Ergodic Theory & Smooth Dynamics, II: Geometric Rigidity & Smooth Dynamics

Goal: Illustrate how tools from dynamical systems can be brought to bear on questions in differential geometry.

* Manifolds without conjugate points
* Negative curvature and marked length spectrum
A metric on $\mathcal{M}$ has no conjugate points if for $x, y \in \mathcal{M}$ there exists a unique geodesic connecting $x$ to $y$.

**Examples**

- $\mathcal{M}$ is nonpositively curved. (Ricci comparison)
  
- Gulliver's $3$ surfaces without conjugate pts (not nonpositively curved)
  
(Q: Does every closed $\mathcal{M}$ without conj. pts admit a metric of nonpos. curvature?)
Theorem (Hopf) Any metric on $\mathbb{T}^2$ without conjugate points is flat.

I will present:

Theorem (Burago - Ivanov, '94) The same holds on $\mathbb{T}^n$. 

Examples w/ Anosov geodesic flow
Marked length spectrum

\[ M = \text{manifold, negative sectional curvature}. \]

\[ \Rightarrow \text{In each free homotopy class of closed curve}\]

\[ \exists \text{! closed geodesic}\]

defines a function

\[ l : \pi_1(M) / \sim \to \mathbb{R}_+ \]

Marked length spectrum

Thm (Otal, Groves) if \( S \times S' \) are negatively
linked metrics or $\Sigma$ with the same marked length spectrum, then $S$ and $S'$ are isometric.
The role of Dynamics

\( M^n = \text{Riem. manifold} \)

\( T^1M = \text{unit tangent bundle} \)

\( \psi: T^1M \times \mathbb{R} \to T^1M \)

Godesic flow

\( \psi_\tau(x) = \dot{x}(\tau) \)

Solution curves to a time-independent ODE.
\[ \Phi_0 = \text{Id} \]
\[ \Phi_{s+t} = \Phi_s \cdot \Phi_t. \]

**Continuous dynamical system**

**Invariant structures:**
- contact 1-form \( \alpha \)
- \( \Delta \) transverse symplectic structure \( da \)
- volume \( \alpha \wedge (da)^{n-1} \)

- In negative curvature: stable & unstable distributions (families of Jacobi fields)
Outline

I. Crash Course in Geodesic flows
   - Symplectic & Riemann Structures on TM, Adapted Coordinate Systems
   - Contact Structure on TM
   - Jacobi fields
   - Conjugate & focal pts
   - Busemann functions
   - the Anosov Property in negative curvature

II. Crash Course in Ergodicity
III

POWRENCÉ, recurrence
the ergodic theorem
and criteria for ergodicity

Crash course in Anosov flows.

- Shadowing
- Structural Stability
- Ergodicity
- Cohomological equations.

IV
Marked Length Spectrum Rigidity

V
The Hopf Conjecture
I. Crash Course in Geodesic Flows

\[ M = \text{Riem mfld. (C^\infty)} \]
\[ \langle , \rangle = \text{metric} \]
\[ \nabla = \text{covariant differentiation operator (connection)} \]
\[ \nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M) \]

1) Torsionless:
\[ \nabla_{fX} Y = f \nabla_X Y \]

2) Leibniz:
\[ \nabla_X (fY) = df(X)Y + f \nabla_X Y \]
3) Metric-preserving:
\[
\mathcal{L}_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle
\]

4) Symmetry:
\[
\nabla_X Y - \nabla_Y X = [X, Y]
\]

Def. A vector field \( X \) along a curve \( c(t) \) is parallel
If \( \nabla_{c'}(t)X \equiv 0 \).

(* small technical issue in defining LHS)

A curve \( Y \) in \( M \) is a geodesic (arc, ray...)

If \( \nabla_{Y'} Y' \equiv 0 \) on its domain of definition.

