Throughout this sheet, we let \( f : A \rightarrow \mathbb{R} \) be a real valued function with domain \( A \subset \mathbb{R} \).

**Remark 10.0.** Theorem 9.21 holds for a function \( f : A \rightarrow \mathbb{R} \) if we change “\( p \in \mathbb{R} \)” to “\( p \in A \)” and “\((p-\delta, p+\delta)\)” to “\( A \cap (p-\delta, p+\delta) \)”.
Similarly, Theorem 9.24 and Definition 9.25 hold for \( f : A \rightarrow \mathbb{R} \) so long as we take \( p \in A \) and change the condition “if \( |x-p| < \delta \)” to “if \( x \in A \) and \( |x-p| < \delta \).”

**Definition 10.1.** Let \( a \in \mathbb{R} \) such that there exists a region \( R \) containing \( a \) with \( R \setminus \{a\} \subset A \). The limit of \( f \) at a point \( a \in \mathbb{R} \) is a number \( L \in \mathbb{R} \) satisfying the following condition: for every \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that

\[
\text{if } x \in A \text{ and } 0 < |x-a| < \delta, \text{ then } |f(x) - L| < \epsilon.
\]

If the limit \( L \) of \( f \) at \( a \) exists, we write this as:

\[
\lim_{x \to a} f(x) = L.
\]

More generally, we can extend the above definition to any \( a \in \mathbb{R} \) such that \( a \) is a limit point of \( A \). In this case, the fact that \( a \) is a limit point guarantees that for all \( \delta > 0 \) there exists an \( x \) such that \( x \in A \) and \( 0 < |x-a| < \delta \). So the “if” part of the statement is never vacuous.

**Proposition 10.2.** Limits are unique: if \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} f(x) = L' \), then \( L = L' \).

**Exercise 10.3.** Give an example of a set \( A \subset \mathbb{R} \), a function \( f : A \rightarrow \mathbb{R} \), and a point \( a \) (such that there exists a region \( R \) with \( R \setminus \{a\} \subset A \)) such that \( \lim_{x \to a} f(x) \) does not exist.

**Theorem 10.4.** A function \( f \) is continuous at \( a \) if and only if:

\[
\lim_{x \to a} f(x) = f(a).
\]

**Exercise 10.5.** (i) For every \( c \in \mathbb{R} \), the constant function \( f : \mathbb{R} \rightarrow \mathbb{R} \), defined by \( f(x) = c \) for all \( x \in \mathbb{R} \), is continuous.

(ii) The identity function, defined by \( g(x) = x \), is continuous.

The following lemma is very useful in proofs involving inequalities. (Hint: the second inequality follows from the first.)

**Lemma 10.6.** For any real numbers \( x \) and \( y \), we have:

- The Triangle Inequality: \( |x + y| \leq |x| + |y| \)
- The Reverse Triangle Inequality: \( ||x| - |y|| \leq |x - y| \).
Exercise 10.7. Show that the absolute value function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ is continuous.

Given real-valued functions $f$ and $g$, we define new functions $f + g$, $fg$ and $\frac{1}{f}$ by:

- $(f + g)(x) = f(x) + g(x)$
- $(fg)(x) = f(x) \cdot g(x)$
- $\frac{1}{f}(x) = \frac{1}{f(x)}$, provided that $f(x) \neq 0$.

We wish to understand the limits of $f + g$, $fg$ and $\frac{1}{f}$ in terms of the limits of $f$ and $g$.

Lemma 10.8. If $|x - x_0| < \frac{\epsilon}{2}$ and $|y - y_0| < \frac{\epsilon}{2}$, then $|(x + y) - (x_0 + y_0)| < \epsilon$.

