Some Algebraic Definitions

Definition 0.0.1  A ring is a set $R$ with two binary operations $+$ and $\cdot$ satisfying rules E1–E3, A1–A5, M1–M2, and D from the definition of the integers. In addition (and independent of each other), the ring is said to be:

- commutative if it also satisfies M3
- with identity if it also satisfies M4
- ordered if it also satisfies O1–O4

A field is a set $F$ with two binary operations $+$ and $\cdot$ satisfying rules E1–E3, A1–A5, M1–M4, and D from the definition of the integers as well as:

M5. (Multiplicative Inverses)
For any $a \in F$ with $a \neq 0$, there is an element $a^{-1} \in F$ such that $a \cdot a^{-1} = 1$ and $a^{-1} \cdot a = 1$.

Definition 0.0.2  A group is a set $G$ with a binary operation $*$ satisfying:

E1. (Reflexivity, Symmetry, and Transitivity of Equality)

Reflexivity of Equality  If $a \in G$, then $a = a$.
Symmetry of Equality   If $a, b \in G$ and $a = b$, then $b = a$.
Transitivity of Equality If $a, b, c \in G$ and $a = b$ and $b = c$, then $a = c$.

E2. (Equality and the Group Law)
If $a, b, c \in G$ and $a = b$, then $a * c = b * c$.

G1. (Closure)
If $a, b \in G$, then $a * b \in G$.

G2. (Associativity)
If $a, b, c \in G$, then $(a * b) * c = a * (b * c)$.

G3. (Identity)
There is an element $e \in G$ such that $a * e = a$ and $e * a = a$ for every $a \in G$.

G4. (Inverses)
For any $g \in G$, there is an element $g^{-1} \in F$ such that $g \cdot g^{-1} = 1$ and $g^{-1} \cdot g = 1$. 