Math 270
Lectures, May 24-28, 2004

Lectures this week focused on two jewels in the crown of complex analysis, the Riemann zeta function \( \zeta(s) \) and the theta function \( \vartheta(t) \). This material is covered in more detail on pp. 111-120 and 168-174 of the textbook.

The zeta function is defined by

\[
\zeta(s) = \sum_{n \geq 1} n^{-s}
\]

The series converges for \( \text{Re} \, s > 1 \) by comparison with the integral \( \int_1^{\infty} x^{-\text{Re} \, s} dx \). (Problem: show the series is analytic for \( \text{Re} \, s > 1 \).) Analytic properties of the zeta function can be related to questions about prime numbers via the infinite product expansion

\[
\zeta(s) = \prod_{p \text{ prime}} \left( \frac{1}{1 - p^{-s}} \right).
\]

(We have not discussed infinite products in the class. You should consider this product for a finite set of primes to convince yourself that the identity is plausible. How might you define a notion of convergence for such infinite products using the function \( \log \)?)

Although (1) only converges for \( \text{Re} \, s > 1 \), we will show that \( \zeta(s) \) admits a meromorphic extension to the whole of \( \mathbb{C} \) which is analytic with the exception of a pole of order 1 at \( s = 1 \). (Problem: show that such an extension if it exists is unique.) To do this, we will use the theta function

\[
\vartheta(t) = \sum_{n \in \mathbb{Z}} e^{-\pi tn^2}.
\]

Fourier Transform

For \( a > 0 \) let \( \mathcal{F}_a \) be the space of functions \( f(z) \) which are defined and analytic on the horizontal strip \(-a < \text{Im} \, z < a\) in \( \mathbb{C} \), and which further satisfy the growth condition in the strip

\[
|f(x + iy)| < \frac{A}{1 + x^2}, \quad |y| < a
\]

Here \( A \) is a constant which may depend on \( f \). We define

\[
\mathcal{F} = \bigcup_{a > 0} \mathcal{F}_a
\]
The example which interests us is \( f(z) = e^{-\pi z^2} \). (Show \( f \in \mathcal{F} \).)

**Theorem 1** (Fourier Transform). For \( f \in \mathcal{F} \), the Fourier transform

\[
\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx
\]

is defined and lies in \( \mathcal{F} \). We have \( \hat{\hat{f}}(w) = f(-w) \).

This theorem is proven in Stein, but we did not discuss the proof in class. We did prove

**Theorem 2** (Poisson Summation). For \( f \in \mathcal{F} \) we have

\[
\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n).
\]

**Proof.** It follows from (4) that \( \sum_{n \in \mathbb{Z}} f(n) \) converges absolutely. (Why?) And then, from the Fourier transform theorem, we get the same absolute convergence for \( \sum_{n \in \mathbb{Z}} \hat{f}(n) \). Suppose \( f \in \mathcal{F}_a \). Choose \( b \) with \( 0 < b < a \). For \( N \geq 1 \) any integer, define \( \Box_N \) to be the rectangle with vertices \( \pm(N + \frac{1}{2}) \pm ib \). Consider the integral

\[
I_N := \int_{\Box_N} \frac{f(z)dz}{e^{2\pi iz} - 1}
\]

We evaluate this integral in two ways. First, using residues, we find

\[
I_N = \sum_{|n| \leq N} f(n).
\]

(Really good exercise: supply the details in this calculation.) On the other hand, it follows from (4) that the integrals over the vertical sides of \( \Box_N \) tend to 0 as \( N \to \infty \). (Details?) We have

\[
\lim_{N \to \infty} I_N = \int_{-\infty - ib}^{+\infty - ib} - \int_{-\infty + ib}^{+\infty + ib}.
\]

For \( \text{Im } z < 0 \) (resp. \( \text{Im } z > 0 \)) we have \( |e^{2\pi iz}| > 1 \) (resp. \( |e^{2\pi iz}| < 1 \)) so we get geometric series expansions

\[
\frac{f(z)dz}{e^{2\pi iz} - 1} = \sum_{n=1}^{\infty} f(z)e^{-2\pi inz}dz \quad \text{(resp. } - \sum_{n=0}^{\infty} f(z)e^{2\pi inz}dz\text{)}.
\]

For any particular value of \( n \) we have

\[
\int_{-\infty - ib}^{+\infty - ib} f(z)e^{2\pi inz}dz = \int_{-\infty + ib}^{+\infty + ib} f(z)e^{2\pi inz}dz = \int_{-\infty}^{+\infty} f(x)e^{2\pi inx}dx
\]
We conclude

\[
\lim_{N \to \infty} I_N = \sum_{n=\infty}^{\infty} \int_{-\infty}^{+\infty} f(x) e^{2\pi inx} \, dx = \sum_{n \in \mathbb{Z}} \hat{f}(-n).
\]

The Poisson summation formula follows by comparing (7) and (11). \(\square\)

**Example 3.** Take \(f(z) = e^{-\pi z^2}\). The Fourier transform is

\[
\hat{f}(\xi) := \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} \, dx = e^{-\pi \xi^2} \int_{-\infty}^{\infty} e^{-\pi(x+i \xi)^2} \, dx =
\]

\[
e^{-\pi \xi^2} \int_{-\infty+i \xi}^{\infty+i \xi} e^{-\pi z^2} \, dz = e^{-\pi \xi^2} \int_{-\infty}^{\infty} e^{-\pi x^2} \, dx = e^{-\pi \xi^2}.
\]

(Justify the equality labeled with "!".)