Theorem 10.9. Suppose that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Then:

$$\lim_{x \to a} (f + g)(x) = L + M.$$ 

Lemma 10.10. If

$$|x - x_0| < \min\left(1, \frac{\epsilon}{2(|y_0| + 1)}\right) \text{ and } |y - y_0| < \frac{\epsilon}{2(|x_0| + 1)},$$

then

$$|xy - x_0y_0| < \epsilon.$$ 

Theorem 10.11. Suppose that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Then:

$$\lim_{x \to a} (fg)(x) = L \cdot M.$$ 

Lemma 10.12. If $x_0 \neq 0$ and

$$|x - x_0| < \min\left(\frac{|x_0|}{2}, \frac{\epsilon|x_0|^2}{2}\right),$$

then $x \neq 0$ and

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| < \epsilon.$$ 

Theorem 10.13. Suppose that $\lim_{x \to a} f(x) = L \neq 0$. Then:

$$\lim_{x \to a} \frac{1}{f(x)} = \frac{1}{L}.$$ 

Corollary 10.14. If $f$ and $g$ are continuous at $a$, then $f + g$, $fg$ and $1/f$ are continuous at $a$ (the latter provided that $f(a) \neq 0$).
Definition 10.15. A polynomial in one variable with real coefficients is a function \( f \) of the form
\[
f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0
\]
for some \( n \in \mathbb{N} \cup \{0\} \), where \( a_i \in \mathbb{R} \) for \( 0 \leq i \leq n \). A rational function in one variable with real coefficients is a function of the form \( h(x) = \frac{f(x)}{g(x)} \) where \( f \) and \( g \) are polynomials in one variable with real coefficients.

Corollary 10.16. Polynomials in one variable with real coefficients are continuous. A rational function in one variable with real coefficients \( h(x) = \frac{f(x)}{g(x)} \) is continuous at all \( a \in \mathbb{R} \) where \( g(a) \neq 0 \).

Now we want to look at limits of the composition of functions. It is not quite true in general that if \( \lim_{x \to a} g(x) = M \) and \( \lim_{y \to M} f(y) = L \), then \( \lim_{x \to a} f(g(x)) = L \), but it is true in some cases.

Proposition 10.17. Let \( a \in \mathbb{R} \). Then
\[
\lim_{x \to a} f(x) = \lim_{h \to 0} f(a + h),
\]
assuming that the limit on the left exists. (Hint: You can think of the right hand side as the composition of \( f \) with the function \( g(h) = a + h \).)

Theorem 10.18. If \( \lim_{x \to a} g(x) = M \) and \( f \) is continuous at \( M \), then \( \lim_{x \to a} f(g(x)) = f(M) \).

Remark 10.19. This theorem can also be rewritten as
\[
\lim_{x \to a} f(g(x)) = f \left( \lim_{x \to a} g(x) \right),
\]
which can be remembered as “limits pass through continuous functions.”

Corollary 10.20. If \( g \) is continuous at \( a \) and \( f \) is continuous at \( g(a) \), then \( f \circ g \) is continuous at \( a \).

We now assume the domain \( A \subset \mathbb{R} \) is open.

Definition 10.21. The derivative of \( f \) at a point \( a \in A \) is the number \( f'(a) \) defined by the following limit:
\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h},
\]
provided the limit on the right hand side exists. If \( f'(a) \) exists, we say that \( f \) is differentiable at \( a \). If \( f \) is differentiable at all points of its domain, we say that \( f \) is differentiable. In this case, the values \( f'(a) \) define a new function \( f' : A \to \mathbb{R} \) called the derivative of \( f \).

Here is a useful reformulation of the definition of the derivative.
Theorem 10.22. If \( f \) is differentiable at \( a \), the derivative of \( f \) at \( a \) is given by the limit:

\[
f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.
\]

Theorem 10.23. If \( f \) is differentiable at \( a \), then \( f \) is continuous at \( a \).

Exercise 10.24. Show that the converse of Theorem 10.23 is not true in general.

Exercise 10.25. (i) Show that for all \( n \in \mathbb{N} \),

\[
x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1}),
\]

or more formally,

\[
x^n - a^n = (x - a) \left( \sum_{i=0}^{n-1} x^{n-1-i}a^i \right).
\]

(ii) Use this to prove that if \( f(x) = x^n \) for some \( n \in \mathbb{N} \), then \( f'(a) = na^{n-1} \).

Exercise 10.26. Suppose that \( f \) and \( g \) are differentiable at \( a \).

(i) Compute \((f + g)'(a)\) in terms of \( f'(a) \) and \( g'(a) \).

(ii) Compute \((fg)'(a)\) in terms of \( f(a) \), \( g(a) \), \( f'(a) \) and \( g'(a) \).

