1. **Inequalities** (21 points, 7 each)

(a) **Solve the inequality** \(-\frac{1}{2} x + 4 \geq 0\), and express your answer in interval notation.
We first subtract 4 from both sides to get \(-\frac{1}{2} x \geq -4\), and then multiply both sides by \(-2\) to get \(x \leq 8\), remembering to reverse the inequality because we multiplied both sides by a negative number. The solution set is expressed in interval form as \((-\infty, 8]\).

(b) **Solve the inequality** \(12 x + 36 < 0\), and express your answer in interval notation.
We first subtract 36 from both sides to get \(12 x < -36\), and then divide both sides by 12 to get \(x < -3\). As an interval, this solution set would be expressed as \((-\infty, -3)\).

(c) **Solve the inequality** \(-\frac{1}{2} x + 4 \leq \frac{12 x + 36}{12 x + 36} < 0\), and express your answer in interval notation.
In order for the quotient to be negative, one or the other of the numerator and denominator must be negative, and the other must be positive. This can happen in two ways. If the numerator is positive (i.e. \(x < 8\)) and the denominator is negative (i.e. \(x < -3\)), we must have \(x < -3\). If the numerator is negative (i.e. \(x > 8\)) and the denominator is positive (i.e. \(x > -3\)), then we must have \(x > 8\). These two cases give us a solution set of \((-\infty, -3) \cup (8, +\infty)\).

2. **Equations and Graphing** (26 points, 8/10/8)

(a) **Graph the line** \(x + 2y - 2 = 0\), being sure to indicate the \(x\)- and \(y\)-intercepts and the slope.
See attached graph. The \(x\)-intercept is 2, the \(y\)-intercept is 1, and the slope is \(-\frac{1}{2}\).

(b) **Graph the parabola** \(y = -\frac{1}{4} x^2 + 4x - 13\), being sure to indicate the \(x\)- and \(y\)-intercepts and the vertex.
See attached graph. The \(x\)-intercepts are \(8 \pm 2\sqrt{3}\), the \(y\)-intercept is \(-13\), and the vertex is \((8, 3)\).

(c) **Find the points of intersection of the line from part (a) and the parabola from part (b).**
To find the points of intersection, we solve the two equations simultaneously. One method for doing this is to substitute the expression for \(y\) from the second equation into the first equation. This yields \(x + 2(-\frac{1}{4} x^2 + 4x - 13) - 2 = 0\), which simplifies to \(-\frac{1}{2} x^2 + 9x - 28 = 0\), which can be further re-written as \(x^2 - 18x + 56 = 0\). To solve this equation, we could use the Quadratic Formula, or we could notice that the left-hand side factors to give \((x - 4)(x - 14) = 0\). Thus, we get the two \(x\)-values 4 and 14. The corresponding \(y\)-values may be obtained by plugging the \(x\)-values into either of the original two equations. If \(x = 4\), we find that \(y = -1\), and if \(x = 14\), we find \(y = -6\). Thus, the two points of intersection are \((4, -1)\) and \((14, -6)\). This can be seen on the attached graph.

3. **Absolute Value** (20 points, 10 each)

(a) **Solve the inequality** \(|5 - 7x| \geq 2\), and graph your solution on the number line.
We translate the inequality to one without absolute values in two cases.

Case 1: If \(5 - 7x \geq 0\) (which gives \(x \leq \frac{5}{7}\)), we have to solve the inequality \(5 - 7x \geq 2\), which yields \(x \leq \frac{3}{7}\). Thus, the Case 1 solutions are \(x \leq \frac{3}{7}\).
Case 2: If $5 - 7x < 0$ (which gives $x > \frac{5}{7}$), we have to solve the inequality $-(5 - 7x) \geq 2$, which yields $x \geq 1$. Thus, the Case 2 solutions are $x \geq 1$.

The complete solution set is $x \leq \frac{2}{7}$ or $x \geq 1$, which can be expressed in interval notation as $(-\infty, \frac{2}{7}) \cup [1, +\infty)$.

