5. Path Integration

Heretofore we have only discussed integration over domains which are in some sense “of maximal dimension”. Thus, we have integrated functions over intervals in $\mathbb{R}$, over rectangles in $\mathbb{R}^2$, over cubes in $\mathbb{R}^3$ etc. Suppose we have a set $S \subset \mathbb{R}^n$ which has “dimension less than $n$”, for example perhaps approximating the volume of $S$ by small $n$-cubes gives 0 as the limit. Can we still integrate over $S$? The simplest example of such an $S$ is:

**Definition 1.** Let $U \subset \mathbb{R}^n$ be an open set. A path in $U$ is a map

$$\phi : [a,b] \rightarrow U; \quad \phi(t) = (\phi_1(t), \ldots, \phi_n(t)).$$

We assume $\phi$ is differentiable in the sense that the derivatives $d\phi_i/dt$ exist and are continuous.

**Example 2.** Let $\phi : [0,2\pi] \rightarrow \mathbb{R}^2; \quad \phi(t) = (\cos(t), \sin(t))$. Then $\phi$ is a path.

What sort of animal can we integrate over a path? Suppose first we are in $\mathbb{R}^1 = \mathbb{R}$. We have $\phi : [a,b] \rightarrow \mathbb{R}$. Let $x$ be the coordinate on $\mathbb{R}$ and let $f(x)$ be a continuous function. We may define

$$\int_\phi f(x)dx := \int_a^b f(\phi(t))\phi'(t)dt.$$  

What about path integrals in $\mathbb{R}^n$? We do the same thing! Consider an expression of the form

$$\omega := f_1(x_1, \ldots, x_n)dx_1 + f_2(x_1, \ldots, x_n)dx_2 + \ldots + f_n(x_1, \ldots, x_n)dx_n$$

Such a thing is called a differential 1-form. Given a path $\phi : [a,b] \rightarrow \mathbb{R}^n$ we define the path integral

$$\int_\phi \omega := \sum_{i=1}^n \int_a^b f_i(\phi_1(t), \ldots, \phi_n(t))\phi_i'(t)dt.$$  

Notice, I haven’t really defined a 1-form. One may say vaguely that a 1-form is “that which can be integrated against a path to give a number”. 
Example 3. Consider the 1-form \( \cos(y)dx + e^{xy}dy \) on \( \mathbb{R}^2 \). Its value on the path \( \phi : [0, 1] \to \mathbb{R}^2; \ x = t, \ y = t^2 \) is given by

\[
\int_{\phi} \cos(y)dx + e^{xy}dy = \int_{0}^{1} (\cos(t^2) + 2e^{t^3}t)dt.
\]

The crucial property of path integrals is their independence of parametrization.

Proposition 4. Let

\[
\omega := f_1(x_1, \ldots, x_n)dx_1 + f_2(x_1, \ldots, x_n)dx_2 + \ldots + f_n(x_1, \ldots, x_n)dx_n
\]

be a differential 1-form on an open set \( U \) in \( \mathbb{R}^n \). Let \( \phi : [a, b] \to U \) be a path. Let \( \psi : [\alpha, \beta] \to [a, b] \) with \( \psi \) differentiable, \( \psi(\alpha) = a, \ \psi(\beta) = b \). Let \( \theta = \phi \circ \psi : [\alpha, \beta] \to U \). Then

\[
\int_{\phi} \omega = \int_{\theta} \omega.
\]

Proof. Let \( u \) be the coordinate on \( [\alpha, \beta] \). We have

\[
\int_{\theta} \omega \overset{\text{def.}}{=} \int_{\alpha}^{\beta} \sum_{i=1}^{n} f_i(\theta_1(u), \ldots, \theta_n(u))\theta'_i(u)du
\]

(5.4)

\[
\overset{\text{why?}}{=} \int_{\alpha}^{\beta} \sum_{i=1}^{n} f_i(\phi_1(\psi(u)), \ldots, \phi_n(\psi(u)))\phi'_i(\psi(u))\psi'(u)du
\]

\[
\overset{\text{why?}}{=} \int_{\alpha}^{b} \sum_{i=1}^{n} f_i(\phi_1(t), \ldots, \phi_n(t))\phi'_i(t)dt =: \int_{\phi} \omega
\]

This proves the proposition.