(iii) Compute \( \left( \frac{1}{f} \right)'(a) \) in terms of \( f'(a) \) and \( f(a) \).

These results are known as the Sum Rule, Product Rule, and (a special case of the) Quotient Rule for Derivatives, respectively.

Exercise 10.27. Suppose \( f \) and \( g \) are differentiable at \( a \) and \( g(a) \neq 0 \).

(i) Show that \( \frac{f(x)}{g(x)} \) is continuous at \( a \).

(ii) Show that \( \frac{f(x)}{g(x)} \) is differentiable at \( a \), and compute \( \left( \frac{f}{g} \right)'(a) \) in terms of \( f(a) \), \( g(a) \), \( f'(a) \) and \( g'(a) \).

This last result is known as the Quotient Rule for Derivatives.

One of the most important results concerning the differentiation of functions is the Chain Rule for the derivative of a composition of functions. Let \( f : B \to \mathbb{R} \), \( g : A \to \mathbb{R} \) be functions such that \( g(A) \subset B \). Thus, the composition \((f \circ g)(x) = f(g(x))\) is defined for all \( x \in A \).

Lemma 10.28. Given \( f \) and \( g \) as above, define a new function \( \varphi : B \to \mathbb{R} \) (note that the domain is the same as the domain of \( f \)) by

\[
\varphi(y) = \begin{cases} 
\frac{f(y) - f(g(a))}{y - g(a)} & \text{if } y \neq g(a), \\
f'(g(a)) & \text{if } y = g(a).
\end{cases}
\]

Then \( \varphi \) is continuous at the point \( g(a) \).
Lemma 10.29. With \( f \) and \( g \) as above,

\[
\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} = f'(g(a))
\]

(Hint: Consider the function \( \varphi \circ g \) and distinguish between the two cases when \( g(x) \neq g(a) \) and when \( g(x) = g(a) \).)

Theorem 10.30 (Chain Rule). Let \( a \in A \), suppose that \( g \) is differentiable at \( a \) and \( f \) is differentiable at \( g(a) \). Then \( f \circ g \) is differentiable at \( a \) and:

\[
(f \circ g)'(a) = f'(g(a)) \cdot g'(a)
\]

We finish this sheet with a discussion of maxima and minima of real-valued functions and the most important theorem in differential Calculus, the Mean Value Theorem.

Definition 10.31. Let \( f : A \to \mathbb{R} \) be a function. If \( f(a) \) is the last point of \( f(A) \), then \( f(a) \) is called the maximum value of \( f \). If \( f(a) \) is the first point of \( f(A) \), then \( f(a) \) is the minimum value of \( f \). We say that \( f(a) \) is a local maximum value of \( f \) if there exists a region \( R \) containing \( a \) such that \( f(a) \) is the last point of \( f(A \cap R) \). We say that \( f(a) \) is a local minimum value of \( f \) if there exists a region \( R \) containing \( a \) such that \( f(a) \) is the first point of \( f(A \cap R) \).

Remark 10.32. Equivalently, \( f(a) \) is a local maximum (resp. minimum) value of \( f \) if there exists \( U \) open in \( A \) such that \( f(a) \) is the last (resp. first) point of \( f(U) \).

Theorem 10.33. Let \( f : A \to \mathbb{R} \) be differentiable at \( a \). Suppose that \( f(a) \) is the maximum value or minimum value of \( f \). Then \( f'(a) = 0 \).

Corollary 10.34. Let \( f : A \to \mathbb{R} \) be differentiable at \( a \). Suppose that \( f(a) \) is a local maximum or local minimum value of \( f \). Then \( f'(a) = 0 \).

Theorem 10.35 (Rolle’s Theorem). Suppose that \( f : [a,b] \to \mathbb{R} \) is continuous, differentiable on \((a,b)\) and that \( f(a) = f(b) = 0 \). Then there exists a point \( c \in (a,b) \) such that \( f'(c) = 0 \).

Corollary 10.36 (The Mean Value Theorem). Suppose that \( f : [a,b] \to \mathbb{R} \) is continuous and differentiable on \((a,b)\). Then there exists a point \( c \in (a,b) \) such that:

\[
f(b) - f(a) = f'(c)(b - a).
\]