(b) Solve the equation $-2 + |3x - 8| = x^2$.

Because of the absolute value, we have two cases:

Case 1: If $3x - 8 \geq 0$ (that is, if $x \geq \frac{8}{3}$), then $|3x - 8| = 3x - 8$, and we need to solve the equation $-2 + (3x - 8) = x^2$. This simplifies to $0 = x^2 - 3x + 10$. However, the discriminant of this quadratic is $b^2 - 4ac = (-3)^2 - 4(1)(10) = -31$. Since this is negative, there are no real solutions in this case.

Case 2: If $3x - 8 < 0$ (that is, if $x < \frac{8}{3}$), then $|3x - 8| = 8 - 3x$, and we need to solve the equation $-2 + (8 - 3x) = x^2$. This simplifies to $0 = x^2 + 3x - 6$. The right-hand side does not factor conveniently, but we can use the Quadratic Formula to solve this equation. We get $x = \frac{-3 \pm \sqrt{33}}{2}$. Both of these satisfy the conditions of Case 2.

Overall, the solution set consists of the two values $x = \frac{-3 \pm \sqrt{33}}{2}$.

4. Linear Equations. (20 points, 5 each)

Consider the two lines below:

<table>
<thead>
<tr>
<th>Line 1</th>
<th>Line 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x + 3y - 3 = 0$</td>
<td>$-4x + 9y - 4 = 0$</td>
</tr>
</tbody>
</table>

(a) Find the point of intersection of Line 1 and Line 2.

To find the point of intersection, we need to find values $x$ and $y$ which satisfy these equations simultaneously. The standard approach is to try to eliminate one of the variables. If we multiply the equation for Line 1 by 3 on each side, we get $4x + 12y - 12 = 0$. Adding this to the equation for Line 2, we get $21y - 16 = 0$. We can now solve for $y$, and we obtain $y = \frac{16}{21}$. Plugging this value of $y$ back into either of the original equations, we obtain $x = \frac{15}{21} = \frac{5}{7}$.

(b) Are the two lines perpendicular? Explain.

The slope of Line 1 is $m_1 = -\frac{1}{3}$, and the slope of Line 2 is $m_2 = \frac{4}{9}$. Because we do not have $m_1 \cdot m_2 = -1$, the two lines are not perpendicular.

(c) Find the equation of the line parallel to Line 2 that passes through the $x$-intercept of Line 1.

First, we note that the $x$-intercept of Line 1 occurs $x = 3$. In other words, it is the point $(3, 0)$. Any line parallel to Line 2 must also have slope $m = \frac{4}{9}$. To find the line through the $x$-intercept of Line 1 that is parallel to Line 2, we use Point-Slope Form to get $y - 0 = \frac{4}{9}(x - 3)$. Rearranged into Slope-Intercept Form (which was not required), this becomes $y = \frac{4}{9}x - \frac{4}{3}$.

(d) Graph Line 1, Line 2, and the line from part (c) on one set of axes.

See attached graph.

5. The Sum of Cubes. (18 points, 10/8)

(a) Use the rules of algebra to verify the formula $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

Starting from the right-hand side, we perform the following steps with the listed justifications:

$$
(a + b)(a^2 - ab + b^2) = a(a^2 - ab + b^2) + b(a^2 - ab + b^2) \quad \text{by D}
$$
$$
= (a^3 - a^2b + ab^2) + (a^2b - ab^2 + b^3) \quad \text{by D, M2, and defn. of exponents}
$$
$$
= a^3 + (a^2b + a^2b) + (ab^2 - ab^2 + b^3) \quad \text{by A1 and A2}
$$
$$
= a^3 + 0 + 0 + b^3 \quad \text{by A4}
$$
$$
= a^3 + b^3 \quad \text{by A3}
$$
(b) Noting that \(1729 = 10^3 + 9^3\) and that \(1729 = 12^3 + 1^3\), use the formula from part (a) to find the prime factorization of the number 1729.

