Instructions: This exam has a total of 100 points, 120 including extra credit. You have 50 minutes. Try to show your work, so that if you make a mistake you will at least get partial credit.

(10) 1. Calculate the following:
   - \( \sin \pi/4 \)
   - \( \cos 2\pi \)
   - \( \tan \pi \)
   - \( \csc \pi/2 \)
   - \( \cot 5\pi/4 \)

Solution:
   - \( \sin \pi/4 = 1/\sqrt{2} \)
   - \( \cos 2\pi = 1 \)
   - \( \tan \pi = 0 \)
   - \( \csc \pi/2 = 1 \)
   - \( \cot 5\pi/4 = 1 \)

(10) 2. Prove \( \tan^2 x + 1 = \sec^2 x \).

Solution:
\[
\tan^2 x + 1 = \frac{\sin^2 x}{\cos^2 x} + 1 = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = 1/\cos^2 x = \sec^2 x,
\]
where we use the pythagorean identity \( \sin^2 x + \cos^2 x = 1 \).

(20) 3. Use the fact that \( \sin(a + b) = \sin a \cos b + \cos a \sin b \) to prove that \( f(x) = \sin x \) is continuous everywhere. You may assume that \( \lim_{x \to 0} \sin x = 0 \) and that \( \lim_{x \to 0} \cos x = 1 \).

Solution:
We need to show that for any \( c \lim_{x \to c} \sin x = \sin c \). Note that \( \lim_{x \to c} \sin x = \lim_{h \to 0} \sin(c + h) = \lim_{h \to 0} \sin c \cos h + \cos c \sin h = \sin c \lim_{h \to 0} \cos h + \cos c \lim_{h \to 0} \sin h = \sin c(1) + \cos c(0) = \sin c \). So we are done.

(10) 4. Calculate \( \lim_{t \to 0} \frac{\sin 5t}{8t} \).

Solution:
\[
\lim_{t \to 0} \frac{\sin 5t}{8t} = \lim_{t \to 0} \frac{5 \sin 5t}{8t} = \frac{5}{8} \lim_{t \to 0} \frac{\sin 5t}{5t} = \frac{5}{8} (1) = \frac{5}{8}.
\]
(10) 5. Calculate, using the definition of derivative, the derivative of \( f(x) = \cos x \) at \( c = 0 \).

**Solution:**

By the definition of derivative, 
\[
 f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\cos x - \cos 0}{x - 0} = \lim_{x \to 0} \frac{\cos x - 1}{x} = 0.
\]

(10) 6. Find the maxima and minima of \( f(x) = \cos^2 x \) on the interval \([0, \pi]\), and identify where the function is increasing and where it is decreasing.

**Solution:**

\( f'(x) = -2 \cos x \sin x \) by the chain rule. \( f'(x) = 0 \) at \( x = \pi/2 \) (remember, we only consider the derivative at interior points of the interval), so the points we need to check are the \( 0, \pi/2, \pi \) (endpoints and stationary point). At \( x = 0 \) and \( \pi \), \( f(x) = \cos^2 x = 1 \), and at \( x = \pi/2 \), \( f(x) = 0 \). So \( f \) achieves its maximum value 1 at 0, \( \pi \) and its minimum value 0 at \( x = \pi/2 \). \( f'(x) < 0 \) for \( x \in (0, \pi/2) \) and \( f'(x) > 0 \) for \( x \in (\pi/2, \pi) \) so by the monotonicity theorem, \( f \) is decreasing on the first interval and increasing on the second.

(10) 7. Find the maxima and minima of \( f(x) = |x| \) on the interval \([-1, 2]\), and identify where the function is increasing and where it is decreasing.

**Solution:**

\( f'(x) = 1 \) for \( x > 0 \), \( f'(x) = -1 \) for \( x < 0 \) and \( f'(0) \) does not exist. So the points we must check are \(-1, 0, 2\), endpoints and a singular point. \( f(-1) = 1 \), \( f(0) = 0 \), and \( f(2) = 2 \), so \( f \) achieves its maximum value 2 at \( x = 2 \) and its minimum value 0 at \( x = 0 \). It is increasing on \((0, 2)\) and decreasing on \((-1, 0)\).

(10) 8. Suppose we are given 20 meters of fencing, and we want to make a rectangular fence. What is the maximum area we can enclose?

**Solution:**

Let us suppose the length of one side is \( x \) meters. Then the length of the other side must be \( y = 10 - x \) meters, since the perimeter has to be 20 meters, and so \( 2x + 2y = 20 \), i.e., \( x + y = 10 \), or \( y = 10 - x \). Thus the area can be written \( A(x) = x(10 - x) \), where \( x \) ranges from 0 to 10. So we want to maximize \( A(x) \) on the interval \([0, 10]\). \( A'(x) = 10 - 2x \), so \( A'(x) = 0 \) when \( x = 5 \). So we need to check \( A(x) \) for \( x = 0, 5, 10 \). \( A(0) = A(10) = 0 \), and \( A(5) = 25 \). Thus the maximum area we can enclose is 25 square meters.

(10) 9. Suppose we drive from Chicago to New York City, a distance of approximately 900 miles, in a smooth manner (i.e., continuous and differentiable), and it takes us 15 hours. What does the mean value theorem tell us about our speed?

**Solution:**

The MVT tells us that at some point in our journey we were traveling exactly at \( 900/15 = 60 \) miles per hour.

(20) 10. (Extra credit) Suppose that \( f \) is continuous on \([a, b]\), and differentiable on \((a, b)\), with \( f'(x) > 0 \) for \( x \in (a, b) \). Prove that \( f \) is strictly increasing on \([a, b]\). Hint: Use the mean value theorem.
Solution:
To show $f$ is strictly increasing on $[a, b]$ we must show that given any $x_1, x_2 \in [a, b], x_1 < x_2$, that $f(x_1) < f(x_2)$. Consider the interval $[x_1, x_2]$. $f$ is continuous on this interval, and differentiable on the open interval $(x_1, x_2)$. So, by the MVT, there is a point $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$ 

We know $f'(c) > 0$, and that $x_2 - x_1 > 0$, so we get that $f(x_2) - f(x_1) > 0$, i.e., that $f(x_2) > f(x_1)$ as desired.