Exercise 0.0.1  Find the unique prime factorizations of the following positive integers. Feel free to use technological assistance.

- 1,000,000
- 1,000,001
- 4,999,999
- 123,456,789

Exercise 0.0.2  Use the Euclidean Algorithm to find the greatest common divisors \((a, b)\) for the following \(a\) and \(b\):

- \(a = 233\) and \(b = 377\)
- \(a = 3,657,329\) and \(b = 1,348,867\)
- \(a = d - 1\) and \(b = d + 1\) for some \(d \in \mathbb{Z}\).

Exercise 0.0.3  Let \(\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\} \). Define addition and multiplication in \(\mathbb{Z}[\sqrt{-5}]\) as follows:

\[
(a + b\sqrt{-5}) + (c + d\sqrt{-5}) = (a + c) + (b + d)\sqrt{-5}
\]

\[
(a + b\sqrt{-5}) \cdot (c + d\sqrt{-5}) = (ac - 5bd) + (ad + bc)\sqrt{-5}
\]

Define the norm map \(N : \mathbb{Z}[\sqrt{-5}] \to \mathbb{Z}\) as follows:

\[
N(a + b\sqrt{-5}) = a^2 + 5b^2
\]

We say that an element \(z \in \mathbb{Z}[\sqrt{-5}]\) is *prime* if there do not exist \(x, y \in \mathbb{Z}[\sqrt{-5}]\), with both \(N(x) > 1\) and \(N(y) > 1\), such that \(z = xy\).

1. Show that \(\mathbb{Z}[\sqrt{-5}]\) is a commutative ring with identity (see Algebra script if you need the definition).

2. Show that \(N(xy) = N(x)N(y)\) for any \(x, y \in \mathbb{Z}[\sqrt{-5}]\).

3. Show that the only elements \(x \in \mathbb{Z}[\sqrt{-5}]\) with \(N(x) \leq 1\) are \(x = -1, 0, 1\).

4. Show that \(\mathbb{Z}[\sqrt{-5}]\) does not have unique factorization into primes by showing that \(x = 6\) can be factored into primes in two distinct ways.