The initial value problem:

\[
X'(0) = u \quad u \in TM
\]

\[
X''(t) := \nabla_{X'}X'(t) = 0
\]
has a unique solution \( x = \Phi u \), defined \( \forall \cdot t \in \mathbb{R} \), if the curvatures of \( \langle \cdot \rangle \) are bounded by the existence and uniqueness of solutions to ODEs.

Hence we have a flow

\[
\Phi : TM \times \mathbb{R} \rightarrow TM \\
\Phi_t(\nu) = \nu'(t) \\
\Phi_0 = Id, \quad \Phi_{s+t} = \Phi_s \circ \Phi_t
\]
**Def** \( \Phi_t \) is called the geodesic flow for \( M \) (on \( TM \)).

**Prop** The geodesic flow preserves length, i.e. \( t \to \| \Phi_t(u) \| \) is const.

**Proof**

\[
\frac{d}{dt} \langle \Phi_t(u), \Phi_t(u) \rangle = L_{\Phi_t(u)} \langle \Phi_t(u), \Phi_t(u) \rangle = \langle \Phi_t(u), \nabla_{\Phi_t(u)} \Phi_t(u) \rangle = 0
\]
for each $t$, the map

$N \mapsto \Phi^t(N)$

$TM \mapsto TM$

is a diffeomorphism, called the time-$t$ map of $\Phi$.

Special coordinates on $T(TM)$.

The connection $\nabla$ defines a distribution $H \subseteq T(TM)$.
defined by: for \( v \in TM \),

\[ H(v) = \{ \exists z \in TuTM : \nabla_{DuT}(z) = 0 \} \]

where \( \Pi : TM \rightarrow M \).

\( H \) is called the horizontal bundle of \( TTM \). Note:

\[ \text{D}v : H(v) \cong T_{\Pi(v)}M \]

Let \( K_v : TuTM \rightarrow \ker D\Pi \)
be the projection with $\ker k_u = H(n)$

Hence we have an isomorphism:

$\kappa_u \times D_uT : T_uTM \to \ker D_uT + \overline{T_{\pi(n)} M}$

But note:

$\ker D_uT = T_u(\ker \pi)$

$= T_u(\overline{T_{\pi(n)} M})$

$\subseteq T_{\pi(n)} M$

natural isomorphism
Hence we have shown:

**Proposition:** The connection determines an isomorphism

\[ TuTM \cong \prod_{\pi(u)} M \times \prod_{\pi'(u)} M \]

\[ \forall u \in TM. \text{ In other words:} \]

\[ TTM = TM \times (TM \otimes TM) \]

(bundle product over \( n \)).

**Exercise:** Show that if \( \beta(t) \in TM \) is a curve, \( b(t) = \prod_{\pi} \beta(t) \)

then (in these coordinates)

\[ \beta' = (\beta, b', \nabla_{b'} \beta) \]
Structures on TM

- Riemann structure (called the Sasaki metric)
  For $\tilde{\xi}_i = (v_i, w_i) \in T_u TM$,
  Let $i = 1, 2,$

  $\langle \tilde{\xi}_1, \tilde{\xi}_2 \rangle_u =$

  $\langle v_1, w_1 \rangle_{\Pi(u)} + \langle v_2, w_2 \rangle_{\Pi(u)}$

  (sum $\delta \not\prec \not\prec \not\prec \not\prec$ on horizontal + vertical bundles)
1-form $\alpha$:

for $\xi = (u,v,w) \in T_u TM$, set

$$\alpha(\xi) = - \langle u, v \rangle$$

Symplectic form $d\alpha$:

$\xi_i = (u_i, v_i, w_i) \in T_u TM$:

$$d\alpha(\xi_i, \xi_j) = \langle v_i, w_j \rangle - \langle w_i, v_j \rangle$$

**Exercise**: Do $\alpha$ as claimed and is nondegenerate.
Geodesic Spray \( \mathfrak{g} \)

For \( u \in TM \), let
\[
\mathfrak{g}(u) = (u, u, 0) \in T_u TM
\]

