Calculus Placement Exam Answer Key

Harris School of Public Policy

September 23, 2013

1

A

\[
\lim_{x \to 4} \frac{x^2 - 16}{x^2 - 7x + 12} = \lim_{x \to 4} \frac{(x - 4)(x + 4)}{(x - 4)(x - 3)}
= \lim_{x \to 4} \frac{x + 4}{x - 3}
= 8
\]

B

\[
\lim_{x \to 2} \frac{\sqrt{x + 1} - \sqrt{3}}{x - 2} = \lim_{x \to 2} \frac{(\sqrt{x + 1} - \sqrt{3})(\sqrt{x + 1} + \sqrt{3})}{(x - 2)(\sqrt{x + 1} + \sqrt{3})}
= \lim_{x \to 2} \frac{x - 2}{(x - 2)(\sqrt{x + 1} + \sqrt{3})}
= \lim_{x \to 2} \frac{1}{\sqrt{x + 1} + \sqrt{3}}
= \frac{1}{2\sqrt{3}}
\]

C

\[
\lim_{x \to -\infty} \frac{1 + e^{-x}}{1 - x} = +\infty
\]

D

\[
\lim_{x \to -1} e^{x+1}(4 - x^2) = 3
\]
\[ \lim_{x \to a} \frac{x^4 - a^4}{x - a} = \lim_{x \to a} \frac{(x^2 - a^2)(x^2 + a^2)}{x - a} = \lim_{x \to a} \frac{(x - a)(x + a)(x^2 + a^2)}{x - a} = \lim_{x \to a} (x + a)(x^2 + a^2) = 4a^3 \]

B

\[ \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(x) \]

In other words, the limit evaluated in (2.1) is the derivative of the function \( f \). It represents the instantaneous rate of change of the function at any point.

3

A

\[ h'(x) = 3x^2 \cdot \ln(1 + \sqrt{x}) + x^3 \cdot \frac{1}{1 + \sqrt{x}} \cdot \frac{1}{2\sqrt{x}} \]

B

\[ j'(x) = \frac{1}{3} \left( \frac{1 - 3x}{1 + 3x} \right)^{\frac{2}{3}} \cdot \left( \frac{-3(1 + 3x) - 3(1 - 3x)}{(1 + 3x)^2} \right) \]

C

\[ k'(x) = \ln(2) \cdot 2^{-x^2 + 3x} \cdot (-2x + 3) \]

There are vertical asymptotes where \( f(x) \) is undefined. That is, where \( x = -4 \) and \( x = -10 \):

\[ \lim_{x \to -4^-} \frac{(3x - 6)(x + 6)}{(x + 4)(x + 10)} = +\infty \]

\[ \lim_{x \to -4^+} \frac{(3x - 6)(x + 6)}{(x + 4)(x + 10)} = -\infty \]
\[
\lim_{x \to -10^-} \frac{(3x - 6)(x + 6)}{(x + 4)(x + 10)} = +\infty
\]
\[
\lim_{x \to -10^+} \frac{(3x - 6)(x + 6)}{(x + 4)(x + 10)} = -\infty
\]

Since the numerator and denominator of \( f(x) \) are of the same degree, the horizontal asymptote will occur where \( y \) equals the ratio of the leading coefficients. That is, where \( y = 3 \):

\[
\lim_{x \to \pm\infty} \frac{(3x - 6)(x + 6)}{(x + 4)(x + 10)} = 3
\]

5

First, find \( f'(x) \) to identify critical points:

\[
f'(x) = 3x^2 + 12x - 15
\]
\[
= 3(x - 1)(x + 5)
\]

Critical points can occur in three places: where \( f'(x) = 0 \), where \( f'(x) \) is undefined, and at end points. In this instance:

- \( f'(x) = 0 \) at \( x = -5 \) and \( x = 1 \)
- \( f'(x) \) is undefined nowhere
- End points occur at \( x = -10 \) and \( x = 3 \)

Next, we evaluate whether the derivative is positive or negative before and/or after these critical points to determine whether they represent maxima or minima:

<table>
<thead>
<tr>
<th></th>
<th>( (x - 1) )</th>
<th>( (x + 5) )</th>
<th>( f'(x) )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-10, -5)</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>Inc</td>
</tr>
<tr>
<td>(-5, 1)</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>Dec</td>
</tr>
<tr>
<td>(1, 3)</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>Inc</td>
</tr>
</tbody>
</table>

Therefore, minima occur at \( x = -10 \) and \( x = 1 \), and maxima occur at \( x = -5 \) and \( x = 3 \). To determine which are local minima / maxima and which are global, plug these \( x \) values into the function \( f \):

\[
f(-10) = -322
\]
\[
f(1) = -80
\]
\[
f(-5) = 28
\]
\[
f(3) = -36
\]
Finally, we find that a global minimum occurs at \( x = -10 \), and a local minimum at \( x = 1 \). A global maximum occurs at \( x = -5 \), and a local maximum at \( x = 3 \).

6

A

A local minimum occurs at \( x = 6 \), and a local maximum at \( x = -2 \).

B

There is an inflection point at \( x = -4 \). (Note that \( x = 3 \) is an asymptote, and therefore neither a maximum or minimum nor an inflection point.)

C

\[
g'(x) = \frac{3}{2}(12 - 2x)^{\frac{1}{2}} \cdot -2 = -3\sqrt{12 - 2x}
\]

7

A

Critical points can occur in three places: where \( g'(x) = 0 \), where \( g'(x) \) is undefined, and at end points. In this instance:
• $g'(x) = 0$ at $x = 6$
• $g'(x)$ is undefined nowhere
• $g(x)$ has an end point at $x = 6$ (The natural domain of $g(x)$ is $x \leq 6$.)

C

<table>
<thead>
<tr>
<th>Table 2: default</th>
<th>$\sqrt{12 - 2x}$</th>
<th>$g'(x)$</th>
<th>$g(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty, 6)$</td>
<td>+</td>
<td>-</td>
<td>Dec</td>
</tr>
</tbody>
</table>

Therefore, $g(x)$ is always decreasing.

D

Given $g(x)$ is always decreasing, there is no local maxima. A local minimum occurs at $x = 6$.

E

$$g''(x) = \frac{-3}{2} (12 - 2x)^{-\frac{3}{2}} \cdot -2$$
$$= \frac{3}{\sqrt{12 - 2x}}$$

F

Inflection points can occur in two places: where $g''(x) = 0$ and where $g''(x)$ is undefined. In this instance:
• $g''(x) = 0$ nowhere
• $g''(x)$ is undefined when $x = 6$

Therefore, an inflection point may occur at $x = 6$.

G

<table>
<thead>
<tr>
<th>Table 3: default</th>
<th>$\sqrt{12 - 2x}$</th>
<th>$g''(x)$</th>
<th>$g(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty, 6)$</td>
<td>+</td>
<td>+</td>
<td>Up</td>
</tr>
</tbody>
</table>

$g(x)$ is always concave up.

H

Given $g(x)$ is always concave up, there are no inflection points.
I

There are no vertical, horizontal, or slant asymptotes.

J

\[ \frac{\partial f}{\partial x} = 2x + y^3 \]

\[ \frac{\partial f}{\partial y} = 3xy^2 - 6 \]

C

\[
\begin{align*}
2x + y^3 &= 0 \\
2x + 2^3 &= 0 \\
x &= -4
\end{align*}
\]