Have you ever asked yourself, “What is $dx$?” In the middle of doing your math homework, do you ever feel like the Coyote in Roadrunner cartoons who’s just chased the bird off a cliff and looked down to realize there is no solid earth beneath his feet? What does an expression like $f(x)dx$ mean anyway?

Let $U \subset \mathbb{R}^n$ be an open set. Let $f : U \to \mathbb{R}$ be a differentiable function. Consider the following problem: find a 1-form (which we will call) $d(f)$ on $U$ such that for any path $\phi : [a, b] \to U$, the path integral is given by

\begin{equation}
\int_\phi d(f) = f(\phi(b)) - f(\phi(a)).
\end{equation}

**Example 1.** Suppose $n = 1$. I claim we may take $d(f) := f'(x)dx$. Indeed, with this definition, the path integral just becomes

\begin{equation}
\int_\phi d(f) = \int_\phi f'(x)dx = \int_a^b f'(\phi(t))\phi'(t)dt = \int_a^b \frac{df}{dt}(\phi(t))dt = f(\phi(b)) - f(\phi(a)).
\end{equation}

The case for general $\mathbb{R}^n$ is not much more difficult. We simply define

\begin{equation}
d(f) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}dx_i.
\end{equation}

**Proposition 2.** Let $\phi : [a, b] \to \mathbb{R}^n$ be a path. Then

\begin{equation}
\int_\phi d(f) = \int_\phi \sum_{i=1}^n \frac{\partial f}{\partial x_i}dx_i = f(\phi(b)) - f(\phi(a)).
\end{equation}

**Proof.** Write $\phi(t) = (x_1(t), x_2(t), \ldots, x_n(t))$. Then

\begin{equation}
\int_\phi \sum_{i=1}^n \frac{\partial f}{\partial x_i}dx_i = \int_a^b \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i(t)dt = \int_a^b \frac{d}{dt}(f(\phi(t)))dt = f(\phi(b)) - f(\phi(a)).
\end{equation}
Example 3. In $\mathbb{R}^2$, let $f(x,y) = \sqrt{x^2 + y^2}$. Then
\[d(f) = \frac{x \, dx + y \, dy}{\sqrt{x^2 + y^2}}.\]

We have for any path $\phi$,
\[
\int_{\phi} \frac{x \, dx + y \, dy}{\sqrt{x^2 + y^2}} = \sqrt{\phi_1(b)^2 + \phi_2(b)^2} - \sqrt{\phi_1(a)^2 + \phi_2(a)^2}.
\]

It is important to note that the path integral condition (8.1) uniquely determines $d(f)$. Indeed, if $D(f)$ were another solution satisfying (8.1) for every path $\phi$ we could write $d(f) - D(f) = \sum g_i(x_1, \ldots, x_n)dx_i$ and conclude that
\[0 = \sum_i \int_{\phi} g_i(\phi(t))\phi'_i(t)dt.\]

If we take for $\phi$ the path $\phi(t) = (c_1, \ldots, c_{i-1}, t, c_{i+1}, \ldots, c_n) \quad (t \text{ in the } i\text{-th coordinate, } c_j \text{ constants})$ we see that
\[
\int_a^b g_i(c_1, \ldots, c_{i-1}, t, c_{i+1}, \ldots, c_n)dt = 0
\]
Since the constants $c_j$ and the endpoints $a$ and $b$ are arbitrary, we conclude that all the $g_i(x_1, \ldots, x_n) = 0$, so $D(f) = d(f)$.

The following definition is convenient.

Definition 4. A 0-form on $U \subset \mathbb{R}^n$ is a differentiable function $f : U \to \mathbb{R}$.

Thus, we have defined a map
\[
(8.3) \quad d : \{0 \text{-forms}\} \to \{1 \text{-forms}\}
\]
\[
d(f) := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.
\]

Finally, the wiley reader will have detected disaster ahead. After all, we have a 1-form $dx_i$ already. But $x_i$ is a function on $U \subset \mathbb{R}^n$, so we may define in our new sense a 1-form $d(x_i)$. Fortunately, our coyote is on solid ground, because
\[
d(x_i) = \sum_j \frac{\partial x_i}{\partial x_j} dx_j = dx_i.
\]

For the philosophically inclined in the peanut gallery, this gives a rigorous meaning to the “infinitesimal” $dx$. 