First, we use the formula from part (a) with \(a = 10\) and \(b = 9\). We get \(1729 = 10^3 + 9^3 = (10 + 9)(10^2 - 10 \cdot 9 + 9^2) = 19 \cdot 91\). Second, we use the formula from part (a) with \(a = 12\) and \(b = 1\). We get \(1729 = 12^3 + 1^3 = (12 + 11)(12^2 - 12 \cdot 1 + 1^2) = 13 \cdot 133\).

From the first factorization, we see that 19 is prime factor of 1729. From the second factorization, we see that 13 is prime factor of 1729. Hence 13 must divide evenly into 91, yielding the other prime factor of 7. (Or, we could observe that 19 must divide 133, which also yields 7.)

The prime factorization is \(1729 = 7 \cdot 13 \cdot 19\).

6. An Equation with Radicals. (24 points, 6 each) Consider the equation in one variable \(x\) given by

\[8 + 7\sqrt{x} = x\]

(a) Substitute \(x = y^2\) to write the equation as a polynomial in the variable \(y\).

We get \(8 + 7|y| = y^2\). (Because \(\sqrt{y^2} = |y|\).)

(b) Solve the equation from part (a) for the variable \(y\).

If \(y \geq 0\), we need to solve \(8 + 7y = y^2\). This is re-written as \(0 = y^2 - 7y - 8\), which factors as \(0 = (y - 8)(y + 1)\) and hence has solutions \(y = 8\) and \(y = -1\). Only \(y = 8\) is positive.

If \(y < 0\), we need to solve \(8 - 7y = y^2\). This is re-written as \(0 = y^2 + 7y - 8\), which factors as \(0 = (y + 8)(y - 1)\) and hence has solutions \(y = -8\) and \(y = 1\). Only \(y = -8\) is negative.

(c) Solve the original equation for the variable \(x\).

From either \(y = \pm 8\), we get \(x = y^2 = 64\).

(d) Check your answer(s) from part (c) in the original equation and explain.

If \(x = 64\), then \(\sqrt{x} = 8\), and we do have a solution because \(8 + 7 \cdot 8 = 64\). (The possible value of \(y = \pm 1\) yielded a false solution of \(x = 1\) because the square root is always positive, and so it could not return the value of \(7\sqrt{1} = -7\).)

7. Exponentials and Logarithms. (20 points, 10 each)

(a) Solve the logarithmic equation: \(\log_3(x + 2) - \log_3(x + 1) = 2\).

We use the laws of logarithms to re-write the left-hand side and get \(\log_3 \left( \frac{x + 2}{x + 1} \right) = 2\). Now we use the definition of the logarithm to see that we need \(\frac{x + 2}{x + 1} = 3^2 = 9\). Multiplying both sides by \(x + 1\), we get the equation \(x + 2 = 9(x + 1)\). Distributing and re-arranging, we get \(-7 = 8x\) or \(x = -\frac{7}{8}\).

(b) Put the following five real numbers in increasing order:

\[
\frac{1}{25}, \quad 3\sqrt{-\frac{1}{27}}, \quad \log_{10} \left( \frac{1}{100} \right), \quad 4^{-3}, \quad \sqrt{11}
\]

Working one at a time, we have

\[
\frac{1}{25} = \frac{1}{32}, \quad 3\sqrt{-\frac{1}{27}} = -\frac{1}{3}, \quad \log_{10} \left( \frac{1}{100} \right) = -2, \quad 4^{-3} = \frac{1}{64}, \quad \sqrt{11} \approx 3.3
\]

Since the second and thirs are negative, they come first, and since the fifth number is the only one larger than 1, it comes last. Our final arrangement in increasing order is

\[
\log_{10} \left( \frac{1}{100} \right), \quad 3\sqrt{-\frac{1}{27}}, \quad 4^{-3}, \quad \frac{1}{25}, \quad \sqrt{11}
\]

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