**Exercise** The vector field \( \mathfrak{g} \in \mathcal{X}(TM) \) generates the geodesic flow: that is:
\[
\forall u \in TM, \forall t_0 \in \mathbb{R}:
\]
\[
\dot{\mathfrak{g}}(\mathfrak{g}_t^0(u)) = \frac{d}{dt} \mathfrak{g}_t^0(u) \bigg|_{t=t_0}
\]

Properties of these structures

1) \( \alpha(\mathfrak{g}(u)) = \langle u, u \rangle = -\|u\|^2 \)
2) If $\xi = (u, v, w) \in T_u M$, then
\[
d\alpha (\Phi (u), \xi) = \langle u, w \rangle.
\]

**Proposition** The geodesic flow $\Phi^t$ preserves $d\alpha$:
\[
(\Phi^t)^* d\alpha = d\alpha \quad \forall t \in \mathbb{R}
\]

**Proof:**
\[
\frac{d}{dt} (\Phi^t)^* d\alpha = L_{\dot{\Phi}} d\alpha
\]
\[
= i_{\dot{\Phi}} (d^2 \alpha) + d(i_{\dot{\Phi}} d\alpha)
\]
\[
= 0 \quad \text{(by 2: exercise)}
\]
3) Let \( \beta(t) \) be a curve in \( \mathbb{T} \text{M} = \{u \mid u(0) = t\} \).

Let \( b(t) = \Pi \circ \beta(t) \) then

\[
\mathbf{O} = \frac{d}{dt} \langle b(t), \beta'(t) \rangle = \sum_{t} b(t) \left< \beta(t), \beta'(t) \right> = 2 \cdot \left< \beta(t), \nabla_{b'} b(t) \right>
\]

\[
\mathbf{T}(\mathbb{T} \text{M}) = \{ (u, v, w) : \|u\| = 1, \langle u, w \rangle = 0 \}
\]

**Proposition:** The restriction \( \Phi \) to the unit tangent bundle \( \text{T} \text{M} \) preserves the 1-form \( \alpha \):
Let $\alpha \in \mathcal{A}_R$

\[ \varphi^*(\alpha|_{\mathcal{T}^1M}) = \alpha|_{\mathcal{T}^1M} \]

(Write $\alpha$ for $\alpha|_{\mathcal{T}^1M}$ in the sequel).

Moreover, $\alpha$ is a contact form on $\mathcal{T}^1M$, meaning $|\alpha \wedge (d\alpha)^n|$ defines a volume on $\mathcal{T}^1M$.

Thus $\varphi$ preserves the 

**Hermite volume**:

\[ \lambda = |\alpha \wedge (d\alpha)^n| \]

**Proof Exercise**
Summary

$\varphi_t : T'M \rightarrow T'M$ preserves:

- Contact 1-form $\alpha$
  (satisfying $\alpha \wedge (d\alpha)^n$ nonvanishing)
  $\alpha (\varphi_t) = -1$.

$\ker \alpha = \varphi_t^{-1}$

$= \{ (u,v,w) : \| u \| = 1 \ ; \ \langle u,v \rangle = \langle u,w \rangle = 0 \}$

$n$-dim'le subspace $\varphi_t^{-1}(u) = \ker \alpha$

Picture in $T_u T'M$

Geodesic flow direction $\dot{\varphi}(u)$

- Liouville volume given by form $\lambda = \alpha \wedge (d\alpha)^n$
This volume form defines a Lebesgue measure \( m \) on \( T'M \) by:

\[
m(A) = \int_A \nu
\]

(\( A \subseteq T'M \) any Borel set) \( M \) compact \( \Rightarrow m(T'M) < \infty \).