Exercise 5. Compute the following path integrals \( \int_{\phi} \omega \):

i. \( \phi : [0, 1] \to \mathbb{R}^2; \ \phi(t) = (-t, t), \ \omega = dx_2 \).

ii. \( \phi : [0, 2\pi] \to \mathbb{R}^2; \ \phi(t) = (\cos(t), \sin(t)), \ \omega = x_1dx_2 - x_2dx_1 \).

iii. \( \phi : [a, b] \to \mathbb{R}^n; \ \phi(t) = (t, t^2, \ldots, t^n), \ \omega = \sum_{i=1}^{n} x_idx_{n+1-i} \).

The point of view that a 1-form is that which can be integrated over a path may seem a bit vague. But actually it is quite suggestive. For example, what relations should we impose on 1-forms. Suppose to start again we are on an open set \( U \subset \mathbb{R}^1 \). Let \( x \) and \( y \) be two different coordinate functions on \( U \). Suppose we have 1-forms \( \omega = f(x)dx \) and \( \tau = g(y)dy \) on \( U \). When should we say \( \omega = \tau \)? Well, our philosophy suggests \( \omega = \tau \) if and only if for every path \( \phi : [a, b] \to U \), we have \( \int_{\phi} \omega = \int_{\phi} \tau \). Now, the statement that \( x \) and \( y \) are both coordinates on
U means that e.g. \( y = \psi(x) \) with \( \psi'(x) \) nonvanishing on \( U \). If the path \( \phi \) is given by \( x = \phi(t) \), then \( y = \psi(\phi(t)) \). We have

\[
\int_{\phi} \tau := \int_{a}^{b} g(\psi(\phi(t))) \frac{d}{dt}(\psi(\phi(t)))dt = \int_{a}^{b} g(\psi(\phi(t))) \psi'(\phi(t))\phi'(t)dt
\]

\[
\int_{\phi} \omega = \int_{a}^{b} f(\phi(t))\phi'(t)dt
\]

The two path integrals are thus equal if

\[
g(\psi(\phi(t))) \psi'(\phi(t)) = g(\psi(\phi(t))) \psi'(\phi(t)).
\]

Substituting \( x = \phi(t) \) and \( y = \psi(x) \) we find

\[
(5.5) \quad f(x)dx = g(y)dy \quad \text{if} \quad f(x) = g(y(x)) \frac{dy}{dx}
\]

Put another way, we should impose the relation on 1-forms

\[
(5.6) \quad g(y(x)) \frac{dy}{dx} dx = g(y)dy.
\]

What about 1-forms in \( \mathbb{R}^2 \)? Suppose we have two sets of coordinates, say \( x_1, x_2 \) and \( y_1, y_2 \). We have \( y_i = y_i(x_1, x_2) \), and staring at (5.6) suggests the relations

\[
(5.7) \quad dy_1 = \frac{\partial y_1}{\partial x_1} dx_1 + \frac{\partial y_1}{\partial x_2} dx_2; \quad dy_2 = \frac{\partial y_2}{\partial x_1} dx_1 + \frac{\partial y_2}{\partial x_2} dx_2.
\]

Thus

\[
(5.8) \quad g_1(y_1, y_2)dy_1 + g_2(y_1, y_2)dy_2 =
\]

\[
\left( g_1(y_1, y_2) \frac{\partial y_1}{\partial x_1} + g_2(y_1, y_2) \frac{\partial y_2}{\partial x_1} \right) dx_1 + \left( g_1(y_1, y_2) \frac{\partial y_1}{\partial x_2} + g_2(y_1, y_2) \frac{\partial y_2}{\partial x_2} \right) dx_2
\]

Exercise 6.  

i. Rewrite the differential 1-form \( x_1 dx_1 + x_2 dx_2 \) in terms of coordinates \( t, u \) if \( x_1 = t + u, \ x_2 = t - u \).

ii. Suppose given 1-forms \( f_1 dx_1 + f_2 dx_2 \) and \( g_1 dy_1 + g_2 dy_2 \) where the \( f \)'s and \( g \)'s are related as in (5.8). Show the integrals of these 1-forms along any path agree.

iii. What is the analog of (5.8) for 1-forms on \( \mathbb{R}^n \)?