Solve the following problems, writing your answers clearly and explaining them as completely as possible.

1. If $F$ is a field, show that

$$F[x, x^{-1}] = \{ \sum_{k=-N}^{N} a_k x^k | a_k \in F, N \geq 0 \}$$

is a Euclidean domain.

2. Suppose $F \subset K$ are fields, and $K$ is $n$-dimensional as a vector space over $F$. Given $\lambda \in K$, show that the image of the homomorphism $\varphi : F[x] \rightarrow K$ given by $\varphi(q(x)) = q(\lambda)$ is a subfield of $K$.

3. How many conjugacy classes of elements of order 5 are there in the group $GL_4(\mathbb{F}_2)$? What are the rational canonical forms of the matrices in these conjugacy classes?

4. (a) Determine the number of isomorphism classes of $\mathbb{Z}[i]$-modules with 26 elements.

(b) Determine the number of isomorphism classes of $\mathbb{Z}[i]$-modules with 35 elements.

5. (a) Let $F$ be a field, and $\alpha_1, \ldots, \alpha_n$ be $n$ distinct elements of $F$. Show that

$$\frac{F[x]}{(\prod_{i=1}^{n} (x - \alpha_i))} \cong \bigoplus_{1}^{n} \frac{F[x]}{(x - \alpha_i)}$$

as $F[x]$-modules. (Note: the Chinese Remainder Theorem says this holds as an equality of rings.)

(b) Show that if $T$ is a linear operator on a finite dimensional vector space $V$ over $F$, and if the minimal polynomial of $T$ splits into a product of distinct linear factors, then $T$ is diagonalisable.

6. Find the number of solutions of the equation $x^7 + 1 = 0$ in the finite fields $\mathbb{F}_{71}$ and $\mathbb{F}_{89}$. 