Solutions

Exercise 1

1. Give the Taylor formula for a function of two variables up to order 2.

\[ f(x, y) = f(x_0, y_0) + \partial_x f(x_0, y_0)(x - x_0) + \partial_y f(x_0, y_0)(y - y_0) \\
+ \frac{1}{2} \partial_x^2 f(x_0, y_0)(x - x_0)^2 + \frac{1}{2} \partial_y^2 f(x_0, y_0)(y - y_0)^2 \\
+ \partial_{xy} f(x_0, y_0)(y - y_0)(x - x_0) + R_3(x, y) \]

with \( |R_3(x, y)| \leq M((x - x_0)^2 + (y - y_0)^2)^{3/2} \) when \((x - x_0)^2 + (y - y_0)^2 \leq r^2\).

2. Give the chain rule to compute the derivative of \( g(t) = f(x(t), y(t)) \).

\[ \frac{dg}{dt}(t) = \frac{\partial f}{\partial x}(x, y) \frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(x, y) \frac{dy}{dt}(t) \]

3. Classify all the critical points of the function given by \( f(x, y) = xy \) on \( \mathbb{R}^2 \).

We have \( \nabla f = (y, x) \), hence the only critical point is \((0, 0)\). At the origine, we have

\[ \partial_x^2 f(0, 0)\partial_y^2 f(0, 0) - \partial_{xy} f(0, 0)^2 = 0 - 1 = -1 < 0 \]

thus we have a saddle point at the origin.

Exercise 2

1. Find the rectangle of fixed perimeter \( l \) that has the shortest diagonal.

Consider a rectangle whose side are of length \( x \) and \( y \), the perimeter is given by \( l = 2x + 2y \) and the diagonal is of length \( \sqrt{x^2 + y^2} \). Thus we
want to minimize the function \( \sqrt{x^2 + y^2} \) or equivalently \( x^2 + y^2 \) under the constraint \( 2x + 2y = l \).

We use the Lagrange multipliers method, and consider the function \( f(x, y) = x^2 + y^2 - 2\lambda(x + y) \) whose critical points are given by

\[
\nabla f = (2x - 2\lambda, 2y - 2\lambda) = 0.
\]

This gives \( x = \lambda, y = \lambda \) which together with the constraint \( 2x + 2y = 4\lambda \) gives \( \lambda = l/4 \) and \( x = l/4, y = l/4 \). If \( \lambda = l/4 \) we have

\[
f(x, y) = x^2 + y^2 - l(x + y)/2 = (x - l/4)^2 + (y - l/4)^2 - l^2/8 \\
geq -l^2/8 = f(l/4, l/4)
\]

thus \( f \) has a minimum at \( x = l/4, y = l/4 \). The rectangle of fixed perimeter \( l \) that has the shortest diagonal is the square of size \( l/4 \).

2. Find the rectangle of maximum area that can be inscribed in the ellipse of equation \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).

Consider a rectangle inscribed in the ellipse of equation \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), if the coordinates of the summits are \((x, y), (x, -y), (-x, y)\) and \((-y, -x)\), with \( x \geq 0, y \geq 0 \). Its area is \( 4xy \), hence we want to maximize the function \( 4xy \) under the constraint \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \). We use the Lagrange multipliers method and consider the function \( f(x, y) = 4xy - \lambda(\frac{x^2}{a^2} + \frac{y^2}{b^2}) \). The critical points are given by

\[
\nabla f = (4y - 2\lambda x/a^2, 4x - 2\lambda y/b^2) = 0
\]

This gives \( y = \lambda x/2a^2, x = \lambda y/2b^2 \) thus \( \lambda = 2ab \) (the equation gives \( \lambda = \pm 2ab \) but we have assumed that \( x \geq 0 \) and \( y \geq 0 \)), which together with the constraint \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) further gives \( x = a/\sqrt{2} \) and \( y = b/\sqrt{2} \). If we take \( \lambda = 2ab \) then

\[
f(x, y) = 4xy - 2ab(\frac{x^2}{a^2} + \frac{y^2}{b^2}) = -2ab(x/a - y/b)^2 \\
\leq 0 = f(a/\sqrt{2}, b/\sqrt{2})
\]

which proves that the maximum is given at the points \( x = a/\sqrt{2} \) and \( y = b/\sqrt{2} \).
Exercise 3

1. 

\[ A = \iint_{x^2+y^2 \leq R^2} dx \, dy = \int_0^R \int_0^{2\pi} r \, dr \, d\theta = \pi R^2 \]

2. 

\[ V = \iiint_{x^2+y^2+z^2 \leq R^2} dx \, dy \, dz \]

\[ = \int_0^R \int_0^{2\pi} \int_0^\pi \sin \theta \, d\theta \, d\varphi \, r^2 \, dr = 4\pi R^3 / 3 \]

3. 

\[ \mathcal{V} = \iiint_{x^2+y^2 \leq R^2} \mathcal{V} \, dx \, dy \, dz = \int_0^H \int_0^R \int_0^{2\pi} d\theta \, r \, dr \, dz = \pi H R^2 \]

4. 

\[ W = \iiint_{\frac{x^2+y^2}{R^2} \leq \frac{z^2}{H^2}} dx \, dy \, dz \]

\[ = \int_0^H \pi \frac{R^2 z^2}{H^2} \, dz = \frac{\pi R^2 H}{3} \]