Reading. Rudin, Chapter 3; Probability notes, Sections 3 and 4.

Problems Rudin: Chapter 3: 4, 5, 9, 11, 16, 23, 25
(Note: Problem 5 is really a problem about random variables, and you might want to “translate” the problem into probability notation.)

Probability Notes: Exercise 4.1.

Exercise 1 Let $X$ be a random variable with $E[|X|] < \infty$. Let $X_n = X 1\{|X| \leq n\}$. Show that if $\alpha > 1$,
$$
\lim_{n \to \infty} \frac{E[|X_n|^\alpha]}{n^{\alpha-1}} = 0.
$$

Exercise 2 Suppose $X_1, X_2, \ldots$ are independent random variables each with mean zero. Let $S_n = X_1 + \cdots + X_n$.

Prove that for every $\lambda > 0$,
$$
P \left\{ \omega : \max_{1 \leq j \leq n} S_j(\omega) \geq \lambda \right\} \leq \frac{E[S_n^2]}{\lambda^2}.
$$

(Hint: Let $T = \min\{j : S_j \geq \lambda\}$, let $A_j$ be the event $\{T = j\}$. Show that for every $j \leq n$,
$$
E [S_n^2 1_{A_j}] \geq E [S_j^2 1_{A_j}] \geq \lambda^2 P(A_j).
$$

Exercise 3 Suppose $X_1, X_2, \ldots$ are independent, identically distributed random variables with mean zero and let $\hat{X}_n = X_n 1\{|X| \leq n\}$,
$$
\hat{S}_n = \hat{X}_1 + \cdots + \hat{X}_n.
$$

The goal of this exercise is to show
$$
\sum_{n=1}^{\infty} 2^{-2n} \operatorname{Var}[S_{2^n}] < \infty,
$$
by proving the following steps.

1. For each $n$,
$$
\operatorname{Var}[S_{2^n}] \leq 2^n E \left[ X_1^2 1\{|X| \leq 2^n\} \right].
$$

2.
$$
\sum_{n=1}^{\infty} 2^{-2n} \operatorname{Var}[S_{2^n}] \leq E \left[ X_1^2 \sum_{n=1}^{\infty} 2^{-n} 1\{|X| \leq 2^n\} \right].
$$

3.
$$
\sum_{n=1}^{\infty} 2^{-n} 1\{|X| \leq 2^n\} \leq \frac{1}{|X_1|}.
$$