**Corollary (Poincare Recurrence)**

Suppose \( m(T'M) < \infty \). Then for \( m \)-a.e. \( u \in T'M \):

\[
\liminf_{t \to \pm \infty} d(\Phi_t(u), u) = 0
\]

(in the Sasaki metric)
Proof: Exercise
Variations of Geodesics and Jacobi Fields

We would like to determine the effect of the derivative of the geodesic flow, equivalently, the action of $\mathfrak{g}$ on curves in $TM$, up to first order. Let $\mathfrak{g} \in T_uTM$
\[ \vec{\xi} = (u_0, v_0, w_0); \quad v_0, w_0 \in \pi_{u_0}(M) \]

Let \( X(s) \) be any curve in \( TM \) tangent to \( \vec{\xi} \) at \( s = 0 \), and let \( \alpha(s) = \Pi \circ X(s) \).

Then
\[
U_0 = X(0), \quad V_0 = \frac{\partial \alpha}{\partial s} \bigg|_{s=0}, \quad W_0 = \nabla_{V_0} X \bigg|_{s=0}.
\]

Extend to a function
\[ \alpha : \mathbb{R} \times \mathbb{R} \rightarrow M \]
by
\[ \alpha(s, t) = X_{X(s)}(t) \]
\( \alpha \) is called a variation of geodesics. Notice that

\[
\alpha(s, t) = \Pi(\varphi_t(\alpha(s)))
\]

Hence the vector field

\[
X(s) = \frac{d\alpha}{dt}(s, t_0)
\]

along the curve \( \alpha(s, t_0) \) is the image of \( X(s) \) under the time-\( t_0 \) map \( \varphi_t \).
This implies that:

$$D\Phi(t) = \left( \frac{\partial \Phi}{\partial t}(0, t), \frac{\partial \Phi}{\partial s}(0, t), \frac{\partial^2 \Phi}{\partial t^2}(0, t) \right)$$

where

$$\frac{\partial^2 \Phi}{\partial t \partial s} := \nabla \frac{\partial \Phi}{\partial t} \frac{\partial \Phi}{\partial s}$$

$$= \nabla \frac{\partial \Phi}{\partial t} \frac{\partial \Phi}{\partial s}$$

Let

$$J(t) = \frac{\partial \Phi}{\partial s}(0, t).$$

J is a vector field along the geodesic $\gamma(t) = \Phi(0, t)$, called a Jacobi field.
Proposition \( J \) is a Jacobi field along \( Y(t) \) iff:

\[
J''(t) = -R(J(t), \dot{Y}(t)) \dot{Y}(t)
\]

Where \( R \) is the curvature tensor: for vector fields \( A, B \) on \( M \),

\[
R(A, B) = \nabla_A \nabla_B - \nabla_B \nabla_A - [A, B]
\]

\( R(A, B) \) is called the Jacobi equation (along \( Y(t) \)).
\[ \overrightarrow{J(0)} \rightarrow \overrightarrow{J'(0)} \rightarrow J(t) \]

Remark: If \( J(t) \) is a Jacobi field along the geodesic \( \gamma(t) \), then \( D^t_{\gamma_t}(\gamma(0), J(0), J'(0)) = (\gamma(t), J(t), J'(t)) \).

The initial conditions \( \dot{\gamma}(0) = u_0, \ J(0) = v_0, \ J'(0) = w_0 \) uniquely determine the Jacobi field \( J \) (down \( u_0 \)).
Proof. Let \( \alpha(s,t) \) be a variation of geodesics, and let \( J_s(t) = \frac{\partial \alpha}{\partial s}(s,t) \), for fixed \( s \).

\[
J'_s(t) := \nabla_{\frac{\partial \alpha}{\partial t}} \frac{\partial \alpha}{\partial s}(s,t)
\]

\[
= \nabla_{\frac{\partial}{\partial s}} \frac{\partial \alpha}{\partial \alpha}(s,t)
\]

\[
J''_s(t) := \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial \alpha}{\partial s}} \frac{\partial \alpha}{\partial \alpha}(s,t)
\]

\[= \nabla_B \nabla_A \nabla_{B \circ A} \]
Where \[ A = \frac{2\alpha}{\beta t} (s,t) \quad B = \frac{2\alpha}{\beta t} (s,t) \]

But
\[
\nabla_B \nabla_A B = \nabla_A \nabla_B B - R(A,B)B
\]

- \[ \nabla_{[A,B]} B \],
by definition of the curvature tensor.

