1. Determine whether each of the following sets is countable or not.

(a) The set of sequences of rational numbers converging to zero.
(b) The set of quadratic polynomials with rational coefficients.
(c) The set of polynomials with integers as coefficients.
(d) The set of polynomials with algebraic numbers as coefficients.
(e) The set of complex numbers.
(f) The set of transcendental numbers.
(g) The set of power series in $x$ whose coefficients are all in the set \{0, 1\}.
(h) The set of power series in $x$ with integers as coefficients which converge whenever $|x| < 1$.
(i) The set of power series in $x$ with integers as coefficients which converge whenever $|x| < 2$.

2. Give a critique of the following argument that $1 = 0$.

**Argument:** For any natural number $n$,

$$1 = \sum_{k=1}^{n} \frac{1}{n}.$$ 

It follows that

$$1 = \lim_{n \to \infty} 1 = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{n} = \sum_{k=1}^{n} \lim_{n \to \infty} \frac{1}{n} = \sum_{k=1}^{n} 0 = 0.$$ 

3. Suppose that $(f_n)$ is a sequence of functions on the real line which converge uniformly to $f$ and satisfy

$$|f_n(x) - f_n(y)| \leq |x - y|$$

for all $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$. Show that $f$ also satisfies

$$|f(x) - f(y)| \leq |x - y|$$

for all $x, y \in \mathbb{R}$.

4. Is it true that, if $(f_n)$ is a sequence of continuous functions on $[0, 1]$ which converges uniformly on $[0, 1)$, then it converges uniformly on $[0, 1]$? Give a proof or counterexample.
5. Suppose \((f_n)\) is a sequence of functions on \([0, 1]\) with the property that there is a continuous function \(f : [0, 1] \to \mathbb{R}\) such that, for any sequence \((a_n)\) of elements of \([0, 1]\) converging to \(a \in [0, 1]\), we have

\[
\lim_{n \to \infty} f_n(a_n) = f(a).
\]

Show that \((f_n)\) converges uniformly to \(f\).

6. Determine whether each of the following series converges (1) pointwise for any real number and (2) uniformly on the real line.

(a) \[
\sum_{n=1}^{\infty} \frac{(x + 2n)^n}{n^{2n}}
\]

(b) \[
\sum_{n=1}^{\infty} (-1)^n n^{-1} e^{-nx^2}
\]

(c) \[
\sum_{n=1}^{\infty} \frac{x^3}{x^2 + n^2}
\]

Do problems 17 and 18 in Spivak, Chapter 24.