We can make the example more interesting by adding a parameter. For \(t \in \mathbb{R}\) and \(f \in \mathcal{F}\), define \(f_t(z) = f(tz)\). We have

\[
\widehat{f_t}(\xi) = t^{-1} \hat{f}(t^{-1} \xi).
\]

(Exercise: justify this.) For \(t > 0\) it follows that

\[
\hat{e^{-\pi t^2}}(\xi) = t^{-1/2} e^{-t^{-1} \xi^2}.
\]

(Justify.) We apply the Poisson summation formula to the function \((z) e^{-\pi t^2}\) and conclude from (14)

\[
\vartheta(t) := \sum_{n \in \mathbb{Z}} e^{-\pi t n^2} = t^{-1/2} \sum_{n \in \mathbb{Z}} e^{-\pi t^{-1} n^2} = t^{-1/2} \vartheta(t^{-1}).
\]

Here the identity (*) serves to define the theta function \(\vartheta(t)\), and Poisson summation gives us the functional equation

\[
\vartheta(t) = t^{-1/2} \vartheta(t^{-1}).
\]

The functions \(\Gamma(s)\) and \(\xi(s)\)

The gamma function is given by

\[
\Gamma(s) := \int_{0}^{\infty} e^{-t^s} \frac{dt}{t}.
\]

It satisfies the functional equation

\[
\Gamma(s + 1) = s \Gamma(s).
\]

(Use integration by parts to verify this identity.) Note that because of difficulties at \(t = 0\), the integral (17) \textit{a priori} only makes sense for \(\text{Re}\ s > 0\). Using (18), \(\Gamma\) may be extended to a meromorphic function
on all of $\mathbb{C}$ which is holomorphic except for simple (i.e. first order) poles at $s = 0, -1, -2, \ldots$. (Exercise: justify this assertion.)

Lemma 4. We have

$$\int_0^\infty e^{-\pi n^2 t} t^{-s/2} \frac{dt}{t} = \pi^{-s/2} n^{-s} \Gamma(s/2) \tag{19}$$

Proof. Exercise! \hfill \square

Definition 5. We define

$$\xi(s) := \pi^{-s/2} \Gamma(s/2) \zeta(s)$$

where $\zeta$ is the Riemann zeta function (1).

Theorem 6. $\xi(s)$ extends meromorphically to all of $\mathbb{C}$. It is analytic outside simple poles at $s = 0, 1$ and satisfies the functional equation $\xi(s) = \xi(1 - s)$.

Corollary 7. The Riemann zeta function extends meromorphically to $\mathbb{C}$. It is analytic outside a simple pole at $s = 1$. It has first order zeroes at $s = -2, -4, \ldots$ and no other zeroes outside the critical strip $0 < \text{Re } s < 1$. The most famous open conjecture in mathematics is the Riemann hypothesis which asserts that all the zeroes of $\zeta(s)$ in the critical strip lie on the vertical line $\text{Re } s = 1/2$.

(Exercise: use (2) to justify the assertion that $\zeta(s)$ has no zeroes for $\text{Re } s > 1$.)

proof of thm. 6. Define

$$\psi(u) = \frac{\partial(u) - 1}{2} = \sum_{n=1}^\infty e^{-\pi n^2 u^2}. \tag{20}$$

This function satisfies the functional equation

$$\psi(u) = u^{-1/2} \psi(1/u) + \frac{1}{2u^{1/2}} - \frac{1}{2} \tag{21}$$

(Exercise: check this.) We have

$$\xi(s) = \int_0^\infty u^{s/2} \psi(u) \frac{du}{u} = $$

$$\int_0^1 u^{s/2 - 1} \left( u^{-1/2} \psi(1/u) + \frac{1}{2u^{1/2}} - \frac{1}{2} \right) du + \int_1^\infty u^{s/2 - 1} \psi(u) du = $$

$$\frac{1}{s - 1} - \frac{1}{s} + \int_1^\infty (u^{-s/2} + u^{s/2 - 1}) \psi(u) du.$$
The assertions of the theorem follow from the fact that the RHS of (21) is symmetric under $s \mapsto 1 - s$ and the integral is analytic in $s$ for all $s \in \mathbb{C}$. □

(Exercise: Write a formula for $\zeta(1 - s)$ in terms of $\zeta(s)$, $\Gamma$-terms, and powers of $\pi$.)