in L^1(\partial\mathbb{D})$  if and only if  $\alpha_n \in \ell^2$ . Further developments along these lines appeared only recently. The main question is how the sum rule changes when a weight is included in the corresponding Szegő integral, and which  $\alpha_n$  correspond to the integral being finite. In particular, Simon stated in his OPUC book a conjecture pertaining to the case of the weight being a positive trigonometric polynomial and proved it when its degree is 1. In [20] we settle the case of a general degree 2 polynomial. A typical result is that  $(1 - \cos \theta)^2 \log \mu'(\theta) \in L^1(\partial\mathbb{D})$  if and only if  $\alpha_{n+2} - 2\alpha_{n+1} + \alpha_n \in \ell^2$  and  $\alpha_n \in \ell^6$ .

In the paper [21] we address the case of a general positive trigonometric weight and derive the corresponding sum rules. The hardest task in the business is, however, to identify clean  $\ell^p$ -type conditions on the  $\alpha_n$  that are equivalent to the convergence of the relevant sums of coefficients. We are able to do this under the a priori assumption  $\alpha_n \in \ell^4$ .

**Theorem 9** ([21]). *Let  $Q(z) \equiv \sum_{m=0}^M q_m z^m$  be a polynomial,  $\bar{Q}(z) \equiv \sum_{m=0}^M \bar{q}_m z^m$ , and let  $S$  the left shift operator on sequences. Assuming  $\alpha_n \in \ell^4$ , we have*

$$|Q(e^{i\theta})|^2 \log \mu'(\theta) \in L^1(\partial\mathbb{D}) \Leftrightarrow \{\bar{Q}(S)\alpha\}_n \in \ell^2 \quad (9)$$

Our method rests on a surprising new result that is at least as interesting and important as this characterization. Despite the long history of the subject, we provide for the first time exact formulae for the coefficients of orthogonal polynomials in terms of the Verblunsky coefficients  $\alpha_n$ . We also use these to obtain the Fourier coefficients of  $\log \mu'(\theta)$  whenever this function belongs to  $L^1(\partial\mathbb{D})$ , which is the key step in the approach to the sum rules. Finally, we derive new results on the ratio asymptotics of the reversed polynomials  $\Phi_{n+1}^*/\Phi_n^*$ .

**Future Directions.** A natural open question is the general problem of making sense of the coefficient sums in higher-order sum rules. That is, to identify  $\ell^p$  conditions on the  $a_n, b_n$  or the  $\alpha_n$  that are equivalent to the coefficient sums in the sum rules being finite. In the OPUC case there is a hope for further progress in a close examination of the abovementioned new formulae for the coefficients of the orthogonal polynomials. In the OPRL case such formulae are presently not available, and deriving them might be the starting point of a possible approach. The ultimate goal is the characterization, in terms of the corresponding Verblunsky coefficients, of classes of measures whose logarithm is integrable with respect to certain weights (and which satisfy certain Lieb-Thirring conditions in the OPRL case). Further questions involve more general weights (non-polynomial) as well as weights vanishing on an infinite set of points (e.g., an interval or a collection of intervals).

Other related problems include the study of the absolutely continuous spectrum of Schrödinger operators with matrix- and operator-valued potentials (with the goal of obtaining applications to operators in higher dimensions). In the recent years both of these problems have been successfully approached via techniques involving estimates on the integrals of  $\log \mu'(x)$ , with  $\mu$  a relevant spectral measure, and so further progress in this area might be achieved by extending these methods.

## Other Works: Fluid Dynamics and Graph Theory

The paper [22] studies two discrete models of the Euler equations of incompressible fluid dynamics. These are infinite systems of coupled ODEs for functions which can be thought of as wavelet coefficients of the fluid velocity at various scales. One of these models has been proposed and studied by Katz-Pavlović, the second has been recently discussed by Waleffe and goes back to Obukhov's studies of the energy cascade in developed turbulence. These are the only basic models of this type sharing some natural scaling and conservation conditions with the Euler equations. Blowup of general solutions has been shown for the Katz-Pavlović model, which promotes energy transfer from larger to smaller scales, while global regularity has been proved for the Obukhov model, which promotes energy transfer in the opposite direction.

A natural starting point for future research along these lines are questions of blowup and regularity for Navier-Stokes analogs of these models. Global regularity might be possible in both cases due to dissipation, caused by the added Laplacian-like term.

The papers [23–26] study embeddings of graphs in manifolds. Among other results, a characterization of graphs which admit regular embeddings has been obtained.

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