We have developed the calculus of differential forms algebraically, focusing on algebraic manipulations which can be used to calculate the integrals and formulate the basic Stokes's theorem. In doing so, we have never needed to do geometry, i.e. we have never talked about lengths of vectors or angles. (In technical terms, we have worked on a $C^\infty$-manifold, not a Riemannian manifold.)

Recall in $\mathbb{R}^n$ we have the notions of length of a vector and of angle between two vectors. Namely, the length of $v = (v_1, \ldots, v_n)$ is defined to be $|v| := (v_1^2 + \cdots + v_n^2)^{1/2} = (v \cdot v)^{1/2}$. The angle $\theta$ between vectors $v$ and $w$ is determined by

$$\cos \theta = \frac{v \cdot w}{|v||w|}; \quad v \cdot w := v_1w_1 + v_2w_2 + \cdots + v_nw_n.$$

To a differential 1-form $\omega = f_1dx_1 + \cdots + f_n dx_n$ on $\mathbb{R}^n$ we associate the $n$-tuple of functions $(f_1, \ldots, f_n)$ which we interpret geometrically as a vector field on $\mathbb{R}^n$. A vector field assigns to each point $x^0 := (x_1^0, \ldots, x_n^0) \in \mathbb{R}^n$ a vector, namely $(f_1(x^0), \ldots, f_n(x^0))$. We can picture a vector field as simply a collection of arrows emanating from points in $\mathbb{R}^n$. Here are two examples in $\mathbb{R}^2$ with coordinates $x, y$.

**Exercise 1.** (i) Sketch the vector fields $(x, y)$ and $(x + 1, y - 2)$ in $\mathbb{R}^2$.

(ii) Suppose $xf(x, y, z) + yg(x, y, z) + zh(x, y, z) = 0$. Sketch the vector field $(f, g, h)$ in $\mathbb{R}^3$. 

![Vector field (y, -x)](image1)

![Vector field (y, +x)](image2)
An important example of a vector field is the tangent vector field along a path. Suppose \( \phi : I \to \mathbb{R}^n \) is given by \( \phi(t) = (x_1(t), \ldots, x_n(t)) \). Then \( \frac{d\phi}{dt} := (dx_1/dt(t), \ldots, dx_n/dt(t)) \) is the tangent vector field. Notice it is only defined along the path \( \phi \). Here is the picture in \( \mathbb{R}^2 \) when \( \phi(t) = (t, t^2) \).

![Tangent vector field to \( \phi(t) = (t, t^2) \)](image)

(Ignore the fact that the arrows seem to curve. This is an artifact of the computer program.)

We can now write our typical path integral computation

\[
\int_{\phi} \omega = \int_{\phi} f_1 dx_1 + \cdots + f_n dx_n = \\
\int_{a}^{b} (f_1(\phi(t))dx_1/dt + \cdots + f_n(\phi(t))dx_n(t)/dt)dt = \\
\int_{a}^{b} (f_1(\phi(t)), \ldots, f_n(\phi(t))) \cdot \overrightarrow{d\phi/dt}dt
\]

In words, we take the dot product of the vector field associated to the 1-form with the tangent vector field and integrate the resulting function.
of $t$ over $[a, b]$. The picture for $\omega = -ydx + xdy$ and $\phi(t) = (t, t^2)$ looks like

Here the path $\phi$ is in blue, the tangent vector field is in red, and the vector field associated to $\omega$ is in green. (I am quite proud of this illustration, by the way.)

The expression

$$\text{ds} := |\overrightarrow{d\phi/dt}|dt = ((dx_1/dt)^2 + \cdots + (dx_n/dt)^2)^{1/2}dt$$

is the traditional notation for the form computing arc length. From our point of view, this is confusing firstly because $\text{ds}$ is not $d$ of a function. Secondly, $\text{ds}$ depends on the path and is only defined along the path. Our usual situation has the 1-form independent of the path and defined at least in some neighborhood of the path. On the plus side, $\phi \mapsto \int_a^b \text{ds}$ is independent of the parametrization as a path integral should be.
Namely, writing \( t = \psi(u) \), we find

\[
\frac{ds}{dt} = ((dx_1/du)^2 + \cdots + (dx_n/du)^2)^{1/2} du = ((dx_1/du)^2 + \cdots + (dx_n/du)^2)^{1/2} \frac{d\psi}{du} du = ((dx_1(\psi(u))/du)^2 + \cdots + (dx_n(\psi(u))/du)^2)^{1/2} du
\]

Note that the bottom line is the expression for \( ds \) in the \( u \)-coordinate.

In Wade, the unit tangent vector field along \( \phi \) is written

\[
T := \frac{\overrightarrow{d\phi}/dt}{|\overrightarrow{d\phi}/dt|}
\]

The path integral becomes

\[
\int_{\phi} \omega = \int_{a}^{b} (f_1 \circ \phi, \ldots, f_n \circ \phi) \cdot T ds.
\]

**Exercise 2.** Compute the path lengths and the tangent vector fields for the following paths on \([0, 1]\):

\[
\phi(t) = (t, t^3); \quad \phi(t) = (t, t^2, t^3); \quad \phi(t) = (\cos(2\pi t), \sin(2\pi t)).
\]

You may leave the arc length integrals unevaluated if they seem difficult.

