5. More about Path Integration and a Practice Midterm

This is an addendum to the week 4 pdf file entitled "Path Integration". I add one new result and end with a practice midterm exam.

A differential 1-form \( \omega \) on \( \mathbb{R}^n \) (or on an open set \( U \subset \mathbb{R}^n \)) is an expression of the form \( \sum_{i=1}^{n} f_i(x_1, \ldots, x_n) dx_i \) where the \( f_i \) are “nice” \( \mathbb{R} \)-valued functions on \( \mathbb{R}^n \) or on \( U \).

If \( \phi(t) = (\phi_1(t), \ldots, \phi_n(t)) : [a, b] \to \mathbb{R}^n \) (respectively \( \phi(t) = (\phi_1(t), \ldots, \phi_n(t)) : [a, b] \to U \)) is a path, then we can define

\[
\int_{\phi} \omega := \sum_{i=1}^{n} \int_{a}^{b} f_i(\phi(t)) \frac{d\phi_i(t)}{dt} dt.
\]

The point of this definition is that it does not depend on the parametrization of the path. If \( \theta : [\alpha, \beta] \to [a, b] \) then

\[
\int_{\phi \circ \theta} \omega = \int_{\phi} \omega.
\]

Here is a way to write down lots of interesting 1-forms. Let \( g : \mathbb{R}^n \to \mathbb{R} \) be a function, and define

\[
\omega = dg := \frac{\partial g}{\partial x_1} dx_1 + \ldots + \frac{\partial g}{\partial x_n} dx_n.
\]

For example, if \( g(x, y) = y \) then \( dg = dy \). If \( g(x, y) = xy \) then \( dg = ydx + xdy \).

We can view \( d \) as a mapping, the exterior derivative

\[
d : \{ \text{functions} \} \to \{ \text{1-forms} \}.
\]

A 1-form \( dg \) is said to be exact. What happens when we compute the Path integral of an exact 1-form? Good question! Glad you asked!!

Remember the chain rule tells us

\[
\frac{d}{dt} g(\phi(t)) = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(\phi(t)) \frac{d\phi_i(t)}{dt}.
\]

Combining (1) and (2), we find the left hand identity in (3) below:

\[
\int_{\phi} dg = \int_{a}^{b} \frac{d}{dt} g(\phi(t)) dt = g(\phi(b)) - g(\phi(a)).
\]
The right hand identity is the fundamental theorem of calculus! Notice that the expression on the right depends only on the endpoints of $\phi$. We have proven a theorem.

**Theorem 1.** Let $\phi : [a, b] \to \mathbb{R}^n$ be a path, and let $\omega$ be an exact 1-form on $\mathbb{R}^n$. Then $\int_{\phi} \omega$ depends only on the endpoints $\phi(a), \phi(b)$. I.e. if $\sigma : [\alpha, \beta] \to \mathbb{R}^n$ is another path with $\sigma(\alpha) = \phi(a)$ and $\sigma(\beta) = \phi(b)$, then $\int_{\sigma} \omega = \int_{\phi} \omega$. More precisely, if $\omega = dg$, then

$$\int_{\phi} \omega = g(\phi(b)) - g(\phi(a)).$$

**Corollary 2.** Let $\phi : [a, b] \to \mathbb{R}^n$ be a closed path; that is a path such that $\phi(a) = \phi(b)$. Then we have $\int_{\phi} \omega = 0$ for any exact 1-form $\omega$.

**Example 3.** Consider two 1-forms in $\mathbb{R}^2$.

$$\omega = xdy + ydx; \quad \xi = xdy - ydx.$$  

Consider the two paths from $(1, 0)$ to $(0, 1)$:

$$\phi(t) = (\cos t, \sin t), \quad 0 \leq t \leq \pi/2; \quad \sigma(t) = (1 - t, t), \quad 0 \leq t \leq 1.$$  

Here are the path integral computations:

$$\int_{\phi} \omega = \int_{t=0}^{\pi/2} (\cos^2 t - \sin^2 t)dt = 0; \quad \int_{\sigma} \omega = \int_{0}^{1} (1 - t)dt - tdt = 0.$$  

$$\int_{\phi} \xi = \int_{t=0}^{\pi/2} (\cos^2 t + \sin^2 t)dt = \pi/2; \quad \int_{\sigma} \xi = \int_{0}^{1} (1 - t)dt + tdt = 1.$$  

Note that $\omega = xdy + ydx = d(xy)$ while $\xi$ is not an exact 1-form.