Observe that:
\[ [A,B] = \left[ \frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right] = 0 \]
\[ \nabla_B B = \ddot{x}_s = 0 \]

\[ \nabla_B \nabla_A B = - R(A,B)B, \]

i.e.
\[ \mathcal{J}_s''(t) = - R \left( \mathcal{J}_s(t), \dot{x}(A) \dot{x}(t) \right) \]
Properties of Jacobi fields

1. If \( J(0) \perp \dot{x}(0) \) and \( J'(0) \perp \dot{x}(0) \), then \( J(t) \perp \dot{x}(t) \), \( J'(t) \perp \dot{x}(t) \) \( \forall t \)

(Proof: \( \Phi \) preserves \( \ker \alpha = \Phi(t) \) )

\( J \) is called a perpendicular Jacobi field.

2. Similarly, any field \( J(t) = (at+b)\dot{x}(t) \) of the form is a Jacobi field (exercise) called a tangent Jacobi field.
3. Let
\[ K(A, B) = \frac{\langle R(A, B)B, A \rangle}{\| A \wedge B \|^2} \]
be the sectional curvature of the plane spanned by \( A \) & \( B \). If \( K \leq 0 \) along \( y \), then for Jacobi field, we have
\[
(\| J' \|^2)' = 2\langle J, J' \rangle
\]
& \( (\| J' \|^2)'' \geq 2\| J' \|^2 \)

Proof:
\[
\langle J, J \rangle = (2\langle J, J' \rangle)'
= 2 \| J' \|^2 + 2\langle J, J' \rangle
\]
= 2\|J\|^2 - 2\langle R(J, \delta) \delta, J \rangle.
\geq 2\|J\|^2
\]

**Corollary:** If \( K \leq 0 \) along \( \delta \) & \( J \) is a nontrivial field (along \( \delta \), then \( J(t) = 0 \) for at most one value of \( t \).

*(Proof:)* \( J(t) = 0 \Rightarrow (\|J\|^2)' = 0 \)
\& \( (\|J\|^2)'' \geq 0 \)
\Rightarrow \( t \) is a local min for \( \|J\|^2 \). But \( (\|J\|^2)'' \geq 0 \)
\Rightarrow \( \|J\|^2 \) has at most one local max.)
Conjugate Points

**Def:** Say that \( p, q \in M \) are **conjugate points** if \( \gamma \) is a (unit speed) geodesic with \( \gamma(0) = p \), \( \gamma(t) = q \) and a nontrivial Jacobi field \( J \) along \( \gamma \) with \( J(0) = J(t) = 0 \).

**Exercise:** If \( p \) & \( q \) are conjugate points, then \( \exists \xi, \eta \) geodesics \( \xi, \eta \) \( \xi_t = \delta, \eta_t \) from \( p \) to \( q \):

\[
\begin{align*}
\xi_t & = \delta, \\
\eta_t & = 0
\end{align*}
\]
(Hint: Implicit Function Thm)

2) The following are equivalent:

   (i) \( M \) has no conjugate points.
   (ii) \( \forall x \in \tilde{M}, \) the exponential map \( \exp_x : T_x \tilde{M} \to \tilde{M} \) is a diffeomorphism.
   (iii) \( \forall x, y \in \tilde{M}, \exists \) a geodesic segment from \( x \) to \( y \).

(Hint: \( J(t) = D_{tv} \exp_p(tw) \) is a Jacobi field.)
The preceding discussion implies:

**Theorem** If $M$ is nonpositively curved (i.e. $K \leq 0$ everywhere), then $M$ has no conjugate points.

("Cartan-Hadamard Thin")