12. Surface integrals in \( \mathbb{R}^3 \)

There is a classical geometric construction called the cross product of vectors in \( \mathbb{R}^3 \). It is defined by

\[
\overrightarrow{x} \times \overrightarrow{y} := (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).
\]

**Lemma 3.** (i) \( \overrightarrow{x} \cdot (\overrightarrow{x} \times \overrightarrow{y}) = \overrightarrow{y} \cdot (\overrightarrow{x} \times \overrightarrow{y}) = 0 \).

(ii) \( |\overrightarrow{x} \times \overrightarrow{y}| \) equals the (unsigned) area of the parallelogram spanned by \( \overrightarrow{x} \) and \( \overrightarrow{y} \).

**Proof.** (i) is straightforward. Geometrically, this means that \( \overrightarrow{x} \times \overrightarrow{y} \) is perpendicular to the plane spanned by \( \overrightarrow{x} \) and \( \overrightarrow{y} \).

For (ii), note first that even to talk about area in \( \mathbb{R}^3 \) (as opposed to volume) is something intrinsically new for us. Technically, one would say that volume is invariant under the group of \( 3 \times 3 \) matrices with determinant \( \pm 1 \), while area is only invariant under the group \( O_3 \) of \( 3 \times 3 \)-matrices preserving lengths and angles. (This is called the orthogonal group.) Once we have lengths and angles, we can define the area of a parallelogram spanned by two vectors to be the volume of the solid parallelogram obtained by thickening the original figure taking the product with a normal vector of unit length. (“Normal” means perpendicular to \( \overrightarrow{x} \) and \( \overrightarrow{y} \).)
Write $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$. The key algebraic fact which you should check is

$$|\vec{x} \times \vec{y}|^2 = |\vec{x}|^2 |\vec{y}|^2 - (\vec{x} \cdot \vec{y})^2.$$ 

Once you have this, you can prove (ii) axiomatically. Show $\vec{x} \times \vec{x} = 0$ and $(\vec{x} + \vec{z}) \times \vec{y} = \vec{x} \times \vec{y} + \vec{z} \times \vec{y}$. Finally, show $|\vec{x} \times \vec{y}| = |\vec{x}| |\vec{y}|$ when $\vec{x}$ is perpendicular to $\vec{y}$. □

**Exercise 4.** Write out the proof of lemma 3 in full detail.

Consider now a surface integral $\iint \omega$ in $\mathbb{R}^3$. This means $\omega$ is a 2-form and $\varphi : I^2 \to \mathbb{R}^3$. Write

$$\omega = f_1 dx_2 \wedge dx_3 + f_2 dx_3 \wedge dx_1 + f_3 dx_1 \wedge dx_2.$$ 

Substituting as usual $dx_i = \frac{dx_i}{dt} dt + \frac{dx_i}{du} du$ we find

$$\iint \omega = \iint_{I^2} (f_1 \circ \varphi, f_2 \circ \varphi, f_3 \circ \varphi) \cdot \vec{N} dt \wedge du$$

Here

$$\vec{N} = \left( \frac{\partial x_2}{\partial t} \frac{\partial x_3}{\partial u} - \frac{\partial x_3}{\partial t} \frac{\partial x_2}{\partial u}, \frac{\partial x_3}{\partial t} \frac{\partial x_1}{\partial u} - \frac{\partial x_1}{\partial t} \frac{\partial x_3}{\partial u}, \frac{\partial x_1}{\partial t} \frac{\partial x_2}{\partial u} - \frac{\partial x_2}{\partial t} \frac{\partial x_1}{\partial u} \right) = (\frac{\partial x_1}{\partial t}, \frac{\partial x_2}{\partial t}, \frac{\partial x_3}{\partial t}) \times (\frac{\partial x_1}{\partial u}, \frac{\partial x_2}{\partial u}, \frac{\partial x_3}{\partial u}).$$
In the picture, the two red arrows are the tangents \((\partial x_1/\partial t, \partial x_2/\partial t, \partial x_3/\partial t)\) and \((\partial x_1/\partial u, \partial x_2/\partial u, \partial x_3/\partial u)\). The chartreuse arrow is the normal \(\vec{N}\).

Note that \(\vec{N}\) is perpendicular to the tangent vectors by lemma 3(i).

Following Wade, I write \(\vec{n} = \vec{N}/|\vec{N}|\) for the unit normal vector, and \(d\sigma := |\vec{N}|dtdu\).

**Exercise 5.** Argue from lemma 3(ii) that \(d\sigma\) gives a plausible definition for surface area in \(\mathbb{R}^3\), i.e. that one should define

\[
\text{Area}(\varphi(I^2)) := \iint_{I^2} d\sigma.
\]

Finally, our surface integral now reads (assuming we have oriented \(I^2\) so \(\int dt \wedge du > 0\))

\[
\iint_{I^2} \omega = \iint_{I^2} (f_1 \circ \varphi, f_2 \circ \varphi, f_3 \circ \varphi) \cdot \vec{n} d\sigma.
\]

**Exercise 6.** Define \(\varphi(t, u) = (\cos(t) \cos(u), \cos(t) \sin(u), \sin(t))\) for \(0 \leq t \leq \pi, \ 0 \leq u \leq 2\pi\). Write down the surface area integral for this surface. Compute \(\vec{n}(t, u)\) and draw the picture.
Exercise 7. Let \( \varphi(t,u) = (t,u,t^2 + u^2) \). Write down the surface area integral for this surface where \( 0 \leq t,u \leq 1 \). Compute \( \vec{n}(t,u) \) and draw the picture. Compute \( \iint \varphi \, dx \wedge dy + dy \wedge dz + dz \wedge dx \).