Our theorem has an important converse. Let $\omega$ be a 1-form on $\mathbb{R}^n$. Suppose it is the case that the path integral $\int_{\phi} \omega$ depends only on the endpoints of $\phi$. Said another way, suppose that whenever $\phi, \sigma$ have the same endpoints we get $\int_{\phi} \omega = \int_{\sigma} \omega$. Fix a point $p \in \mathbb{R}^n$, and define a function $g : \mathbb{R}^n \to \mathbb{R}$ by

$$g(x) = \int_{p}^{x} \omega,$$

where the integral is computed along any path $\phi$ from $p$ to $x$. (By assumption, it doesn’t depend on the choice of path.) Write $\omega = \sum_i f_i(x_1, \ldots, x_n)dx_i$. I claim $\omega = dg$. Looking back at (2), this amounts to showing $f_i = \partial g/\partial x_i$. Let

$$\psi_\epsilon(t) = (x_1, \ldots, x_{i-1}, x_i + t, x_{i+1}, \ldots, x_n); \quad 0 \leq t \leq \epsilon$$
Writing \( \psi_{\varepsilon}(t) = (\psi_1(t), \ldots, \psi_n(t)) \) we see that \( \psi_j \) is constant for \( j \neq i \) and \( \psi_i(t) = x_i + t \). From the definition (1) it follows that

\[
\frac{\partial g}{\partial x_i} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\psi_{\varepsilon}}^{\psi_{\varepsilon+\varepsilon}} \omega = \frac{1}{\varepsilon} \int_0^{\varepsilon} f_i(x_1, \ldots, x_{i-1}, x_i + t, x_{i+1}, \ldots, x_n) dt = f_i(x).
\]

Thus, \( \omega = dg \). We have proven:

**Theorem 4.** Let \( \omega \) be a 1-form on \( \mathbb{R}^n \) and assume the path integral \( \int_{x}^{y} \omega \) depends only on the endpoints \( x \) and \( y \), and not on the path between them. Then \( \omega \) is an exact 1-form. Indeed, \( \omega = dg \) where \( g(x) := \int_{x}^{y} \omega \) for a fixed \( y \).

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**Math 205**

**Practice Midterm**

**50 Minutes**

1. Let \( D = \{ (x, y) \mid x^2 + y^2 \leq 1 \} \) be the unit disk in \( \mathbb{R}^2 \), and let \( f : D \to \mathbb{R} \) be some nice function. Discuss the computation of

\[
\int \int_D f dx dy
\]

(a) using upper and lower Riemann sums over small rectangles covering \( D \).

(b) using Fubini’s theorem.

State carefully all results that you use.

2. The map \( \phi(r, \theta) = (r \cos 2\pi \theta, r \sin 2\pi \theta) \) maps the unit square to the disk \( D \) as above.

(a) Compute the derivative matrix \( D\phi(r, \theta) \) and the determinant \( \det D\phi(r, \theta) \).

(b) Use 2(a) to give another formula for \( \int \int_D f dx dy \).

(c) Use 2(b) to compute the area of \( D \).

3. Compute the path integrals

\[
\int_{\phi} dx + dy; \quad \int_{\sigma} y dx - x dy
\]

Here \( \phi(t) = (\cos 2\pi t, \sin 2\pi t) \) and \( \sigma(t) = (t + 1, 7 - 2t) \), both on the interval \([0, 1]\).
4. Define 1-forms on $\mathbb{R}^n$. Define what it means for a 1-form to be exact. Show that the 1-form

$$\frac{xdx + ydy}{(x^2 + y^2)^2}$$

is an exact 1-form on $\mathbb{R}^2 - \{(0,0)\}$. 